Graph Labeling and Radio Channel Assignment

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Abstract: The vertex-labeling of graphs with nonnegative integers provides a natural setting in which to study problems of radio channel assignment. Vertices correspond to transmitter locations and their labels to radio channels. As a model for the way in which interference is avoided in real radio systems, each pair of vertices has, depending on their separation, a constraint on the difference between the labels that can be assigned. We consider the question of finding labelings of minimum span, given a graph and a set of constraints. The focus is on the infinite triangular lattice, infinite square lattice, and infinite line lattice, and optimal labelings for up to three levels of constraint are obtained. We highlight how accepted practice can lead to suboptimal channel assignments. © 1998 John Wiley & Sons, Inc.

Keywords: graph labeling, radio channel assignment, optimal labeling, graph distance, minimum span

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1. INTRODUCTION

This article is concerned with labelings of the vertices of graphs using nonnegative integers. Motivation comes from the channel assignment problem in radio systems, where each vertex is taken to be a transmitter location, with the label assigned to it determining the channel on which it transmits. In any particular application, the available channels are uniformly spaced in the spectrum, justifying integer labelings. We shall give special attention to regular lattices, which form the initial framework for many exercises in radio spectrum planning [5, 12]. The physical assumptions involved when using lattices are that all transmitters are identical, and signal propagation is isotropic and independent of frequency. Local variations in signal propagation and restrictions on transmitter placement generally mean that the regular structure is distorted when the plan is actually implemented. Nevertheless, lattice planning remains an important tool for radio engineers and from now on we will accept its inherent physical assumptions.

Suppose that a radio receiver is tuned to a signal on channel \( c_0 \), broadcast by its local transmitter (i.e., the one at closest distance). Reception will be degraded if there is excessive interference from other transmitters in the vicinity. First, there is ‘co-channel’ interference, due to re-use of channel \( c_0 \) at nearby sites; but there are also contributions from sites using channels near \( c_0 \), since in practice neither transmitters nor receivers operate exclusively within the frequencies of their assigned channels.

To ensure acceptable signal quality, constraints are imposed on the allowed channel separations between pairs of potentially interfering transmitters. We will assume that there is a monotonic trade-off between distance and spectral separation, in the sense that, for any two pairs of transmitter sites \( (x_1, y_1) \) and \( (x_2, y_2) \), if \( d(x_1, y_1) \geq d(x_2, y_2) \), then \( \delta_c(x_1, y_1) \leq \delta_c(x_2, y_2) \), where \( d(x, y) \) is the distance between \( x \) and \( y \) in some suitable metric, and \( \delta_c(x, y) \) is the minimum allowed spectral separation of the channels assigned to sites \( x \) and \( y \). Moreover, the physical assumptions made above imply that the spectral constraints are determined only by distances, in the sense that if \( d(x_1, y_1) = d(x_2, y_2) \), then \( \delta_c(x_1, y_1) = \delta_c(x_2, y_2) \).

From now on, we shall take the transmitter sites to correspond to the vertex set \( V(G) \) of a graph \( G \), and use graph distance \( d_G(u, v) \) to specify their distances. Hence, the edge set \( E(G) \) determines all distances between them. Note that \( G \) may have an infinite number of vertices.

**Definition 1.1.** Given a graph \( G \), an \( n \)-labeling of \( G \), where \( n \) is a positive integer, is a function \( \phi : V(G) \rightarrow \{0, 1, \ldots, n - 1\} \).

The labels \( \phi(G) \) will be interpreted as channels assigned to the vertices of the graph. To talk about spectral separation, we need to provide a metric on the set of channels. In this article we use the following ‘cyclic channel distance’.

**Definition 1.2.** For a positive integer \( n \) and \( k, \ell \in \{0, 1, \ldots, n - 1\} \), define \( |k - \ell|_n = \min\{|k - \ell|, n - |k - \ell|\} \).
Constraints may be specified by attaching a list of spectral separations \((k_1, k_2, \ldots, k_p)\) to graph distances \(1, 2, \ldots, p\). Set \(P_G(i) = \{(u, v) | u, v \in V(G), d_G(u, v) = i\}\).

**Definition 1.3.** Given a graph \(G\) and integers \(k_1 \geq k_2 \geq \cdots \geq k_p \geq 1\), we say that an \(n\)-labeling \(\phi\) of \(G\) satisfies the constraints \((k_1, \ldots, k_p)\) if \(|\phi(u) - \phi(v)|_n \geq k_i\), for all \(i \in \{1, \ldots, p\}\) and \((u, v) \in P_G(i)\).

The requirement \(k_1 \geq k_2 \geq \cdots \geq k_p\) in Definition 1.3 arises from the assumption of a monotonic trade-off between distance and spectral separation.

A convenient way to interpret Definition 1.2 is to think of the labels 0, 1, \ldots, \(n - 1\) as consecutive vertices on a cycle \(C_n\). Then, for any distinct pair \(k, \ell \in \{0, 1, \ldots, n - 1\}\), the distance between \(k\) and \(\ell\), \(|k - \ell|_n\), is the length of the shorter of the two paths connecting \(k\) and \(\ell\) on the cycle.

Cyclic channels allow the assignment of a collection of channels \(\phi(u)\), \(\phi(u) + n\), \(\phi(u) + 2n\), \ldots to each site. The possibility of providing multiple coverage in this way is important in, for example, large communication systems that serve many customers simultaneously. This property was noted in [5].

Cyclic channel distance also appears, in a certain sense, in the concept of the star chromatic number of a graph [2, 17]. Following Definition 1.3, we can define the star chromatic number of a graph \(G\) as the infimum of \(n/k_1\) over all possible \(n \geq 2k_1 > 0\) such that there exists an \(n\)-labeling of \(G\) that satisfies the constraint \((k_1)\).

Finally, many schemes for labeling that have been developed for the original ‘linear channel distance’ respect the stronger cyclic channel conditions as well. An example is labeling by arithmetic progression discussed later in this article.

**Definition 1.4.** Given a graph \(G\) and constraints \((k_1, \ldots, k_p)\), the span \(\sigma(G; k_1, \ldots, k_p)\) is the smallest \(n\) such that there exists an \(n\)-labeling of \(G\) satisfying the constraints \((k_1, \ldots, k_p)\). If no such \(n\) exists, we set \(\sigma(G; k_1, \ldots, k_p) = \infty\). If \(n = \sigma(G; k_1, \ldots, k_p)\), then any \(n\)-labeling of \(G\) satisfying the constraints \((k_1, \ldots, k_p)\) is called optimal.

As mentioned before, we are mainly interested in the span of infinite graphs arising from certain regular lattices. Define vectors \(e_1 = (1, 0), e_2 = (0, 1), \) and \(\mathbf{f} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\) in the Euclidean plane. Then the **triangular lattice** \(\Lambda_{\Delta}\) is defined by \(\Lambda_{\Delta} = \{m_1e_1 + m_2e_2 | m_1, m_2 \in \mathbb{Z}\}\) and the **square lattice** \(\Lambda_{\square}\) by \(\Lambda_{\square} = \{m_1e_1 + m_2e_2 | m_1, m_2 \in \mathbb{Z}\}\). The graphs of the triangular lattice and of the square lattice, denoted by \(\Delta\) and \(\square\), respectively, are in their turn defined by \(V(\Delta) = \Lambda_{\Delta}, E(\Delta) = \{uv | u, v \in \Lambda_{\Delta}, d_E(u, v) = 1\}\), \(V(\square) = \Lambda_{\square}\), and \(E(\square) = \{uv | u, v \in \Lambda_{\square}, d_E(u, v) = 1\}\), where \(d_E(u, v)\) denotes the Euclidean distance between \(u\) and \(v\). For brevity, we simply call \(\Delta\) the **triangular lattice** and \(\square\) the **square lattice**, with the metrics induced by the respective edge sets understood.

In order to obtain a better understanding of the span of regular lattices, we also study the span of (the graph of) the one-dimensional regular lattice. For this purpose, define the graph \(\Gamma\) by \(V(\Gamma) = \mathbb{Z}\) and \(E(\Gamma) = \{k\ell | |k - \ell| = 1\}\). We will call \(\Gamma\) the **line lattice**. The problem of determining \(\sigma(\Gamma; k_1, \ldots, k_p)\) is relevant to
channel assignment in one space dimension, which in practice might be a road or rail corridor.

In the next section, we discuss some results for general graphs. Unless otherwise stated, we will assume that the span is finite. This is certainly true if the underlying graph is finite, but may fail for infinite graphs.

In Section 3, we investigate the span of the triangular and square lattice defined above. The triangular lattice is the more important to the radio engineer, since, if the area of coverage (in the Euclidean plane) of each transmitter is a disk of fixed radius \( r \) centered on the transmitter site, then placing those sites at the vertices of a regular triangular lattice (with adjacent sites a distance \( r\sqrt{3} \) apart) covers the whole plane with the smallest possible transmitter density, as shown in Fig. 1. The main result of Section 3 shows that both \( \sigma(\triangle; k_1, k_2) \) and \( \sigma(\Box; k_1, k_2) \) are achieved by assigning channels in arithmetic progression along each line of vertices in the lattice. In order to illustrate some of our methods further, we show in Section 4 that we are able to determine \( \sigma(\Gamma; k_1, k_2, k_3) \) using assignments that are periodic though not always in arithmetic progression. Section 5 looks further at periodic assignments, and Section 6 contains proofs that were omitted from Sections 3 and 4 in the interest of clarity.

