GREEN FUNCTION, PAINLEVÉ VI EQUATION, AND EISENSTEIN SERIES OF WEIGHT ONE

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ABSTRACT. We study the problem: *How many singular points of a solution $\lambda(t)$ to the Painlevé VI equation with parameter $(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})$ might have in $\mathbb{C} \setminus \{0,1\}$?* Here $t_0 \in \mathbb{C} \setminus \{0,1\}$ is called a singular point of $\lambda(t)$ if $\lambda(t_0) \in \{0,1,t_0,\infty\}$. Based on Hitchin's formula, we explore the connection of this problem with Green function and the Eisenstein series of weight one. Among other things, we prove:

(i) There are only three solutions which have no singular points in $\mathbb{C} \setminus \{0,1\}$. (ii) For a special type of solutions (called real solutions here), any branch of a solution has at most two singular points (in particular, at most one pole) in $\mathbb{C} \setminus \{0,1\}$. (iii) Any Riccati solution has singular points in $\mathbb{C} \setminus \{0,1\}$. (iv) For each $N \geq 5$ and $N \neq 6$, we calculate the number of the real j-values of zeros of the Eisenstein series $\mathfrak{E}_1^N(\tau; k_1, k_2)$ of weight one, where (k_1, k_2) runs over $[0, N-1]^2$ with $\gcd(k_1, k_2, N) = 1$.

The geometry of the critical points of the Green function on a flat torus E_τ, as τ varies in the moduli $\mathcal{M}_1$, plays a fundamental role in our analysis of the Painlevé IV equation. In particular, the conjectures raised in [22] on the shape of the domain $\Omega_5 \subset \mathcal{M}_1$, which consists of tori whose Green function has extra pair of critical points, are completely solved here.

CONTENTS

1. INTRODUCTION

1.1. **Painlevé property.** In the literature, a nonlinear differential equation in one complex variable is said to possess the *Painlevé property* if its solutions have neither movable branch points nor movable essential singularities. For the class of second order differential equations

$$\lambda''(t) = F(t, \lambda, \lambda'), \quad t \in \mathbb{CP}^1, \tag{1.1}$$

where $F(t, \lambda, \lambda')$ is meromorphic in t and rational in both λ and λ', Painlevé (later completed by Gambier, [11, 29]) obtained the classification of those nonlinear ODEs which possess the Painlevé property. They showed that there were fifty canonical equations of the form (1.1) with this property, up to Möbius transformations. Furthermore, of these fifty equations, forty-four are either integrable in terms of previously known functions (such as elliptic functions), equivalent to linear equations, or are reduced to one of six new nonlinear ODEs which define new transcendental functions (see eg. [17]). These six nonlinear ODEs are called Painlevé equations. Among them, Painlevé VI is often considered to be the *master equation*, because others can be obtained from Painlevé VI by the *confluence*. Due to its connection with many different disciplines in mathematics and physics, Painlevé VI has been extensively studied in the past several decades. See [1, 3, 8, 10, 12, 13, 15, 20, 24, 25, 27, 28, 35] and the references therein.

Painlevé VI (PVI) is written as

$$\begin{aligned} \frac{d^2\lambda}{dt^2} =& \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} \\ &+ \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{\lambda^2} + \gamma\frac{t-1}{(\lambda-1)^2} + \delta\frac{t(t-1)}{(\lambda-t)^2}\right], \end{aligned} \tag{1.2}$$

where $\alpha, \beta, \gamma, \delta$ are four complex constants. From (1.2), the Painlevé property says that any solution $\lambda(t)$ is a multi-valued meromorphic function in $\mathbb{C}\backslash\{0,1\}$. To avoid the multi-valueness of $\lambda(t)$, it is better to lift solutions of (1.2) to its universal covering. It is known that the universal covering of $\mathbb{C}\backslash\{0,1\}$ is the upper half plane $\mathbb{H} = \{\tau \mid \operatorname{Im}\tau > 0\}$. Then t and the solution $\lambda(t)$ can be lifted through the covering map $\tau \mapsto t$ by

$$t(\tau) = \frac{e_3(\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}, \qquad \lambda(t) = \frac{\wp(p(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}, \tag{1.3}$$

where $\wp(z) = \wp(z|\tau)$ is the Weierstrass elliptic function defined by

$$\wp(z|\tau) := \frac{1}{z^2} + \sum_{\omega \in \Lambda_\tau \backslash \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right), \tag{1.4}$$

and $\Lambda_\tau := \{m + n\tau \mid m, n \in \mathbb{Z}\}$ is the lattice generated by $\omega_1 = 1$ and $\omega_2 = \tau$. Also $\omega_3 = 1 + \tau$ and $e_i(\tau) = \wp(\frac{\omega_i}{2}|\tau)$ for $i = 1, 2, 3$. Consequently,

$p(\tau)$ satisfies the following elliptic form of PVI:

$$\frac{d^2 p(\tau)}{d\tau^2} = \frac{-1}{4\pi^2} \sum_{i=0}^{3} \alpha_i \wp' \left(p(\tau) + \tfrac{\omega_i}{2} | \tau \right), \tag{1.5}$$

where $\omega_0 = 0$ and

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \left(\alpha, -\beta, \gamma, \tfrac{1}{2} - \delta \right). \tag{1.6}$$

This elliptic form was first discovered by Painlevé [30]. For more recent derivations of it, see [1, 25].

1.2. **Hitchin solutions.** In this paper, we consider the special case $\alpha_i = \frac{1}{8}$ for $0 \leq i \leq 3$, i.e.,

$$\frac{d^2 p(\tau)}{d\tau^2} = \frac{-1}{32\pi^2} \sum_{i=0}^{3} \wp' \left(p(\tau) + \tfrac{\omega_i}{2} | \tau \right), \tag{1.7}$$

which is the elliptic form of $\mathrm{PVI}_{(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})}$. Equation (1.7) has connections with some geometric problems. The well-known example is related to the construction of Einstein metrics in four dimension; see [15]. In the seminal work [15], Hitchin obtained his famous formula to express a solution $p(\tau)$ of (1.7) with some complex parameters r, s:

$$\wp(p(\tau)|\tau) = \wp(r + s\tau|\tau) + \frac{\wp'(r + s\tau|\tau)}{2(\zeta(r + s\tau|\tau) - (r\eta_1(\tau) + s\eta_2(\tau)))}. \tag{1.8}$$

Here $\eta_i(\tau) = 2\zeta(\frac{\omega_i}{2}|\tau)$, $i = 1, 2$, are quasi-periods of the Weierstrass zeta function $\zeta(z|\tau) = -\int^z \wp(\xi|\tau) d\xi$.

By (1.8), he could construct an Einstein metric with positive curvature if $r \in \mathbb{R}$ and $s \in i\mathbb{R}$, and an Einstein metric with negative curvature if $r \in i\mathbb{R}$ and $s \in \mathbb{R}$. He also obtained an Einstein metric with zero curvature, but the corresponding solution of (1.7) is given by another formula other than (1.8). Indeed, this corresponds to the Riccati solutions of (1.7); see §3.

For simplicity, we denote $p_{r,s}(\tau)$ (equivalently, $\lambda_{r,s}(t)$ via (1.3)) to be the solution of (1.7) with the expression (1.8). It is obvious that if

$$(r, s) \in \tfrac{1}{2}\mathbb{Z}^2 := \left\{ (0,0), (0, \tfrac{1}{2}), (\tfrac{1}{2}, 0), (\tfrac{1}{2}, \tfrac{1}{2}) \right\} + \mathbb{Z}^2,$$

then either $\zeta(r + s\tau|\tau) - (r\eta_1(\tau) + s\eta_2(\tau)) \equiv \infty$ or $\zeta(r + s\tau|\tau) - (r\eta_1(\tau) + s\eta_2(\tau)) \equiv 0$ in $\mathbb{H}$. Hence for any complex pair $(r, s) \notin \frac{1}{2}\mathbb{Z}^2$, $p_{r,s}(\tau)$ is always a solution to (1.7), or equivalently, $\lambda_{r,s}(t)$ is a (multi-valued) solution to $\mathrm{PVI}_{(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})}$. We say that *two solutions $\lambda_{r,s}(t)$ and $\lambda_{r',s'}(t)$ give (or belong to) the same solution if $\lambda_{r',s'}(t)$ is the analytic continuation of $\lambda_{r,s}(t)$ along some closed loop in $\mathbb{C} \setminus \{0, 1\}$*. In §4, we will prove that $\lambda_{r,s}$ and $\lambda_{r',s'}$ give the same solution to $\mathrm{PVI}_{(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})}$ if and only if $(s', r') \equiv (s, r) \cdot \gamma \mod \mathbb{Z}^2$ for some matrix

$$\gamma \in \Gamma(2) = \{ \gamma \in SL(2, \mathbb{Z}) \mid \gamma \equiv I_2 \mod 2 \}.$$

In this paper, we are mainly concerned with the question of smoothness of solutions to $\mathrm{PVI}_{(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})}$, and the location of its singular points. Notice that for a solution $\lambda(t)$ and a point $t_0 \in \mathbb{C}\backslash\{0,1\}$, the RHS of Painlevé VI (1.2) has a singularity at $(t_0, \lambda(t_0))$ provided that $\lambda(t_0) \in \{0,1,t_0,\infty\}$. Therefore, in this paper, we say $\lambda(t)$ is *smooth at* t_0 if $\lambda(t_0) \notin \{0,1,t_0,\infty\}$. Furthermore, a singular point $t_0 \in \mathbb{C}\backslash\{0,1\}$ is called of *type* 0 *(*1, 2, 3 *respectively) if* $\lambda(t_0) = \infty$ *(*$\lambda(t_0) = 0, 1, t_0$ *respectively)*.

We take $\mathrm{PVI}_{(0,0,0,\frac{1}{2})}$ as an initial example for our discussion, because it can be transformed to $\mathrm{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ of our concern by a Bäcklund transformation (cf. [28]). In the literature, the Bäcklund transformation plays a very useful role in the study of Painlevé VI; for example, for finding the algebraic solutions, see [10, 26, 24]. Conventionally, solutions of $\mathrm{PVI}_{(0,0,0,\frac{1}{2})}$, the so-called *Picard solutions*, can be expressed in terms of Gauss hypergeometric functions. It was first found by Picard [31]. Let

$$\omega_1(t) = -i\pi F(\tfrac{1}{2}, \tfrac{1}{2}, 1; 1-t), \qquad \omega_2(t) = \pi F(\tfrac{1}{2}, \tfrac{1}{2}, 1; t) \tag{1.9}$$

be two linearly independent solutions of the Gauss hypergeometric equation

$$t(1-t)\omega''(t) + (1-2t)\omega'(t) - \tfrac{1}{4}\omega(t) = 0. \tag{1.10}$$

Then Picard solution of $\mathrm{PVI}_{(0,0,0,\frac{1}{2})}$ can be expressed as

$$\hat{\lambda}_{\nu_1,\nu_2}(t) = \wp(\nu_1\omega_1(t) + \nu_2\omega_2(t) \mid \omega_1(t), \omega_2(t)) + \frac{1+t}{3}, \tag{1.11}$$

for some $(\nu_1, \nu_2) \notin \frac{1}{2}\mathbb{Z}^2$, where $\wp(\cdot|\omega_1(t), \omega_2(t))$ is the Weierstrass elliptic function with periods $\omega_1(t)$ and $\omega_2(t)$. See [12] for a proof. Obviously, $\hat{\lambda}_{\nu_1,\nu_2}(t)$ is smooth for all $t \in \mathbb{C}\backslash\{0,1\}$ if and only if $(\nu_1, \nu_2) \in \mathbb{R}^2\backslash\frac{1}{2}\mathbb{Z}^2$ (see (A.1)). The lifting of $\hat{\lambda}_{\nu_1,\nu_2}(t)$ by (1.3) is given by

$$\hat{p}_{\nu_1,\nu_2}(\tau) = \nu_1 + \nu_2\tau, \tag{1.12}$$

which of course is a solution of the elliptic form of $\mathrm{PVI}_{(0,0,0,\frac{1}{2})}$: $\frac{d^2p(\tau)}{d\tau^2} = 0$. Then the Bäcklund transformation takes $\hat{\lambda}_{\nu_1,\nu_2}(t)$ into solution $\lambda_{r,s}(t)$ of $\mathrm{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ with $(r,s) = (\nu_1, \nu_2)$. Thus, in the elliptic form the Bäcklund transformation seems comparably *simple*. This fact and (1.12) are well known to experts. But it is difficult to find references for the proof. For the reader's convenience, we present a rigorous proof in Appendix A.