2. SPANS OF GRAPHS

Concepts related to the span of a graph, using the cyclic channel condition, can be found in the literature (see, e.g., [9] and references therein). Several authors have studied the problem of determining for a graph \( G \) the minimum \( n \) for which there exists an integer \( n \)-labeling of \( V(G) \) with adjacent vertices having labels that differ by at least two, and vertices at distance two having labels that differ by at least one \([4, 7, 8, 19]\). A generalization to larger differences between labels can be found in [6]. Definition 1.3 differs from these concepts in that we allow constraints on larger distances and we use a cyclic channel condition.

We first give, without proofs, some straightforward properties of the span. By \((k)^p\) we mean the \(p\)-tuple \((k, k, \ldots, k)\).

![Figure 1. The triangular lattice, \( \Lambda_\triangle \), with the regions of coverage of the transmitters on the left and the corresponding graph, \( \triangle \), on the right.](image)
Proposition 2.1. For all graphs $G$ and constraints $(k_1, \ldots, k_p)$ and $(k_1', \ldots, k_p')$ with $p \geq p'$ and $k_i \geq k_i'$ for $i \in \{1, \ldots, p\}$ we have $\sigma(G; k_1, \ldots, k_p) \geq \sigma(G; k_1', \ldots, k_p')$.

Corollary 2.1. For all graphs $G$ and constraints $(k_1, \ldots, k_p)$,
$$\max\{\sigma(G; (k_i)^i) \mid i = 1, \ldots, p\} \leq \sigma(G; k_1, \ldots, k_p) \leq \sigma(G; (k_1)^p).$$

The chromatic number of a graph $G$, denoted $\chi(G)$, is the minimum number of colors needed to color the vertices of the graph such that adjacent vertices have a different color. The clique number of $G$, denoted $\omega(G)$, is the largest $n$ such that $G$ contains the complete graph $K_n$ as a subgraph. If $p$ is a positive integer, then the $p$-th power of $G$, denoted $G^p$, is the graph with vertex set $V(G^p) = V(G)$ and edge set $E(G^p) = \{uv \mid u, v \in V(G), d_G(u, v) \leq p\}$. With these definitions we immediately obtain the following.

Proposition 2.2. For all graphs $G$ and positive integers $k$ and $p$, $\sigma(G; (1)^p) = \chi(G^p)$ and $k \cdot \omega(G^p) \leq \sigma(G; (k)^p) \leq k \cdot \chi(G^p)$.

Although Proposition 2.2 suggests a link between the span of a graph and its chromatic number, this connection seems to exist only for very special cases. There are classes of graphs for which the chromatic number seems much easier to compute than the span. For instance, for the $n$-dimensional cube $Q_n$, we know $\chi(Q_n) = 2$, but the span of $Q_n$ with constraints at distance 1 and 2 seems to be very hard to determine (see [7]).

Theorem 2.1. Let $G$ be a graph, $(k_1, \ldots, k_p)$ a list of constraints, and $d$ a positive integer. Then
$$d \cdot (\sigma(G; k_1, \ldots, k_p) - 1) + 1 \leq \sigma(G; d \cdot k_1, d \cdot k_2, \ldots, d \cdot k_p) \leq d \cdot \sigma(G; k_1, \ldots, k_p).$$

Proof. Let $\phi$ be an $n$-labeling of $G$ satisfying the constraints $(k_1, \ldots, k_p)$. Then the labeling $\phi' : V(G) \to \{0, 1, \ldots, dn - 1\}$, defined by $\phi'(u) = d \phi(u)$, satisfies the constraints $(d \cdot k_1, \ldots, d \cdot k_p)$, thus proving the second inequality.

On the other hand, if $\psi$ is an $n$-labeling of $G$ satisfying the constraints $(d \cdot k_1, \ldots, d \cdot k_p)$, then $\psi' : V(G) \to \{0, 1, \ldots, \lfloor n/d \rfloor - 1\}$, defined by $\psi'(u) = \lfloor \psi(u)/d \rfloor$, is a labeling satisfying $(k_1, \ldots, k_p)$. Since $d \cdot \lfloor n/d \rfloor \leq n + d - 1$, we obtain the first inequality.

Theorem 2.1 admits no sharpening, as can be seen by considering $G = C_5$. Straightforward analysis shows that $\sigma(C_5; 1) = 3, \sigma(C_5; 2) = 5, \sigma(C_5; 3) = 8,$ and $\sigma(C_5; 4) = 10$. This means that $2(\sigma(C_5; 1) - 1) + 1 = \sigma(C_5; 2)$ (equality on the left-hand side), $\sigma(C_5; 4) = 2\sigma(C_5; 2)$ (equality on the right-hand side), and $3(\sigma(C_5; 1) - 1) + 1 < \sigma(C_5; 3) < 3\sigma(C_5; 1)$ (strict inequality on both sides).

That $\sigma(G; d \cdot k_1, \ldots, d \cdot k_p)$ may be strictly less than $d \cdot \sigma(G; k_1, \ldots, k_p)$ is surprising in view of the following result. Given a graph $G$ and constraints $(k_1, \ldots, k_p)$, we define the linear span $\sigma_L(G; k_1, \ldots, k_p)$ of $G$ as the smallest $n$ such that there exists an $(n + 1)$-labeling of $G$ satisfying $|\phi(u) - \phi(v)| \geq k_i$, for all $i \in \{1, \ldots, p\}$ and $(u, v) \in P_G(i)$. 

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Proposition 2.3. Let $G$ be a graph, $(k_1, \ldots, k_p)$ a list of constraints, and $d$ a positive integer. Then

$$\sigma_L(G; d \cdot k_1, d \cdot k_2, \ldots, d \cdot k_p) = d \cdot \sigma_L(G; k_1, \ldots, k_p).$$

Proof. This proposition is a more general version of [8, Lemma 2.1] and [6, Lemma 2.2] (where the case $p = 2$ is considered), and the proof runs completely similar.

It is straightforward to show the following relation between the cyclic and linear spans of a graph:

$$\sigma(G; k_1, \ldots, k_p) - k_1 \leq \sigma_L(G; k_1, \ldots, k_p) \leq \sigma(G; k_1, \ldots, k_p) - 1.$$

In general, without more specific knowledge about the constraints $(k_1, \ldots, k_p)$, this result is the best one can do. This should be kept in mind when comparing our results with the results on linear spans as they appear in, e.g., [6].

The significance of the last result in this section is in showing that the span of an infinite graph is determined by the span of (sufficiently large) finite subgraphs.

Theorem 2.2. Let $G$ be a graph and $(k_1, \ldots, k_p)$ a list of constraints. If for every induced, finite subgraph $G'$ of $G$ we have $\sigma(G'; k_1, \ldots, k_p) \leq m$ for some integer $m$, then $\sigma(G; k_1, \ldots, k_p) \leq m$.

Proof. The proof is a variation of the proof of the famous De Bruijn–Erdős Theorem [3] that an infinite graph is $k$-colorable if and only if every finite subgraph is $k$-colorable. Let $\{G_i \mid i \in I\}$ be the collection of induced, finite subgraphs of $G$ and, for each $i \in I$, let $\phi_i$ be an $s$-labeling of $G_i$ satisfying the constraints $(k_1, \ldots, k_p)$. Following Rado’s Selection Principle ([15], see also [13, Chapter 4]) there exists an $s$-labeling $\phi$ of $G$ such that for each $i \in I$ there exists a $j_i \in I$ with $V(G_i) \subseteq V(G_{j_i})$ and $\phi(v) = \phi_{j_i}(v)$ for all $v \in V(G_i)$. We must prove that $\phi$ satisfies the constraints $(k_1, \ldots, k_p)$. Let $(u, v) \in P_G(m)$ for some $m \in \{1, \ldots, p\}$. Then clearly there exists an induced, finite subgraph $G_i$ of $G$ such that $(u, v) \in P_{G_i}(m)$. Since $G_{j_i}$ satisfies $V(G_i) \subseteq V(G_{j_i})$, and also $(u, v) \in P_{G_{j_i}}(m)$, we have $|\phi_{j_i}(u) - \phi_{j_i}(v)|_s \geq k_m$. Combined with $\phi(u) = \phi_{j_i}(u)$ and $\phi(v) = \phi_{j_i}(v)$, this proves $|\phi(u) - \phi(v)|_s \geq k_m$.

3. SPANS OF 2-DIMENSIONAL LATTICES

In this section we study the spans of the triangular and the square lattice.

The proof of the next lemma can be found in Section 6. Recalling the definition of the power $G^p$ of a graph $G$, we have the following.

Lemma 3.1. For positive integers $p$, $\omega(\Delta^p) = \chi(\Delta^p) = \lceil \frac{3}{4}(p+1)^2 \rceil$ and $\omega(\Box^p) = \chi(\Box^p) = \lceil \frac{1}{2}(p+1)^2 \rceil$.