It is surprising to us that after the Bäcklund transformation from $\mathrm{PVI}_{(0,0,0,\frac{1}{2})}$ to $\mathrm{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$, $\mathrm{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ has only three solutions which are smooth in $\mathbb{C}\backslash\{0,1\}$.

Theorem 1.1. *There are only three solutions $\lambda(t)$ to $PVI_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ such that $\lambda(t)$ is smooth for all $t \in \mathbb{C}\backslash\{0,1\}$. They are precisely $\lambda_{\frac{1}{4},0}(t)$, $\lambda_{0,\frac{1}{4}}(t)$ and $\lambda_{\frac{1}{4},\frac{1}{4}}(t)$.*

Theorem 1.1 shows that the Bäcklund transformation does not preserve the smoothness of solutions. Thus, Theorem 1.1 can not be proved by applying Picard solutions and the Bäcklund transformation. We remark that the Bäcklund transformation is complicated due to not only the complicated form of birational maps between solutions but also the fact that it transforms *a pair of solutions of the Hamiltonian system* (equivalently, the pair $(\lambda(t), \lambda'(t))$), but not the solution $\lambda(t)$ only.

To prove Theorem 1.1, we start from the formula (1.8). Of course, (1.8) does not give the complete set of solutions to (1.7). The missing ones are solutions obtained from Riccati equations. For such Riccati solutions, we have some expressions like (1.8). By employing these expressions, we will prove in §6 that any Riccati solution has singularities in $\mathbb{C}\backslash\{0,1\}$. Hence our strategy for the proof of Theorem 1.1 is to study the smoothness of $\lambda_{r,s}(t)$ for any complex pair $(r,s) \notin \frac{1}{2}\mathbb{Z}^2$.

From (1.8), it is easy to see that if (r,s) is not a real pair, then $\lambda_{r,s}(t)$ always possesses a singularity $t_0 \notin \{0,1,\infty\}$ (indeed, infinitely many singularities), because there always exist infinitely many $\tau_0 \in \mathbb{H}$ such that $r+s\tau_0$ is a lattice point of the torus $E_{\tau_0} := \mathbb{C}/\Lambda_{\tau_0}$. So for the proof of Theorem 1.1 we could restrict ourselves to consider only $(r,s) \in \mathbb{R}^2\backslash\frac{1}{2}\mathbb{Z}^2$. In this case, we introduce the *Green function* and the *Hecke form* to study it.

1.3. **Green function and Hecke form.** Let $G(z|\tau)$ be the Green function on the torus E_τ:

$$\begin{cases} -\Delta G\left(z|\tau\right) = \delta_0\left(z\right) - \frac{1}{|E_\tau|} \text{ in } E_\tau, \\ \int_{E_\tau} G\left(z|\tau\right) dz = 0, \end{cases} \tag{1.13}$$

where δ_0 is the Dirac measure at 0 and $|E_\tau|$ is the area of the torus E_τ. We recall the analytic description of $G(z|\tau)$ in [22]. Recall the theta function $\vartheta := \vartheta_1$, where

$$\vartheta_1(z;\tau) = -i\sum_{n=-\infty}^{\infty}(-1)^n e^{(n+\frac{1}{2})^2\pi i\tau}e^{(2n+1)\pi iz}.$$

Then the Green function is given by

$$G(z|\tau) = -\frac{1}{2\pi}\log|\vartheta(z;\tau)| + \frac{(\operatorname{Im} z)^2}{2\operatorname{Im}\tau} + C(\tau), \tag{1.14}$$

where $C(\tau)$ is a constant so that $\int_{E_\tau} G = 0$. Recall that $\eta_i(\tau) = 2\zeta(\frac{\omega_i}{2}|\tau)$, $i=1,2$, are quasi-periods of $\zeta(z|\tau)$. Using $(\log\vartheta)_z = \zeta(z) - \eta_1 z$ and the Legendre relation $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$, we have

$$-4\pi G_z(z|\tau) = \zeta(z|\tau) - r\eta_1(\tau) - s\eta_2(\tau), \tag{1.15}$$

where $z = r+s\tau$ with $r,s \in \mathbb{R}$. As mentioned before, $\zeta\left(r+s\tau|\tau\right) - r\eta_1(\tau) - s\eta_2(\tau) \equiv 0$ in $\mathbb{H}$ whenever $(r,s) \in \frac{1}{2}\mathbb{Z}^2 \setminus \mathbb{Z}^2$. Thus for $(r,s) \in \mathbb{R}^2\backslash\frac{1}{2}\mathbb{Z}^2$, (1.15) shows that $r+s\tau$ is a non half-period critical point of $G(z|\tau)$ (we call such critical point a *nontrivial critical point*) if and only if $\zeta(r+s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau) = 0$. Naturally, we ask the question: *How many nontrivial critical points might $G(z|\tau)$ have?* Note that the nontrivial critical points must appear in pair because $G(z|\tau)$ is an even function in z. This question was answered in the following surprising result:

Theorem A. [22] *For any torus E_τ, $G(z|\tau)$ has at most one pair of nontrivial critical points.*

Theorem B. [23] *Suppose that $G(z|\tau)$ has one pair of nontrivial critical points. Then the three half-periods are all saddle points of $G(z|\tau)$, i.e., the Hessian satisfies $\det D^2G(\frac{\omega_k}{2}|\tau) \leq 0$ for $k = 1,2,3$.* [1]

For any $(r,s) \in \mathbb{R}^2 \backslash \frac{1}{2}\mathbb{Z}^2$, we define $Z = Z_{r,s}$ by

$$Z_{r,s}(\tau) := \zeta(r + s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau), \quad \forall \tau \in \mathbb{H}. \tag{1.16}$$

Clearly $Z_{r,s}$ is a holomorphic function in $\mathbb{H}$. If (r,s) is an N-torsion point, i.e., $(r,s) = (\frac{k_1}{N}, \frac{k_2}{N})$ with $0 \leq k_1, k_2 < N$ and $\gcd(k_1,k_2,N) = 1$, it was proved by Hecke in [14] that $Z_{r,s}(\tau)$ is a modular form of weight 1 with respect to $\Gamma(N) = \{\, A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv I_2 \pmod N \,\}$. This modular form is called the *Hecke form* in [21]. Indeed, it is the Eisenstein series of weight 1 with characteric (r,s) if (r,s) is an N-torsion point. Following [32, p.59], the Eisenstein series of weight 1 is defined by

$$\mathfrak{E}_1^N(\tau, s; k_1, k_2) := (\operatorname{Im}\tau)^s \sum_{(m,n)} (m\tau + n)^{-1} |m\tau + n|^{-2s},$$

where (m,n) runs over $\mathbb{Z}^2$ under the condition $0 \neq (m,n) \equiv (k_1,k_2) \bmod N$. It is known that $\mathfrak{E}_1^N(\tau,s;k_1,k_2)$ is a meromorphic function in the s-plane and holomorphic at $s = 0$. Set $\mathfrak{E}_1^N(\tau;k_1,k_2) := \mathfrak{E}_1^N(\tau,0;k_1,k_2)$. By using the Fourier expansions of both $Z_{r,s}(\tau)$ and $\mathfrak{E}_1^N(\tau;k_1,k_2)$ (see [32, p.59] and [9, p.139]), we have

$$Z_{r,s}(\tau) = N\mathfrak{E}_1^N(\tau;k_1,k_2), \text{ if } (r,s) \equiv (\tfrac{k_1}{N}, \tfrac{k_2}{N}) \bmod 1. \tag{1.17}$$

Hence, (1.15) yields that $G(z|\tau_0)$ has *a critical N-torsion point $\frac{k_1}{N} + \frac{k_2}{N}\tau_0$ with $N \geq 3$ if and only if $\mathfrak{E}_1^N(\tau_0;k_1,k_2) = 0$.*

Now we see the connection of the Hecke form with the solution $p_{r,s}(\tau)$ (or $\lambda_{r,s}(t)$) of (1.7): $Z_{r,s}(\tau)$ appears in the denominator of the RHS of (1.8). When $(r,s) \in \mathbb{R}^2\backslash\frac{1}{2}\mathbb{Z}^2$, the formula (1.8) implies that t_0 is a type 0 singularity, i.e., $\lambda_{r,s}(t_0) = \infty$ if and only if $Z_{r,s}(\tau_0) = 0$, $t_0 = t(\tau_0)$, or equivalently, the Green function $G(z|\tau_0)$ has *a nontrivial critical point $r + s\tau_0$*. By Theorem A, it means that $G(z|\tau_0)$ have exactly five critical points in the torus E_{τ_0}: $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, $\frac{\omega_3}{2}$ and $\pm(r + s\tau_0)$. This connection and (1.8) together with the Painlevé property say that the Eisenstein series $\mathfrak{E}_1^N(\tau;k_1,k_2)$ of weight 1 has only simple zeros; see Theorem 4.1. The simplicity of zeros was also proved by Dahmen [7] as a consequence of his counting formula of algebraic integral

[1] Theorem B is used in the proof of Theorem 1.2 (ii) to be stated later. After establishing Theorem 1.2, we have a stronger version of Theorem B: $\det D^2G(\frac{\omega_k}{2}|\tau) < 0$ for $k = 1,2,3$ if $G(z|\tau)$ has one pair of nontrivial critical points; see Proposition 6.2 in §6.

Lamé equations by the method of dessins d'enfants. In §7 we will discuss the position and the number of those zeros of $\mathfrak{E}_1^N(\tau_0; k_1, k_2)$.

Recall the group action of $SL(2,\mathbb{Z})$ on the upper half plane $\mathbb{H}$:

$$\tau' = \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}, \ \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Then we have the transformation law (see (4.4) in §4):

$$Z_{r',s'}(\tau') = (c\tau + d) Z_{r,s}(\tau) \ \text{ where } (s',r') = (s,r)\cdot \gamma^{-1}. \tag{1.18}$$

From here, we see that $G(z|\tau')$ has five critical points whenever $G(z|\tau)$ has five critical points. Let $\mathcal{M}_1 := \mathbb{H}/SL(2,\mathbb{Z})$ and

$$\begin{aligned} \Omega_5 &:= \{\tau \in \mathcal{M}_1 \mid G(z|\tau) \text{ has five critical points}\}, \\ \Omega_3 &:= \{\tau \in \mathcal{M}_1 \mid G(z|\tau) \text{ has three critical points}\}. \end{aligned}$$

Then we have $\Omega_3 \cup \Omega_5 = \mathcal{M}_1$ by Theorem A. Moreover, from the proof of Theorem A in [22], we know that $\Omega_5 \subset \mathcal{M}_1$ is open and Ω_3 is closed. In this paper we determine the geometry of Ω_3 and Ω_5 as conjectured in [22]:

Theorem 1.2 (Geometry of Ω_3 and Ω_5)**.**

(i) *Both Ω_5 and $\bar{\Omega}_3 = \Omega_3 \cup \{\infty\}$ are simply connected in $\overline{\mathcal{M}_1} \cong S^2$.*
(ii) *$C = \partial\Omega_5 = \partial\bar{\Omega}_3 \cong S^1$. $C\backslash\{\infty\} \cong \mathbb{R}$ is smooth. It consists of points τ so that some half-period is a degenerate critical point of $G(z|\tau)$.*

The proof is given in §5 and §6. We actually prove a stronger result on $\partial\Omega_5$: For any $\tau \in \partial\Omega_5$, there is only one half period whose Hessian $\det D^2G$ vanishes.