Corollary 3.1. For constraints $(k_1, \ldots, k_p)$,
max\{k_i \cdot \left\lceil \frac{3}{2}(i+1)^2 \right\rceil \mid i = 1, \ldots, p\} \leq \sigma(\triangle; k_1, \ldots, k_p) \leq k_1 \cdot \left\lceil \frac{3}{2}(p+1)^2 \right\rceil,
max\{k_i \cdot \left\lceil \frac{1}{2}(i+1)^2 \right\rceil \mid i = 1, \ldots, p\} \leq \sigma(\square; k_1, \ldots, k_p) \leq k_1 \cdot \left\lceil \frac{1}{2}(p+1)^2 \right\rceil.

**Proof.** The result follows by combining Corollary 2.1, Proposition 2.2, and Lemma 3.1.

**Corollary 3.2.** For positive integers \(k\) and \(p\), \(\sigma(\triangle; (k)^p) = k \cdot \left\lceil \frac{3}{2}(p+1)^2 \right\rceil\), and \(\sigma(\square; (k)^p) = k \cdot \left\lceil \frac{1}{2}(p+1)^2 \right\rceil\).

The main result of this section is the following.

**Theorem 3.1.** For constraints \((k_1, k_2)\),

\[
\sigma(\triangle; k_1, k_2) = \begin{cases} 
3k_1 + 3k_2, & \text{if } k_1 \geq 2k_2, \\
9k_2, & \text{if } k_1 \leq 2k_2 \text{ and } 2k_1 \geq 3k_2, \\
4k_1 + 3k_2, & \text{if } 2k_1 \leq 3k_2,
\end{cases}
\]

and \(\sigma(\square; k_1, k_2) = 2k_1 + 3k_2\).

The result for \(\sigma(\square; k_1, k_2)\) has a counterpart in the values of the linear spans of the products of finite graphs given in [6, Section 4]. The use of the linear channel separation leads to several cases depending on the relative sizes of \(k_1\) and \(k_2\), much the same as our result for \(\sigma(\triangle; k_1, k_2)\).

In a moment we will show that there exist labelings of \(\triangle\) and \(\square\) satisfying \((k_1, k_2)\) and using the number of labels indicated in Theorem 3.1, thus giving upper bounds for \(\sigma(\triangle; k_1, k_2)\) and \(\sigma(\square; k_1, k_2)\). The proof that the same values are also lower bounds can be found in Section 6, thus completing the proof of the theorem.

**Definition 3.1.** We say that an \(n\)-labeling \(\phi\) of \(\triangle\) is a labeling by arithmetic progression if there exist nonnegative integers \(a\) and \(b\) such that \(\phi(m_1 \mathbf{e}_1 + m_2 \mathbf{f}) = am_1 + bm_2 \pmod{n}\) for all \(m_1 \mathbf{e}_1 + m_2 \mathbf{f} \in V(\triangle)\). Similarly, an \(n\)-labeling \(\phi\) of \(\square\) is a labeling by arithmetic progression if there exist nonnegative integers \(a\) and \(b\) such that \(\phi(m_1 \mathbf{e}_1 + m_2 \mathbf{f}_2) = am_1 + bm_2 \pmod{n}\) for all \(m_1 \mathbf{e}_1 + m_2 \mathbf{f}_2 \in V(\square)\).

If we want to specify the parameters of a labeling by arithmetic progression, we will speak of an \((n; a, b)\)-labeling.

**Proposition 3.1.** Given constraints \((k_1, k_2)\), the \((n; a, b)\)-labeling with parameters

\[
n = 3k_1 + 3k_2, \quad a = 2k_1 + k_2, \quad b = k_1, \quad \text{if } k_1 \geq 2k_2, \\
n = 9k_2, \quad a = 5k_2, \quad b = 2k_2, \quad \text{if } k_1 \leq 2k_2 \text{ and } 2k_1 \geq 3k_2, \\
n = 4k_1 + 3k_2, \quad a = 2k_1 + k_2, \quad b = k_1, \quad \text{if } 2k_1 \leq 3k_2
\]
gives a labeling of \(\triangle\) satisfying the constraints \((k_1, k_2)\).
The \((n; a, b)\)-labeling with parameters \(n = 2k_1 + 3k_2, a = k_1, b = k_1 + k_2\) is a labeling of \(\square\) satisfying the constraints \((k_1, k_2)\).

**Proof.** Let \(\phi\) be an \((n; a, b)\)-labeling of \(\triangle\). Then we immediately have that 
\[
|\phi(v) - \phi(u)|_n = |\phi(v + w) - \phi(u + w)|_n \quad \text{for all } u, v, w \in V(\triangle),
\]

hence 
\[
|\phi(v) - \phi(u)|_n = |\phi(v - u) - \phi(0)|_n \quad \text{and } |\phi(v) - \phi(0)|_n = |\phi(-v) - \phi(0)|_n.
\]

Together with the fact that \(\phi(0) = 0\), this means that, if we want to check whether \(\phi\) satisfies the constraints \((k_1, k_2)\), then we have to check only that \(k_1 \leq \phi(v) \leq n - k_1\) for \(v \in \{e_1, f, -e_1 + f\}\), and \(k_2 \leq \phi(v) \leq n - k_2\) for \(v \in \{2e_1, e_1 + f, 2f, -e_1 + 2f, -2e_1 + 2f, -2e_1 + f\}\). This check is straightforward and left to the reader. \(\square\)

An important conclusion from Theorem 3.1 and Proposition 3.1 is that it is possible to find labelings by arithmetic progression for both the triangular and the square lattice, satisfying the constraints \((k_1, k_2)\), which are optimal. This observation motivates the following definition.

**Definition 3.2.** Given constraints \((k_1, \ldots, k_p)\), the arithmetic span \(\sigma_{AP}(\triangle; k_1, \ldots, k_p)\) is the smallest \(n\) such that there exists an \((n; a, b)\)-labeling of \(\triangle\) satisfying \((k_1, \ldots, k_p)\). The arithmetic span of \(\square, \sigma_{AP}(\square; k_1, \ldots, k_p)\), is defined analogously.

However, it is no longer true that \(\sigma(\triangle; k_1, \ldots, k_p) = \sigma_{AP}(\triangle; k_1, \ldots, k_p)\) for \(p > 2\). For instance, from Corollary 3.2 we obtain \(\sigma(\triangle; 1, 1, 1) = 12\), whereas a straightforward check reveals that there exist no \(a, b\) such that the \((12; a, b)\)-labeling of \(\triangle\) satisfies \((1, 1, 1)\). Since the \((13; 1, 4)\)-labeling and the \((13; 2, 7)\)-labeling of \(\triangle\) satisfy \((1, 1, 1)\), we obtain \(\sigma_{AP}(\triangle; 1, 1, 1) = 13\). For the square lattice we have a similar example with \(\sigma(\square; 2, 1, 1) = 9\) and \(\sigma_{AP}(\square; 2, 1, 1) = 10\).

A second observation from Theorem 3.1 and Proposition 3.1 is that \(\sigma_{AP}(\triangle; d \cdot k_1, d \cdot k_2) = d \cdot \sigma_{AP}(\triangle; k_1, k_2)\) for all positive integers \(d\) and constraints \((k_1, k_2)\). Again, this is no longer true if we add a constraint at distance three. A computer search shows that \(\sigma_{AP}(\triangle; 5, 2, 1) = 25\) and \(\sigma_{AP}(\triangle; 10, 4, 2) = 49\).

In the remainder of this section we look at some questions related to the computation of spans of infinite graphs. If \(G\) is a finite graph, then the span can be determined by exhaustive search. But it is not clear how to proceed if \(G\) is an infinite graph, and, in particular, how to determine \(\sigma(\triangle; k_1, \ldots, k_p)\) and \(\sigma(\square; k_1, \ldots, k_p)\) for given constraints \((k_1, \ldots, k_p)\). Any labeling satisfying the constraints gives an upper bound for the span. To obtain lower bounds, one can determine \(\sigma(G'; k_1, \ldots, k_p)\) for a finite subgraph \(G'\) of \(G\) and use the obvious fact that \(\sigma(G'; k_1, \ldots, k_p) \leq \sigma(G; k_1, \ldots, k_p)\). We calculated lower bounds on \(\sigma(\triangle; k_1, k_2, k_3)\) for small values of \(k_1, k_2, k_3\) by taking \(G'\) to be \(\triangle_{19}\), the subgraph of \(\triangle\) induced by the 19 vertices at distance at most three from the origin. In fact, having found optimal assignments of \(\triangle_{19}\), in most cases we were able to reveal a pattern that could be extended to an assignment of \(\triangle\) with the same span. Such assignments are clearly optimal for \(\triangle\). Nevertheless, the following general question remains.
**Question 3.1.** Does there exist a finite algorithm to decide the following decision problems:

**INSTANCE:** A positive integer \( n \) and a list of constraints \( (k_1, \ldots, k_p) \).

**QUESTION 1:** Is \( \sigma(\triangle; k_1, \ldots, k_p) \leq n \)?

**QUESTION 2:** Is \( \sigma(\square; k_1, \ldots, k_p) \leq n \)?

There are several ways that a positive answer for this question might arise. At present the proof of Theorem 2.2 is nonconstructive, using Rado’s Selection Principle, which in turn depends on the Axiom of Choice. If it is possible to give a constructive proof of Theorem 2.2 in the case \( G \cong \triangle \) or \( G \cong \square \) and such that, given the list of constraints, we also obtain an upper bound on the size of the needed induced, finite subgraph, then an exhaustive search on this finite subgraph would give the span of the lattice.