Theorem 1.1 is clearly closely related to the following question: *What is the set of pairs (r,s) such that $Z_{r,s}(\tau)$ has no zeros?* We should write an alternative form of (i) in Theorem 1.2 to answer this question. We note that the following two statements hold:

$$Z_{r,s}(\tau) = \pm Z_{r',s'}(\tau) \Longleftrightarrow (r,s) \equiv \pm(r',s') \pmod{\mathbb{Z}^2}, \tag{1.19}$$

$$\lambda_{r,s}(\tau) = \lambda_{r',s'}(\tau) \Longleftrightarrow (r,s) \equiv \pm(r',s') \pmod{\mathbb{Z}^2}. \tag{1.20}$$

The statement (1.19) is trivial while (1.20) was proved in [6]. From both (1.19) and (1.20), we could assume $(r,s) \in [0,1] \times [0,\frac{1}{2}]\backslash \frac{1}{2}\mathbb{Z}^2$. Then (i) of Theorem 1.2 can be stated more precisely. For this purpose, we consider

$$F_0 = \{\, \tau \in \mathbb{H} \mid 0 \le \operatorname{Re}\tau \le 1, \, |\tau - \tfrac{1}{2}| \ge \tfrac{1}{2} \,\}. \tag{1.21}$$

It is elementary to prove that F_0 is a fundamental domain for $\Gamma_0(2)$ (c.f. Remark 5.1). Notice that F_0 is one half of the fundamental domain of $\Gamma(2)$. The following theorem will imply (i) of Theorem 1.2.

Theorem 1.3. *Let $(r,s) \in [0,1] \times [0,\frac{1}{2}]\backslash\frac{1}{2}\mathbb{Z}^2$. Then $Z_{r,s}(\tau) = 0$ has a solution $\tau \in F_0$ if and only if*

$$(r,s) \in \triangle_0 := \{(r,s) \mid 0 < r,s < \tfrac{1}{2}, \, r+s > \tfrac{1}{2}\}.$$

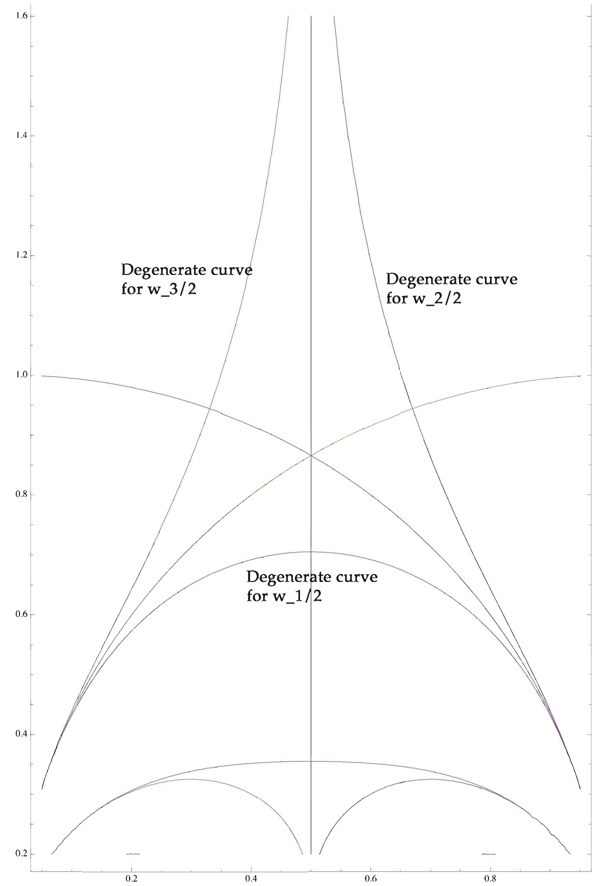

FIGURE 1. The lifted domain $\tilde{\Omega}_5 \subset F_0$ of Ω_5 is the domain bounded by the 3 curves corresponding to the loci of degenerate critical points.

Moreover, the solution $\tau \in F_0$ is unique for any $(r,s) \in \triangle_0$.

We will see that Theorem 1.1 is a consequence of the non-existence part of Theorem 1.3 in §5. Indeed, the existence part of Theorem 1.3 has applications as well; see the next subsection, where we will discuss the singular points of a *real solution* $\lambda(t)$.

1.4. **Real solution.** It is well known that Painlevé VI governs the isomonodromic deformations of some linear ODE. In the elliptic form it is convenient to choose the ODE to be a generalized Lamé equation (c.f. (2.4)). A solution $\lambda(t)$ of $\mathrm{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ is called a *real solution* if its associated monodromy of the generalized Lamé equation is unitary. In [6] it was proved that a solution $\lambda(t)$ is a real solution if and only if $\lambda(t) = \lambda_{r,s}(t)$ for some $(r,s) \in \mathbb{R}^2\backslash\frac{1}{2}\mathbb{Z}^2$. We call such a solution of $\mathrm{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ real because any solution with unitary monodromy representation must come from blowup solutions of the mean field equation; see [6]. We remark that real solutions do not mean *"real-valued solutions along the real-axis of t"*. Indeed, for (1.7) there are no real-valued solutions; see the discussion in Appendix B.

The reasons we are studying real solutions are: (i) any algebraic solution is a real solution; (ii) any real solution is smooth for $t \in \mathbb{R}\backslash\{0,1\}$ (see [6]); (iii) any real solution has no essential singularity even at $0,1$ and ∞ (see Appendix B).

It is known (see §2) that $t(\tau) = \frac{e_3(\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}$ maps any fundamental domain of $\Gamma(2)$ one-to-one and onto $\mathbb{C} \setminus \{0,1\}$. Then by the transformation (1.3), we see that any solution $\lambda(t(\tau))$ is single-valued and meromorphic whenever τ is restricted on a fundamental domain of $\Gamma(2)$. In this paper, *a branch of a solution* $\lambda(t)$ to (1.2) means a solution $\lambda(t(\tau))$ defined for τ in a fundamental domain of $\Gamma(2)$ (e.g. F given by (2.1)).

Recall a singular point $t_0 \notin \{0,1,\infty\}$ of $\lambda(t)$ means $\lambda(t_0) \in \{0,1,t_0,\infty\}$. Denote $\mathbb{C}_\pm = \{t \mid \operatorname{Im} t \gtrless 0\}$. Then for real solutions we have:

Theorem 1.4. *Suppose $\lambda(t)$ is a real solution. Then any branch of $\lambda(t)$ has at most two singular points in $\mathbb{C} \setminus \{0,1\}$, and they must be different type singular points if the branch has exactly two singular points. Furthermore, the set*

$$\Omega_-^{(0)} := \{t \in \mathbb{C}_- \mid t \text{ is a type } 0 \text{ singular point of some real solution}\} \tag{1.22}$$

is open and simply connected and $\partial\Omega_-^{(0)}$ consists of three smooth curves connecting $0,1,\infty$ respectively.

Remark 1.5. Theorem 1.4 shows that for each $k \in \{0,1,2,3\}$, any branch of a real solution has at most one type k singular point in $\mathbb{C} \setminus \{0,1\}$. Theorem 1.4 will be proved in §6, where we will see that, the curve of $\partial\Omega_-^{(0)}$ connecting ∞ and 0 (resp. connecting 1 and ∞, connecting 0 and 1) is the image of the degenerate curve of $\frac{\omega_1}{2}$ (resp. $\frac{\omega_2}{2}$, $\frac{\omega_3}{2}$) of Green function $G(z|\tau)$ in F_0 under the map $t(\tau)$. Similarly, we can define

$$\Omega_\pm^{(k)} := \{t \in \mathbb{C}_\pm \mid t \text{ is a type } k \text{ singular point of some real solution}\}. \tag{1.23}$$

Then $\Omega_-^{(k)} = \Omega_-^{(0)}$ and $\Omega_+^{(k)} = \Omega_+^{(0)} = \{t \mid t^{-1} \in \Omega_-^{(0)}\}$ for $k \in \{1,2,3\}$, and any real solution is smooth in $\mathbb{C} \setminus (\Omega_-^{(0)} \cup \Omega_+^{(0)} \cup \{0,1\})$, which consists of three connected components that contain $(-\infty,0)$, $(0,1)$ and $(1,+\infty)$ respectively. See the proof in §6.

1.5. **Algebraic solution.** A solution $\lambda(t)$ to PVI is called an algebraic solution if there is a polynomial $h \in \mathbb{C}[t,x]$ such that $h(t,\lambda(t)) \equiv 0$. It is equivalent to that $\lambda(t)$ has only a finite number of branches. By our classification theorem for (1.7), $\lambda(t)$ is an algebraic solution of $\text{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ if and only if $\lambda(t) = \lambda_{r,s}(t)$ for some $(r,s) \in Q_N$ with $N \geq 3$, where

$$Q_N := \left\{ \left(\tfrac{k_1}{N}, \tfrac{k_2}{N}\right) \middle| \gcd(k_1,k_2,N) = 1, 0 \leq k_1,k_2 \leq N-1 \right\} \tag{1.24}$$

is the set of N-torsion points of exact order N. The classification of the algebraic solutions for $\mathrm{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ could be deduced from the Bäcklund transformation and Picard solutions, as shown in [26]. It is therefore natural to ask the following question:

Is any singular point t_0 of an algebraic solution $\lambda(t)$ an algebraic number? Is the lifting τ_0 of t_0 a transcendental number?

The first question is equivalent to asking whether the j-value of any zero of $\mathfrak{C}_1^N(\tau;k_1,k_2)$ is an algebraic number. Here $j(\tau)$ is the classical modular function, the j-invariant of τ, under the action by $SL(2,\mathbb{Z})$; see (1.25) below.

This question can be answered easily from the aspect of Painlevé VI or from the q-expansion principle in the theory of modular forms (c.f. [19]). It is well known from the addition theorem of $\wp$ function that there is a polynomial $\Psi_N \in \mathbb{Z}[x,y,g_2,g_3]$ such that if (x,y) is an N-torsion point of the elliptic curve $y^2 = 4x^3 - g_2x - g_3$, then $\Psi_N(x,y) = 0$. The degree of Ψ_N is $\frac{N^2-1}{2}$, and y appears only with odd powers in $\Psi_N(x,y)$ if N is even; y appears only with even powers in $\Psi_N(x,y)$ if N is odd. See [16, p.272].

Now we come back to (1.9) and (1.11). Suppose that $\hat{\lambda}(t) = \hat{\lambda}_{\nu_1,\nu_2}(t)$ is a solution of $\mathrm{PVI}_{(0,0,0,\frac{1}{2})}$, where (ν_1,ν_2) is an N-torsion point. Then by the above result and the formulae for $\tilde{e}_k := \wp(\frac{\omega_k(t)}{2}|\omega_1(t),\omega_2(t))$ (here $\omega_3 = \omega_1+\omega_2$, see [26]):

$$\tilde{e}_1 = -\frac{1+t}{3}, \quad \tilde{e}_2 = 1-\frac{1+t}{3}, \quad \tilde{e}_3 = t-\frac{t+1}{3},$$

we see that there is a polynomial $\hat{P} \in \mathbb{Q}[t,x]$ such that

$$\hat{P}(t,\hat{\lambda}(t)) \equiv 0.$$

This polynomial seems too complicated to be computed in general. By the Bäcklund transformation, we conclude that for any algebraic solution $\lambda(t)$, there is a polynomial $P \in \mathbb{Q}[t,x]$ such that $P(t,\lambda(t)) \equiv 0$. Hence any singular point t of λ must be a root of a polynomial with integral coefficients, which implies that t is an algebraic number.