In contrast with the problems of determining \( \sigma(\triangle; k_1, \ldots, k_p) \) or \( \sigma(\square; k_1, \ldots, k_p) \), we have the following result if we restrict ourselves to assignments by arithmetic progressions.

**Theorem 3.2.** Let \( (k_1, \ldots, k_p) \) be a list of constraints. Then \( \sigma_{AP}(\triangle; k_1, \ldots, k_p) \) can be determined in polynomial time (in \( k_1 \) and \( p \)). A similar statement holds for \( \sigma_{AP}(\square; k_1, \ldots, k_p) \).

**Proof.** Let \( \phi \) be an \((n; a, b)\)-labeling of \( \triangle \). Then \( |\phi(v) - \phi(u)|_n = |\phi(v - u)|_n \) for all \( u, v \in V(\triangle) \). For \( i \in \{1, \ldots, p\} \), define \( Q(i) \) as the set of vertices in \( V(\triangle) \) at graph distance \( i \) from \( 0 \). By the observation above and the fact that \( \phi(0) = 0 \), in order to check if \( \phi \) satisfies \( (k_1, \ldots, k_p) \) we need to test only that \( k_i \leq \phi(v) \leq n - k_i \) for all \( v \in Q(i) \) and \( i \in \{1, \ldots, p\} \). Straightforward counting gives \( |Q(i)| = 6i \), hence we can verify whether \( \phi \) satisfies the constraints \( (k_1, \ldots, k_p) \) in a number of tests \( f_1(p) \) which is polynomial in \( p \). We are left to prove that the number of triples \( n, a, b \) we have to consider is also polynomial. Given \( n \), it is enough to consider integers \( a, b \) with \( 0 \leq a \leq b \leq n - 1 \), hence at most \( n^2 \) pairs \( a, b \) need be considered. We are done if we can show that the number of values for \( n \) we need to look at is polynomial. Let \( \phi \) be the \((n; a, b)\)-labeling with parameters \( n = (p + 1)^2 k_1, a = k_1 \), and \( b = (p + 1)k_1 \). If \( \mathbf{v} = m_1 \mathbf{e}_1 + m_2 \mathbf{e}_p \in Q(i) \), then \( |m_1| \leq i \) and \( |m_2| \leq i \). This means \( \phi(\mathbf{v}) \equiv 0 \) (mod \( n \)), unless \( \mathbf{v} = 0 \). But since \( n, a, b \) are multiples of \( k_1 \), we obtain \( k_1 \leq \phi(\mathbf{v}) \leq n - k_1 \), hence \( k_i \leq \phi(\mathbf{v}) \leq n - k_i \) for all \( \mathbf{v} \in Q(i) \) and \( i \in \{1, \ldots, p\} \). We find that \( \phi \) satisfies the constraints \( (k_1, \ldots, k_p) \). This gives \( \sigma_{AP}(\triangle; k_1, \ldots, k_p) \leq (p + 1)^2 k_1 \).

Combining everything, we see that at most \( f_1(p) \cdot ((p + 1)^2 k_1)^2 \) steps are needed to compute \( \sigma_{AP}(\triangle; k_1, \ldots, k_p) \).

4. **SPAN OF THE LINE LATTICE**

In this section we study the span of the line lattice \( \Gamma \). Recall that \( V(\Gamma) = \mathbb{Z} \) and \( E(\Gamma) = \{k\ell \mid |k - \ell| = 1\} \). Many problems and approaches used in the study of
the spans of the 2-dimensional lattices in Section 3 will reappear here, but, because of the simpler nature of the line lattice, we will often be able to pursue our approach further.

Our first result can be compared with Theorem 3.1. First define the regions

\[
A = \{(k_1, k_2, k_3) \mid k_1 \geq k_2 \geq k_3 \geq 1, k_1 \geq k_2 + k_3\},
\]

\[
B = \{(k_1, k_2, k_3) \mid k_1 \geq k_2 \geq k_3 \geq 1, k_1 \leq k_2 + k_3, 2k_1 \geq k_2 + 2k_3, 3k_1 \geq 3k_2 + k_3\},
\]

\[
C = \{(k_1, k_2, k_3) \mid k_1 \geq k_2 \geq k_3 \geq 1, 3k_1 \leq 3k_2 + k_3, k_1 \leq 2k_2 - k_3\},
\]

\[
D = \{(k_1, k_2, k_3) \mid k_1 \geq k_2 \geq k_3 \geq 1, 2k_1 \leq k_2 + 2k_3, k_1 \geq 2k_2 - k_3\}.
\]

Since these regions are specified by linear inequalities in \(k_1, k_2, k_3\), they can be drawn in two dimensions using the coordinates \(k_2/k_1\) and \(k_3/k_1\), as shown in Fig. 2(a).

**Theorem 4.1.** For constraints \((k_1), (k_1, k_2)\) and \((k_1, k_2, k_3)\), \(\sigma(\Gamma; k_1) = 2k_1, \sigma(\Gamma; k_1, k_2) = 2k_1 + k_2\), and

\[
\sigma(\Gamma; k_1, k_2, k_3) = \begin{cases} 
2k_1 + k_2, & \text{if } (k_1, k_2, k_3) \in A, \\
3k_2 + 2k_3, & \text{if } (k_1, k_2, k_3) \in B, \\
3k_1 + k_3, & \text{if } (k_1, k_2, k_3) \in C, \\
2k_1 + 2k_2, & \text{if } (k_1, k_2, k_3) \in D.
\end{cases}
\]

Results for the linear span of finite paths with two constraints \((k_1, k_2)\) were given in [6, Section 3].

We first describe \(n\)-labelings satisfying the various constraints and with \(n\) equal to the spans claimed in Theorem 4.1. The proof that the same values are in fact lower bounds can be found in Section 6.

![Figure 2](https://via.placeholder.com/150)

**FIGURE 2.** The regions for \((k_1, k_2, k_3)\) from Theorems 4.1 and 4.2.
In a similar way to Section 3, we call an \( n \)-labeling \( \phi \) of \( \Gamma \) a \textit{labeling by arithmetic progression} if there exists a nonnegative integer \( a \) such that \( \phi(k) = ak \mod n \), for all \( k \in V(\Gamma) \). Such a labeling will be called an \((n; a)\)-labeling. The \textit{arithmetic span} \( \sigma_{AP}(\Gamma; k_1, \ldots, k_p) \) is the smallest \( n \) such that there exists an \((n; a)\)-labeling of \( \Gamma \) satisfying \((k_1, \ldots, k_p)\).

To begin, define the extra regions
\[
B' = \{(k_1, k_2, k_3) \mid k_1 \geq k_2 \geq k_3 \geq 1, k_1 \leq k_2 + k_3, 3k_1 \geq 3k_2 + k_3\}, \\
C' = \{(k_1, k_2, k_3) \mid k_1 \geq k_2 \geq k_3 \geq 1, 3k_1 \leq 3k_2 + k_3\}.
\]

The regions \( A, B', C' \) are shown in Fig. 2(b).

**Theorem 4.2.** For constraints \((k_1), (k_1, k_2)\) and \((k_1, k_2, k_3)\), \( \sigma_{AP}(\Gamma; k_1) = 2k_1, \sigma_{AP}(\Gamma; k_1, k_2) = 2k_1 + k_2 \) and
\[
\sigma_{AP}(\Gamma; k_1, k_2, k_3) = \begin{cases} 
2k_1 + k_2, & \text{if } (k_1, k_2, k_3) \in A, \\
3k_2 + 2k_3, & \text{if } (k_1, k_2, k_3) \in B', \\
3k_1 + k_3, & \text{if } (k_1, k_2, k_3) \in C'.
\end{cases}
\]

**Proof.** Since for any \( n \) and \( a \), the \((n; a)\)- and \((n; n - a)\)-labelings of \( \Gamma \) satisfy the same constraints, we can always assume \( a \leq \frac{1}{4}n \). The case where we only have a constraint at distance one is straightforward. If the \((n; a)\)-labeling satisfies the constraints \((k_1, k_2)\), then we can rewrite the constraints as \( k_1 \leq a \leq \frac{1}{2}n \) and \( k_2 \leq n - 2a \). It is clear that the smallest \( n \) for which there exists an \( a \) satisfying these constraints is \( n = 2k_1 + k_2 \) with \( a = k_1 \).

We split the set of \((n; a)\)-labelings of \( \Gamma \) satisfying \((k_1, k_2, k_3)\) into two parts: those with \( a < \frac{1}{3}n \) and those with \( \frac{1}{3}n \leq a \leq \frac{1}{2}n \). If \( a < \frac{1}{3}n \), then the constraints may be written
\[
k_1 \leq a < \frac{1}{3}n, \quad k_2 \leq n - 2a, \quad k_3 \leq n - 3a.
\]

Let \( N_1(k_1, k_2, k_3) \) be the smallest \( n \) for which there exists an \( a \) satisfying (1). Recalling that \( k_2 \leq k_1 \leq a \leq \frac{1}{3}n \) is clear that \( N_1(k_1, k_2, k_3) = 3k_1 + k_3 \), with the corresponding \( a \) equal to \( k_1 \).