Let $t = t(\tau)$. Recall the classical modular function $j(\tau)$ of $SL(2,\mathbb{Z})$:

$$j(\tau) := 1728\frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2} = 1728\frac{g_2(\tau)^3}{\Delta(\tau)}, \tag{1.25}$$

where $g_2(\tau)$ and $g_3(\tau)$ are the coefficients of the elliptic curve E_τ: $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$, and the relation between $t(\tau)$ and $j(\tau)$ is

$$j = 256\frac{(t^2-t+1)^3}{t^2(t-1)^2}. \tag{1.26}$$

So if $t(\tau)$ is algebraic, then $j(\tau)$ is algebraic.

Another way to see it is to use a general principle from the theory of modular forms. Since all the coefficients of the Fourier expansion of $Z_{r,s}(\tau)$ are algebraic numbers, $\{j(\tau) \mid Z_{r,s}(\tau) = 0\}$ are algebraic by the so-called

q-expansion principle (c.f. [19]). However, we can prove more. Let us consider

$$Z_{(N)}(\tau) := \prod_{(r,s)\in Q_N} Z_{r,s}(\tau).$$

This is a modular form of weight $|Q_N| := \# Q_N$ with respect to $SL(2,\mathbb{Z})$. For $N \geq 5$, $m := \frac{|Q_N|}{24} \in \mathbb{N}$ and $\frac{Z_{(N)}(\tau)}{\Delta(\tau)^{2m}}$ is invariant under $SL(2,\mathbb{Z})$. Observe that

$$Z_{r,s}(\tau) = \begin{cases} -Z_{1-r,0}(\tau) & \text{if } \; s = 0, \\ -Z_{0,1-s}(\tau) & \text{if } \; r = 0, \\ -Z_{1-r,1-s}(\tau) & \text{if } \; r \neq 0, s \neq 0, \end{cases} \tag{1.27}$$

which implies that any zero of $\frac{Z_{(N)}(\tau)}{\Delta(\tau)^{2m}}$ must be doubled. Since $\frac{Z_{(N)}(\tau)}{\Delta(\tau)^{2m}}$ has no poles in $\mathbb{H}$, we conclude that

$$\frac{Z_{(N)}(\tau)}{\Delta(\tau)^{2m}} = C_{2m}\left(\ell_N(j)\right)^2 \tag{1.28}$$

for some monic polynomial ℓ_N of j and nonzero constant C_{2m}. If N is odd, then $Z_{(N)}(\infty) \neq 0$. Hence $\frac{Z_{(N)}(\tau)}{\Delta(\tau)^{2m}}$ has poles of order $2m$ at $\tau = \infty$, equivalently, $\ell_N(j)$ is a polynomial of degree $m = \frac{|Q_N|}{24}$. If N is even, then $l_N := \deg \ell_N < m$. In any case, we have

$$Z_{(N)}(\tau) = C_{2m}\Delta(\tau)^{2m-2l_n} H(G_4(\tau)^3, \Delta(\tau))^2,$$

where $H(X,Y)$ is a homogeneous polynomial of X, Y and $G_4(\tau) = g_2(\tau)/60$ is the classical Eisenstein series of weight 4. By using the q-expansion of $Z_{r,s}(\tau)$, we can prove that $\ell_N(j)$ has rational coefficients.

Theorem 1.6. *For any $N \geq 5$ with $N \neq 6$, the monic polynomial $\ell_N(j)$ determined by (1.28) has rational coefficients and satisfies*

(i) *for any zero j_0 of $\ell_N(j)$, there is an algebraic solution $\lambda_{r,s}(t)$, $(r,s) \in Q_N$, such that $j_0 = j(\tau_0)$, where $t_0 = t(\tau_0)$ satisfies $\lambda_{r,s}(t_0) = \infty$. Conversely, for any algebraic solution $\lambda_{r,s}(t)$, $(r,s) \in Q_N$, if $\lambda_{r,s}(t_0) = \infty$ for some $t_0 = t(\tau_0)$, then $j_0 = j(\tau_0)$ is a zero of $\ell_N(j)$.*
(ii) *$\ell_N(j)$ has distinct roots.*
(iii) *for any $N_1 \neq N_2$, $\ell_{N_1}(j)$ and $\ell_{N_2}(j)$ have no common zeros.*
(iv)

$$\deg \ell_N = \begin{cases} \frac{|Q_N|}{24} & \textit{if } \; N \textit{ is odd}, \\ \frac{|Q_N|}{24} - \frac{1}{2}\varphi(\frac{N}{2}) & \textit{if } \; N \textit{ is even}, \end{cases} \tag{1.29}$$

where $\varphi(\cdot)$ is the Euler function.

Recall the elementary formulae

$$|Q_N| = N^2 \prod_{p|N,\ p \text{ prime}} \left(1 - \frac{1}{p^2}\right), \quad \varphi(N) = N \prod_{p|N,\ p \text{ prime}} \left(1 - \frac{1}{p}\right). \tag{1.30}$$

Denote the j-value set of zeros of $Z_{(N)}(\tau)$ by

$$J(N) := \{j(\tau) \mid Z_{r,s}(\tau) = 0 \text{ for some } (r,s) \in Q_N\}.$$

If $N = 3$, then $|Q_N| = 8$ and $Z_{(N)}(\tau) = \text{const}\cdot G_4(\tau)^2$. Thus the zero $\tau = \rho := e^{\frac{\pi i}{3}}$ and $J(3) = \{0\}$. By Theorem 1.1 and Lemma 5.2, we see that $J(4) = J(6) = \emptyset$. For $N \geq 5$, $J(N)$ is just the zero set of $\ell_N(j)$. Note that formula (1.29) also holds for $N = 6$, which gives $\deg \ell_6 = 0$, so $\ell_6(j)$ is a non-zero constant. This also proves $J(6) = \emptyset$.