If \( \frac{1}{3}n \leq a \leq \frac{1}{2}n \), the constraints may be written
\[
\max\{\frac{1}{3}n, k_1\} \leq a \leq \frac{1}{2}n, \quad k_2 \leq n - 2a, \quad k_3 \leq 3a - n.
\]

Let \( N_2(k_1, k_2, k_3) \) be the smallest \( n \) for which there exists an \( a \) satisfying (2). The final two inequalities in (2) imply \( a \geq k_2 + k_3 \). It is then clear that \( N_2(k_1, k_2, k_3) = 2k_1 + k_2 \) when \( k_1 \geq k_2 + k_3 \) (with the corresponding \( a \) equal to \( k_1 \)) and \( N_2(k_1, k_2, k_3) = 3k_2 + 2k_3 \) when \( k_1 \leq k_2 + k_3 \) (with the corresponding \( a \) equal to \( k_2 + k_3 \)).

This covers all possible \((n; a)\)-labelings of \( \Gamma \), and so we find in general \( \sigma_{AP}(\Gamma; k_1, k_2, k_3) = \min\{N_1(k_1, k_2, k_3), N_2(k_1, k_2, k_3)\} \), which matches the values given in the theorem.

We see that the values of \( \sigma_{AP}(\Gamma; k_1, k_2, k_3) \) from Theorem 4.2 match the spans claimed in Theorem 4.1, except for the case \((k_1, k_2, k_3) \in D \). (Note that \( B \subseteq B' \) and \( C \subseteq C' \)). For \((k_1, k_2, k_3) \in D \), we need to look at a new type of assignment.
Definition 4.1. An $n$-labeling $\phi$ of $\Gamma$ is called a $t$-periodic $n$-labeling, where $t$ is a positive integer, if $\phi(k + t) = \phi(k)$ for all $k \in V(\Gamma)$.

Note that a labeling by arithmetic progression of $\Gamma$ is a special case of a periodic labeling. In order to describe a periodic labeling of $\Gamma$, we need to provide only $n, t$, and $\phi(k), \phi(k+1), \ldots, \phi(k+t-1)$ for some $k \in V(\Gamma)$ (in fact, by symmetry of the labels, we can always assume $\phi(k) = 0$). This observation makes it straightforward to check the following proposition.

Proposition 4.1. Given constraints $(k_1, k_2, k_3)$ with $(k_1, k_2, k_3) \in D$, the 4-periodic $n$-labeling $\phi$ of $\Gamma$ with $n = 2k_1 + 2k_2, \phi(0) = 0, \phi(1) = k_1, \phi(2) = 2k_1 + k_2$, and $\phi(3) = k_1 + k_2$ gives a labeling satisfying $(k_1, k_2, k_3)$.

At the end of Section 3, we discussed whether a finite algorithm exists to determine the spans of the triangular and square lattice. Corollary 5.1 in the next section shows that the similar question for the line lattice has a positive answer. However, the complexity of finding the span of the line lattice remains open.

5. PERIODIC LABELINGS

The following theorem shows that we always can find an optimal assignment of $\Gamma$ that is $t$-periodic, although possibly with a large period $t$.

Theorem 5.1. Let $(k_1, \ldots, k_p)$ be a list of constraints. Then there exists a $t$-periodic $n$-labeling of $\Gamma$ satisfying the constraints $(k_1, \ldots, k_p)$ with $n = \sigma(\Gamma; k_1, \ldots, k_p)$ and $t \leq pn^t$.

Proof. Let $\psi$ be an $n$-labeling of $\Gamma$ satisfying $(k_1, \ldots, k_p)$ with $n = \sigma(\Gamma; k_1, \ldots, k_p)$. Consider all sequences of the form $(\psi(mp), \psi(mp + 1), \ldots, \psi(mp + p - 1))$, where $m \in V(\Gamma)$. Since there are at most $n^p$ different sequences of this form, there exist $m_1, m_2 \in \mathbb{Z}, m_1 < m_2$, and $m_2 - m_1 \leq n^p$, such that $\psi(m_1p) = \psi(m_2p), \psi(m_1p + 1) = \psi(m_2p + 1), \ldots, \psi(m_1p + p - 1) = \psi(m_2p + p - 1)$. Set $t = (m_2 - m_1)p$, hence $t \leq pn^t$, and let $\phi$ be the unique $t$-periodic $n$-labeling of $\Gamma$ with $\phi(k) = \psi(k)$ for all $k \in \{m_1p, m_1p + 1, \ldots, m_2p - 1\}$. In order to show that $\phi$ satisfies the constraints $(k_1, \ldots, k_p)$, we must show that for all $i \in \{1, \ldots, p\}$ and $k, \ell \in V(\Gamma)$ with $|k - \ell| = i$ we have $|\phi(k) - \phi(\ell)|_n \geq k_i$. By the $t$-periodicity of $\phi$, it is sufficient to show this for $k, \ell \in \{m_1p, m_1p + 1, \ldots, m_1p + t + p - 1\}$. Using that $m_1p + t + p - 1 = m_2p + p - 1$, together with $\psi(m_1p) = \psi(m_2p), \psi(m_1p + 1) = \psi(m_2p + 1), \ldots, \psi(m_1p + p - 1) = \psi(m_2p + p - 1)$, and the definition of $\phi$, we see that, in fact, $\phi(k) = \psi(k)$ for all $k \in \{m_1p, m_1p + 1, \ldots, m_1p + t + p - 1\}$. So the result follows by the fact that $\psi$ satisfies the constraints $(k_1, \ldots, k_p)$.

Corollary 5.1. Let $(k_1, \ldots, k_p)$ be a list of constraints. Then there exists a finite algorithm to determine the span $\sigma(\Gamma; k_1, \ldots, k_p)$.

Proof. Given $n, t$ and a labeling $\phi' : \{0, 1, \ldots, t - 1\} \to \{0, 1, \ldots, n - 1\}$, we can check in finite time if the unique extension of $\phi'$ to a $t$-periodic $n$-labeling of
$\Gamma$ satisfies the constraints $(k_1, \ldots, k_p)$. Since for fixed $n$ and $t$ there are exactly $n^t$ possible choices for $\phi'$, we can determine in finite time if there exists a $t$-periodic $n$-labeling of $\Gamma$ satisfying $(k_1, \ldots, k_p)$. Since trivially $\sigma(\Gamma; k_1, \ldots, k_p) \geq (p + 1)k_p$, we can start with $n = (p + 1)k_p$ and check all values $t = 1, 2, \ldots, pn^p$ to see if there exists a $t$-periodic $n$-labeling of $\Gamma$ satisfying the constraints. Eventually, increasing $n$ one at a time, we will find the smallest $n$ such that there exists a $t$-periodic $n$-labeling of $\Gamma$ with $t \leq pn^p$. By Theorem 5.1 and the fact that $\sigma(\Gamma; k_1, \ldots, k_p) \leq (p + 1)k_1$, this algorithm will terminate, and the value of $n$ at termination is equal to $\sigma(\Gamma; k_1, \ldots, k_p)$.

We now extend the concept of a periodic labeling (Definition 4.1) to 2-dimensional problems. The vectors $e_1, e_2,$ and $f$ are those defined in the Introduction.

**Definition 5.1.** An $n$-labeling $\phi$ of $\triangle$ is called a $(t_1, t_2)$-periodic $n$-labeling, where $t_1, t_2 \in \Lambda_\Delta$ are linearly independent, if $\phi(\mathbf{v} + t_1) = \phi(\mathbf{v} + t_2) = \phi(\mathbf{v})$ for all $\mathbf{v} \in V(\Delta)$. The definition is the same for $\square$, but then with the condition $t_1, t_2 \in \Lambda_\square$. For brevity we call such a labeling a $(t_1, t_2)$-labeling.

In the literature, $(t_1, t_2)$-labelings are sometimes known as regular tilings [1]. Given a point $\mathbf{v}$, the set $C(\mathbf{v}) = \{\mathbf{v} + m_1t_1 + m_2t_2 \mid m_1, m_2 \in \mathbb{Z}\}$ is called the co-channel lattice for $\mathbf{v}$. The set of all co-channel lattices is denoted by $C_\Delta(t_1, t_2)$ or $C_{\square}(t_1, t_2)$. Saying that $\phi$ is a $(t_1, t_2)$-labeling is nothing more than saying that $\phi$ is constant on each of the elements of $C_{\Delta}(t_1, t_2)$ or $C_{\square}(t_1, t_2)$.

In order to describe a $(t_1, t_2)$-labeling of $\triangle$ or $\square$, we must specify an integer $n$, vectors $t_1$ and $t_2$, and a value of $\phi$ for each co-channel lattice in $C_{\Delta}(t_1, t_2)$ or $C_{\square}(t_1, t_2)$. The last condition is equivalent to saying that we need to give the value $\phi(\mathbf{v})$ for a representative $\mathbf{v} \in C$ for each co-channel lattice $C$. A collection of representatives of all co-channel lattices will be called a tile. For an assignment that is known to be a $(t_1, t_2)$-labeling, there are many possible choices for $t_1$ and $t_2$ giving the same labeling, and many different choices for a tile. Conventionally one would choose, by a suitable choice of $(t_1, t_2)$, a tile with a minimum number of vertices, and also choose a tile which induces a connected subgraph of $\triangle$ or $\square$.

The following proposition gives the number of co-channel lattices of a periodic labeling. The proof, which can be found in [10] and [11], is an easy exercise in quadratic forms, and is left to the reader.

**Proposition 5.1.** Let $\phi$ be a $(t_1, t_2)$-labeling of $\triangle$ with $t_1 = a_1e_1 + a_2f$ and $t_2 = b_1e_1 + b_2f$. Then the number of vertices in a tile is given by $|C_{\Delta}(t_1, t_2)| = |a_1b_2 - a_2b_1|$. For a $(t_1, t_2)$-labeling of the square lattice with $t_1 = a_1e_1 + a_2e_2$ and $t_2 = b_1e_1 + b_2e_2$, we get the same formula, i.e., $|C_{\square}(t_1, t_2)| = |a_1b_2 - a_2b_1|$.

It can be shown that all labelings by arithmetic progressions are $(t_1, t_2)$-labelings for suitable choices of $t_1$ and $t_2$ (in fact, we can always choose $t_1, t_2$ such that the labeling is strict, as defined in Definition 5.2 below). It is now natural to ask whether or not the larger class of periodic assignments is sufficiently general to include an optimal answer to all assignment problems on $\triangle$ and $\square$. In practice it is often implicitly assumed that such tilings are all that must be considered when
assigning channels to regular lattices [14, 16]. This question is partially answered by the next proposition, but first we require the following definitions.

**Definition 5.2.** A \((t_1, t_2)\)-periodic labeling is **strict** if every co-channel lattice has a different label.

**Definition 5.3.** Let \(G\) be a graph and \(\phi\) an \(n\)-labeling of \(G\). As usual, we consider the channels \(0, 1, \ldots, n-1\) as the consecutive vertices of an \(n\)-cycle \(C_n\). Let \(S \subseteq V(G)\) be a finite set of vertices with \(|S| = s\) and such that \(\phi(u) \neq \phi(v)\) for all \(u, v \in S, u \neq v\). Fix an orientation on \(C_n\). Then the channel list of \(S\) is a list \((v_1, v_2, \ldots, v_s)\) of the vertices in \(S\) such that \(\phi(v_1), \phi(v_2), \ldots, \phi(v_s)\) occur on \(C_n\) in the order of their indices. We will not distinguish between channel lists arising from the reverse orientation of \(C_n\) or with a different first element. So \((v_1, v_2, \ldots, v_s), (v_s, v_{s-1}, \ldots, v_1)\) and \((v_2, v_3, \ldots, v_s, v_1)\) all denote the same channel list.

**Proposition 5.2.** On the square lattice, there is a 9-labeling satisfying the constraints \((2, 1, 1)\), but there is no strict \((t_1, t_2)\)-labeling of \(\square\) satisfying these constraints and with span less than ten.

**Proof.** First consider the 9-labeling \(\phi\) of \(\square\) given by \(\phi(m_1 \mathbf{e}_1 + m_2 \mathbf{e}_2) = 4(m_1 - m_2) + (-1)^{m_2} (\mod 9), m_1 \mathbf{e}_1 + m_2 \mathbf{e}_2 \in V(\square)\). It is easy to verify that this gives an assignment satisfying the constraints \((2, 1, 1)\) with nine channels.

Now suppose that we have a strict \((t_1, t_2)\)-labeling of \(\square\) with span at most nine (where \(t_1\) and \(t_2\) are chosen such that the number of vertices in a tile is minimum). Because the labeling is strict, a tile has at most nine vertices in it. Also, the shortest graph distance on which we can re-use channels is four. In other words, if \(t_1 = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2\) and \(t_2 = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2\), then

\[ a_1, a_2, b_1, b_2 \in \mathbb{Z}, \quad |a_1| + |a_2| \geq 4, \quad |b_1| + |b_2| \geq 4, \quad |a_1 b_2 - a_2 b_1| \leq 9. \]

In [10] it is shown exactly how to set up and solve these equations. Up to rotational and reflective symmetry, the only possible co-channel lattices are those given by \(t_1 = -2 \mathbf{e}_1 + 2 \mathbf{e}_2, t_2 = 2 \mathbf{e}_1 + 2 \mathbf{e}_2\), or \(t_1 = -2 \mathbf{e}_1 + 2 \mathbf{e}_2, t_2 = \mathbf{e}_1 + 3 \mathbf{e}_2\). In each case, the tile contains eight vertices and can be drawn as in Fig. 3(a).

In this tile, all the vertices must be assigned different channels, so we can look at the channel list on the vertices of the tile. Put an edge between any two vertices in the tile that can be assigned adjacent channels (bearing in mind the interference between vertices in adjacent tiles as well as vertices in the same tile). Both co-channel lattices give rise to the same set of edges on the tile, as shown in Fig. 3(b).

Since the graph induced by these edges consists of two components, when cycling through the vertices in the channel list at least two jumps must be made between vertices that are not joined by an edge. Each jump corresponds to having two consecutive vertices on the channel list that are geographically adjacent and, hence, whose labels are at least two apart. So any \((t_1, t_2)\)-labeling must have at least two unused channels and eight used channels, making a total of at least ten channels. \(\blacksquare\)
In order to prove the left-hand inequality in (3), we give a labeling of \( C \) thus, \( |C| = 3q^2 + 3q + 1 = \frac{3}{4} (2q+1)^2 + \frac{1}{4} = \left[ \frac{3}{4} (p+1)^2 \right] \).

Proposition 5.2 shows that strict \( (t_1, t_2) \)-labelings cannot always provide an optimal solution, contrary to the implicit assumption often made by spectrum managers. The status of general \( (t_1, t_2) \)-labelings remains an open question.

6. PROOFS OF SOME OF THE RESULTS

**Proof of Lemma 3.1.** Since for any graph \( G \), \( \omega(G) \leq \chi(G) \), it is sufficient to prove

\[
\chi(\Delta^p) \leq \left\lceil \frac{3}{4} (p+1)^2 \right\rceil \leq \omega(\Delta^p).
\]

In order to prove the left-hand inequality in (3), we give a labeling of \( \Delta^p \) using the notion of a periodic labeling as defined in Section 5. First suppose that \( p \) is even, \( p = 2q \), say. Set \( t_1 = (q+1)e_1 + qf \) and \( t_2 = -(q+1)e_1 + (2q+1)f \). For \( v \in V(\Delta) \) define the co-channel lattice \( C(v) = \{ \chi_1 \chi_2 \} \chi_1 = m_1 t_1 + m_2 t_2 \ m_1, m_2 \in \mathbb{Z} \} \) and set \( C = \{ C(v) \mid v \in V(\Delta) \} \). Then, since \( d_{\Delta}(0, t_1) = d_{\Delta}(0, t_2) = d_{\Delta}(t_1, t_2) = 2q + 1 = p + 1 \), it follows that, for every \( C(v) \in C \) and every \( u, w \in C(v), d_{\Delta}(u, w) > p \). Hence, \( C(v) \) is an independent set in \( \Delta^p \) and, thus, \( \chi(\Delta^p) \leq |C| \). Using Proposition 5.1 we obtain \( |C| = 3q^2 + 3q + 1 = \frac{3}{4} (2q+1)^2 + \frac{1}{4} = \left[ \frac{3}{4} (p+1)^2 \right] \).

If \( p \) is odd, \( p = 2q+1 \) say, then we take \( t_1 = (q+1)e_1 + (q+1)f \) and \( t_2 = -(q+1)e_1 + (2q+2)f \). The remainder is similar to the case \( p \) even.

For the right-hand inequality in Eq. (3), we again distinguish the cases \( p \) even and \( p \) odd. First suppose that \( p = 2q \). Let \( A(i) \) be the vertices \( v \) in \( V(\Delta) \) that have \( d_{\Delta}(0, v) = i \ (i = 0, 1, \ldots) \) and let \( B(q) = \bigcup_{i=0}^q A(i) \). Then every pair of vertices in \( B(q) \) have distance at most \( 2q = p \), and, hence, \( B(q) \) induces a clique in \( \Delta^p \). Since \( |A(0)| = 1 \) and \( |A(i)| = 6i \) if \( i > 0 \), we obtain \( \omega(\Delta^p) \geq |B(q)| = 1 + \sum_{i=1}^q 6i = \frac{3}{2} (2q+1)^2 + \frac{1}{4} = \left[ \frac{3}{4} (p+1)^2 \right] \).

If \( p = 2q+1 \) is odd, then define \( A'(i) \) as the vertices in \( V(\Delta) \) that have distance exactly \( i \) to one vertex in \( \{0, e_1, f\} \) and distance \( i+1 \) to the other two, and let \( A''(i) \) be the vertices that have distance exactly \( i \) to two vertices in \( \{0, e_1, f\} \) and
We also immediately have hence Then each pair of vertices in \( B'(q) \) has distance at most \( 2q + 1 = p \), hence \( B'(q) \) induces a clique in \( \Delta_p \). Since \( |A'(i)| = 3(i + 1) \) and \( |A''(i)| = 3i \), we obtain \( \omega(\Delta_p) \geq |B'(q)| = \sum_{i=0}^{q} (6i + 3) = \frac{3}{4}(2q + 2)^2 = [\frac{3}{4}(p + 1)^2] \).

This completes the proof of the lemma.

The proofs of Theorems 3.1 and 4.1 exhibit similar techniques. Recall the definition of a channel list in Definition 5.3. Let \( G \) be a graph and \( i \) a positive integer. A set of vertices \( S \subseteq V(G) \) is an \( i \)-clique if \( d_G(u, v) \leq i \) for all \( u, v \in S \).

Let \( \phi \) be an \( n \)-labeling of \( G \) and recall our interpretation of the channels \( 0, 1, \ldots, n - 1 \) as the consecutive vertices of an \( n \)-cycle \( C_n \). If \( k, \ell, m \in \{0, 1, \ldots, n - 1\} \) are three different labels, then there are two paths in \( C_n \) connecting \( k \) and \( \ell \). One of these paths contains \( m \), whereas the other does not. Define \( |k - \ell|_n^{(m)} \) as the length of the path in \( C_n \) connecting \( k \) and \( \ell \) which contains \( m \), and \( |k - \ell|_n^{(m)} \) as the length of the path in \( C_n \) connecting \( k \) and \( \ell \) which does not contain \( m \). A more formal, but less intuitively clear, definition is

\[
|k - \ell|_n^{(m)} = \begin{cases} 
\ell - k, & \text{if } k < m < \ell, \\
\ell - k, & \text{if } \ell < m < k, \\
n - (\ell - k), & \text{if } k < \ell < m \text{ or } m < k < \ell, \\
n - (k - \ell), & \text{if } \ell < k < m \text{ or } m < \ell < k,
\end{cases}
\]

and \( |k - \ell|_n^{(m)} = n - |k - \ell|_n^{(m)} \). Note that \( |k - \ell|_n = \min\{|k - \ell|_n^{(m)}, |k - \ell|_n^{(m)}\} \), hence

\[
|k - \ell|_n^{(m)} \geq |k - \ell|_n \quad \text{and} \quad |k - \ell|_n^{(m)} \geq |k - \ell|_n.
\]

We also immediately have

\[
|k - \ell|_n^{(m)} \geq |k - m|_n + |m - \ell|_n.
\]

**Lemma 6.1.** Let \( G \) be a graph, \( (k_1, \ldots, k_p) \) a list of constraints, and \( \phi \) an \( n \)-labeling of \( G \) satisfying \((k_1, \ldots, k_p)\). Let \( S \subseteq V(G) \) be an \( j \)-clique for some \( j \in \{1, \ldots, p\} \) with \( |S| = s \). Then there exists a channel list \((v_1, v_2, \ldots, v_s)\) of \( S \). Moreover, if \( s \geq 3 \), then

\[
n = \sum_{i=1}^{s} |\phi(v_{i+1}) - \phi(v_i)|_n^{(\phi(v_{i+2}))} \geq \sum_{i=1}^{s} k_{d_G(v_i, v_{i+1})},
\]

where the indices are to be taken modulo \( s \).

**Proof.** From Definition 1.3 and our assumption \( k_1 \geq k_2 \geq \cdots \geq k_p \geq 1 \), it follows that \( \phi(u) \neq \phi(v) \) for all \( u, v \in S, u \neq v \). This means that all elements of \( \phi(S) \) are different and, hence, we can form a channel list \((v_1, v_2, \ldots, v_s)\) of \( S \).

If \( s \geq 3 \), then \( |\phi(v_{i+1}) - \phi(v_i)|_n^{(\phi(v_{i+2}))} \) is the length of the path in \( C_n \) between \( \phi(v_i) \) and \( \phi(v_{i+1}) \) which does not contain any of the other elements of the list. The sum of the lengths of all these paths is equal to the length of the cycle, hence equal.
Let \( \phi(v_{i+1}) - \phi(v_i) \|_{d_i G(v_i, v_{i+1})} \geq 2k_{d_i G(v_i, v_{i+1})} \) by Eq. (4) and Definition 1.3.

**Proof of Theorem 3.1.** Following Proposition 3.1, it remains to show that the value of the spans given in the theorem are lower bounds. For the result on the square lattice, let \( \phi \) be an \( n \)-labeling of \( \square \) satisfying the constraints \((k_1, k_2)\). For \( \mathbf{v} \in V(\square) \), let \( J(\mathbf{v}) \) be the subgraph of \( \square \) induced by the five vertices at distance at most one from \( \mathbf{v} \). Using Lemma 6.1, we find \( n \geq 2k_1 + 3k_2 \), which, together with Proposition 3.1, proves the result for the square lattice.

For the triangular lattice, we proceed by contradiction. Suppose that \( \phi \) is an \( n \)-labeling of \( \sigma \) satisfying the constraints and such that \( n \) is smaller than the values for \( \sigma(\triangle; k_1, k_2) \) given in the theorem. For \( \mathbf{v} \in V(\sigma) \), the subgraph of \( \triangle \) induced by \( \mathbf{v} \) and the six vertices at distance one to \( \mathbf{v} \) will be denoted by \( H(\mathbf{v}) \) (see Fig. 4). Note that \( H(\mathbf{v}) \) is a 2-clique in \( \triangle \). Let \((\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_6)\) be the channel list of \( V(H(\mathbf{v})) \). Since \( d_{\triangle}(\mathbf{v}, \mathbf{v}_1) = d_{\triangle}(\mathbf{v}_6, \mathbf{v}) = 1 \) and \( d_{\triangle}(\mathbf{v}_i, \mathbf{v}_{i+1}) \leq 2 \) for \( i \in \{1, \ldots, 5\} \), Lemma 6.1 immediately gives the lower bound \( n \geq 2k_1 + 5k_2 \). The following series of claims aims to improve this lower bound, through a case-by-case analysis of the possible channel lists.

**Claim 1.** There exists an \( i \in \{1, \ldots, 5\} \) such that \( d_{\triangle}(\mathbf{v}_i, \mathbf{v}_{i+1}) = 1 \), or there exists an \( i \in \{1, 2, 3, 4\} \) such that \( d_{\triangle}(\mathbf{v}_i, \mathbf{v}_{i+2}) = 1 \).

If \( d_{\triangle}(\mathbf{v}_i, \mathbf{v}_{i+1}) = 1 \) for some \( i \in \{1, \ldots, 5\} \), then Lemma 6.1 gives \( n \geq 3k_1 + 4k_2 \), whereas if \( d_{\triangle}(\mathbf{v}_i, \mathbf{v}_{i+2}) = 1 \) for some \( i \in \{1, 2, 3, 4\} \), then using Lemma 6.1 on the set \( V(H(\mathbf{v})) \setminus \{\mathbf{v}_{i+1}\} \) with channel list \((\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_i, \mathbf{v}_{i+2}, \ldots, \mathbf{v}_6)\) gives \( n \geq 3k_1 + 3k_2 \). However, \( k_1 \geq 2k_2 \), this contradicts the choice of \( \phi \) as a counterexample. Hence, we have the following.

**Claim 2.** \( k_1 < 2k_2 \).

An edge \( uu' \in E(\triangle) \) will be called thin if \( |\phi(u) - \phi(u')|_n < 2k_2 \) and thick otherwise. Note that for a thin edge \( uu' \in E(\triangle) \) we still have \( |\phi(u) - \phi(u')|_n \geq k_1 \).

**Claim 3.** Let \((\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_6)\) be the channel list of \( V(H(\mathbf{v})) \) for some \( \mathbf{v} \in V(\triangle) \). Then one of \( \mathbf{vv}_1 \) or \( \mathbf{vv}_6 \) is a thin edge.

**Proof.** If both \( \mathbf{vv}_1 \) and \( \mathbf{vv}_6 \) are thick edges, then by Lemma 6.1 and Eq. (4) we obtain \( n \geq 2 \cdot 2k_2 + 5k_2 = 9k_2 \). If \( 3k_2 \geq 2k_1 \), then \( 9k_2 \geq 4k_1 + 3k_2 \), so we always contradict the supposition that \( \phi \) is a counterexample.

**Claim 4.** Let \( u, u' \in V(H(\mathbf{v})) \) for some \( \mathbf{v} \in V(\triangle) \). If \( uu' \in E(\triangle) \) is a thin edge, then \( u \) and \( u' \) are consecutive vertices in the channel list of \( V(H(\mathbf{v})) \).

![Figure 4. The graph H(v).](image-url)
Proof. If \( uu \) and \( uu' \) are not consecutive in the channel list of \( V(H(y)) \), then we can find two other vertices \( w, w' \in V(H(y)) \) such that \( (uu, uu', w, w') \) is a sublist of the channel list. Using (5) this means

\[
|\phi(uu) - \phi(uu')|_n = \min\{|\phi(u) - \phi(u')|_n, |\phi(u) - \phi(u')|_n\} \\
\geq \min\{|\phi(u) - \phi(w)|_n + |\phi(w) - \phi(u')|_n, \\
|\phi(u) - \phi(w')|_n + |\phi(w') - \phi(u')|_n\} \\
\geq 2k_2,
\]

contradicting the choice of \( uu' \) as a thin edge. \( \Box \)

From Claim 4 we immediately obtain the following.

Claim 5. There do not exist three thin edges in \( \triangle \) that have an end vertex in common or that induce a triangle in \( \triangle \).

Claim 6. Let \( uu', vv' \in E(H(y)) \) for some \( y \in V(\triangle) \) with \( y \notin \{u, u'\} \) and let \( (y, y_1, \ldots, y_6) \) be the channel list of \( V(H(y)) \). If \( uu' \) is a thin edge, then both \( vv_1 \) and \( vv_6 \) are also thin edges.

Proof. Suppose that at least one of \( vv_1, vv_6 \) is thick. Then by Claim 4 and Lemma 6.1 we obtain \( n \geq 2k_2 + k + 4k_2 + k_1 = 2k_1 + 3k_2 \). If \( 2k_1 \geq 3k_2 \), then \( 2k_1 + 4k_2 \geq 9k_2 \), while if \( 3k_2 \geq 2k_1 \), then \( 2k_1 + 4k_2 \geq 4k_1 + 3k_2 \). This contradicts the choice of \( \phi \) as a counterexample.

Claim 7. Let \( uu', ww' \in E(H(y)) \) for some \( y \in V(\triangle) \) with \( y \notin \{u, u', w, w'\} \) and \( uu' \neq ww' \). Then not both \( uu' \) and \( ww' \) are thin edges.

Proof. If both \( uu' \) and \( ww' \) are thin edges, then by Claim 4 and Lemma 6.1 we get \( n \geq 2k_1 + 3k_2 + 2k_1 = 4k_1 + 3k_2 \). If \( 2k_1 \geq 3k_2 \), then \( 2k_1 + 4k_2 \geq 9k_2 \), so we always get a contradiction.

By Claim 3 there exists at least one thin edge in \( \triangle \), say \( uu' \). Let \( y \) be one of the two vertices in \( \triangle \) at distance one to both \( uu \) and \( uu' \) and let \( (y, y_1, \ldots, y_6) \) be the channel list of \( V(H(y)) \). Then Claim 6 gives that both \( vv_1 \) and \( vv_6 \) are thin edges. Moreover, by Claims 5 and 7, \( uu', vv_1, vv_6 \) are the only thin edges in \( H(y) \). Let \( u_1, u_2, \ldots, u_6 \) be the neighbors of \( y \) in \( \triangle \) such that \( u_1 = u, u_2 = uu', u_3, u_4 \) is a 6-cycle in \( \triangle \). Of the fifteen possible choices of the pair \( y_1, y_6 \), many are ruled out. We cannot have \( \{y_1, y_6\} = \{u_1, u_2\} \) (i.e., \( uu' \)) because of Claim 5. If \( \{y_1, y_6\} = \{u_1, u_2\} \), then we contradict Claim 7 by considering \( H(u_3) \). A similar contradiction occurs if \( \{y_1, y_6\} = \{u_1, u_2\} \) is equal to \( \{u_2, u_1\}, \{u_3, u_4\}, \{u_5, u_6\}, \{u_1, u_5\}, \{u_2, u_6\} \). So the only possible choices for \( \{y_1, y_6\} \) are the following:

1. \( \{y_1, y_6\} = \{u_1, u_4\} \) or \( \{y_1, y_6\} = \{u_2, u_5\} \);
2. \( \{y_1, y_6\} = \{u_1, u_6\} \) or \( \{y_1, y_6\} = \{u_2, u_3\} \);
3. \( \{y_1, y_6\} = \{u_1, u_3\}, \{y_1, y_6\} = \{u_1, u_5\}, \{y_1, y_6\} = \{u_2, u_6\} \);
4. \( \{y_1, y_6\} = \{u_1, u_5\} \).
FIGURE 5. The possibilities for thin edges in $H(\mathbf{v})$. Thin edges are indicated by the dashed lines; the other edges are thick. In every picture, $u_1$ is the top left vertex and $u_5$ is the top right vertex.

These choices are sketched in Fig. 5.

Claim 8. Case (1) never occurs.

Proof. Suppose that $\{v_1, v_6\} = \{u_1, u_4\}$ and let $(u_3, v'_1, \ldots, v'_6)$ be the channel list of $V(H(u_3))$ and $(u_5, v''_1, \ldots, v''_6)$ the channel list of $V(H(u_5))$. Then according to Claim 6, the edges $u_3v'_1, u_3v'_2, u_5v''_1,$ and $u_5v''_2$ must be thin. But for each possible choice of these thin edges, we violate Claim 5 or 7. By symmetry, the case $\{v_1, v_6\} = \{u_2, u_3\}$ is also impossible.

Claim 9. Case (2) never occurs.

Proof. Suppose that $\{v_1, v_6\} = \{u_1, u_5\}$ and let $(u_6, v'_1, \ldots, v'_6)$ be the channel list of $V(H(u_6))$. Then according to Claim 6, $u_6v'_1$ and $u_6v'_6$ must be thin edges. We, therefore, need at least one thin edge incident on $u_6$, apart from $u_6v'_1$. But for each possible choice, we violate Claim 5, 7, or 8. The case $\{v_1, v_6\} = \{u_2, u_3\}$ is done by symmetry.

Claim 10. Case (3) never occurs.

Proof. Suppose that $\{v_1, v_6\} = \{u_1, u_4\}$ and let $(u_3, v'_1, \ldots, v'_6)$ be the channel list of $V(H(u_3))$. Then both $u_3v'_1$ and $u_3v'_6$ must be thin edges. We, therefore, need at least one thin edge incident on $u_3$, apart from $u_3v'_1$. Then for each possible choice, we violate Claim 5, 7, 8, or 9. The other possibilities for Case (3) are ruled out similarly.

Claim 11. Case (4) never occurs.

Proof. Let $(u_4, v'_1, \ldots, v'_6)$ be the channel list of $V(H(u_4))$. Then both $u_4v'_1$ and $u_4v'_6$ must be thin. But for each possible choice of these thin edges, we violate Claim 5, 7, 8, 9, or 10.
All possibilities for a counterexample have been ruled out and the proof is now complete.

**Proof of Theorem 4.1.** As with Theorem 3.1, we need only show that the values of \( \sigma(\Gamma; k_1, \ldots, k_p) \) claimed in the theorem give lower bounds for the span of the line lattice.

The case in which we only have a constraint at distance one is trivial, so suppose that we have constraints \((k_1, k_2)\), satisfied by an \( n \)-labeling \( \phi \) of \( \Gamma \). Then the set \( \{0, 1, 2\} \subseteq V(\Gamma) \) forms a 2-clique. The only possible channel list of this 2-clique is \( (0, 1, 2) \). Since \( d_\Gamma(0, 1) = d_\Gamma(1, 2) = 1 \) and \( d_\Gamma(0, 2) = 2 \), the result follows by Lemma 6.1.

Now suppose that we have a list of constraints \((k_1, k_2, k_3)\). Since \( \sigma(\Gamma; k_1, k_2, k_3) \geq \sigma(\Gamma; k_1, k_2) \), we are done if \((k_1, k_2, k_3) \in A\). So assume \((k_1, k_2, k_3) \in B \cup C \cup D\) and let \( \phi \) be an \( n \)-labeling of \( \Gamma \) satisfying the constraints. The set \( \{0, 1, 2, 3\} \subseteq V(\Gamma) \) is a 3-clique and the only possible channel lists are \((0, 1, 2, 3)\), \((0, 1, 3, 2)\), and \((0, 2, 1, 3)\). We consider each of these in turn.

**Case 1:** The channel list is \((0, 1, 2, 3)\). Using Lemma 6.1 we obtain \( n \geq 3k_1 + k_3 \), which is sufficient for region \( C \). Since \( 3k_1 + k_3 \geq 3k_2 + 2k_3 \), if \( 3k_1 \geq 3k_2 + k_3 \) (region \( B \)) and \( 3k_1 + k_3 \geq 2k_1 + 2k_2 \) if \( k_1 + k_3 \geq 2k_2 \) (region \( D \)), we are done.

**Case 2:** The channel list is \((0, 1, 3, 2)\). Again using Lemma 6.1 we see that \( n \geq 2k_1 + 2k_2 \), which is sufficient for region \( D \). This time we see that \( 2k_1 + 2k_2 \geq 3k_2 + 2k_3 \) if \( 2k_1 \geq k_2 + 2k_3 \) (region \( B \)) and \( 2k_1 + 2k_2 \geq 3k_1 + k_3 \) if \( 2k_2 \geq k_1 + k_3 \) (region \( C \)), which completes the proof for this case.

**Case 3:** The channel list is \((0, 2, 1, 3)\). Lemma 6.1 now gives \( n \geq k_1 + 2k_2 + k_3 \), which is not necessarily greater than or equal to the results in Theorem 4.1. But consider also the set \( \{1, 2, 3, 4\} \subseteq V(\Gamma) \), which is again a 3-clique in \( \Gamma \). The only possible choices for the channel list of \( \{1, 2, 3, 4\} \) are \((1, 2, 3, 4)\), \((1, 2, 4, 3)\), and \((1, 3, 2, 4)\). If the channel list is \((1, 2, 3, 4)\) or \((1, 2, 4, 3)\), we are done following Case 1 or 2, respectively. If the channel list is \((1, 3, 2, 4)\), it follows that

\[
|\phi(2) - \phi(1)|_n \geq |\phi(2) - \phi(4)|_n + |\phi(4) - \phi(1)|_n \geq k_2 + k_3.
\]

Returning to the channel list \((0, 2, 1, 3)\), we now have \( n \geq k_2 + (k_2 + k_3) + k_2 + k_3 = 3k_2 + 2k_3 \), which is sufficient for region \( B \). Noting that \( 3k_2 + 2k_3 \geq 3k_1 + k_3 \) if \( 3k_2 + k_3 \geq 3k_1 \) (region \( C \)) and \( 3k_2 + 2k_3 \geq 2k_1 + 2k_2 \) if \( k_2 + 2k_3 \geq 2k_1 \) (region \( D \)) completes the proof for this last case.

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