The computation of ℓ_N seems to be difficult in general. However, by applying PVI, it is considerably easier for small N. Here are some examples:

$$J(3) = \{0\},\ J(5) = \left\{\tfrac{5\cdot 2^{12}}{3^5}\right\},\ J(8) = \left\{\tfrac{207646}{3^8}\right\}, \tag{1.31}$$

$$J(7) = \left\{\tfrac{2^{11}}{5^7\cdot 3^4}(-333009 \pm 175519\sqrt{21})\right\}. \tag{1.32}$$

For $N = 9$, the polynomial is

$$\begin{aligned}\ell_9(j) =& j^3 + \frac{86191391040000000}{78815638671875}j^2 + \frac{1988564811286944 1536}{78815638671875}j \\ & - \frac{7205712225604271603712}{78815638671875}.\end{aligned}$$

So $J(9) = \{a, b, \bar{b}\}$, where $a \in \mathbb{R}$ and $b \notin \mathbb{R}$. Numerically,

$$a \approx 186.3,\ \ b \approx -639.9 + 285.0 \times \sqrt{-1}. \tag{1.33}$$

It seems that except for $N = 3$, all elements in $J(N)$ are not *algebraic integers*. If this would be true, then by a classical result of Siegal and Schneider, all τ such that $\lambda(t(\tau)) = \infty$ for an algebraic solution $\lambda(t)$ are transcendental. A natural question is how to determine their location in the fundamental domain F of $SL(2, \mathbb{Z})$, where

$$F = \{\tau \in \mathbb{H} \mid 0 \leq \operatorname{Re}\tau < 1,\ |\tau| \geq 1,\ |\tau - 1| > 1\} \cup \{\rho = e^{\frac{\pi i}{3}}\}. \tag{1.34}$$

The above examples show that there is at least one zero of $\ell_N(j)$ of where the corresponding τ is on the circular arc $\{\tau \in \mathbb{H} \mid |\tau| = 1\}$. Define

$$J_N^- = \{(r,s) \in Q_N \mid 2r + s = 1 \text{ and } \tfrac{1}{3} < s < \tfrac{1}{2}\},$$
$$J_N^+ = \{(r,s) \in Q_N \mid 2r + s = 1 \text{ and } 0 < s < \tfrac{1}{3}\}.$$

Then we have the following interesting result.

Theorem 1.7. *For any $N \geq 5$ with $N \neq 6$, $\ell_N(j)$ has exactly $\#J_N^+$ real zeros in $(0, 1728)$ and exactly $\#J_N^-$ real zeros in $(-\infty, 0)$. Furthermore, $\ell_N(j)$ has no zeros in $\{0\} \cup [1728, +\infty)$.*

Notice that in the fundamental domain F of $SL(2, \mathbb{Z})$, the corresponding τ of any positive zero of $\ell_N(j)$ is on the circular arc $\{\tau \in F \mid |\tau| = 1\}$; while the corresponding τ of any negative zero of $\ell_N(j)$ is on the line $\{\tau \in F \mid$

$\operatorname{Re}\tau = \frac{1}{2}\}$. We can use (1.31)-(1.33) to check the validity of Theorem 1.7 for small values of N. For example,

$$J_5^+ = \{(\tfrac{2}{5}, \tfrac{1}{5})\},\ J_5^- = \emptyset; \quad J_7^+ = \{(\tfrac{3}{7}, \tfrac{1}{7})\},\ J_7^- = \{(\tfrac{2}{7}, \tfrac{3}{7})\};$$

$$J_8^+ = \{(\tfrac{3}{8}, \tfrac{2}{8})\},\ J_8^- = \emptyset; \quad J_9^+ = \{(\tfrac{4}{9}, \tfrac{1}{9})\},\ J_9^- = \emptyset.$$

The proof of Theorems 1.6 and 1.7 will be given in §7. In §8 we will give some further remarks about Theorems 1.4 and 1.7. The explicit relation between Picard solutions and Hitchin's solutions will be given in Appendix A. Finally, Appendix B is denoted to the asymptotics of real solutions at $\{0, 1, \infty\}$, which are needed in the computation of $\ell_N(j)$.

2. Painlevé VI: Overviews

In this section, we start with the discussion of Painlevé VI (1.2):

$$\begin{aligned}\frac{d^2\lambda}{dt^2} =& \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt} \\ &+ \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{\lambda^2} + \gamma\frac{t-1}{(\lambda-1)^2} + \delta\frac{t(t-1)}{(\lambda-t)^2}\right].\end{aligned}$$

It is well-known that (1.2) possesses the *Painlevé property*, which says that any solution $\lambda(t)$ has no branch points and no essential singularities at any $t \in \mathbb{C}\backslash\{0, 1\}$.

2.1. **Multi-valueness via single-valueness.**

The Painlevé property implies that although a solution $\lambda(t)$ is multi-valued in $\mathbb{C}$, $\lambda(t)$ is a single-valued meromorphic function if t is restricted in $\mathbb{C}_\pm = \{z = x + iy | y \gtrless 0\}$. That means if $\lambda(t)$ is analytically continued along a closed curve $t = t(\epsilon)$, $t(0) = t(1)$, in $\mathbb{C}_+$ (or $\mathbb{C}_-$), then $\lambda(t(0)) = \lambda(t(1))$.

Due to the multi-valueness of a solution of (1.2), it is convenient to lift solutions and the equation to the universal covering. The universal covering space of $\mathbb{C}\backslash\{0, 1\}$ is the upper half plane $\mathbb{H}$. The covering map $t(\tau)$ is given in (1.3), by which, Painlevé VI (1.2) is transformed into the elliptic form (1.5).

It is elementary that $t(\tau)$ is invariant under the action of $\gamma \in \Gamma(2)$, where

$$\Gamma(2) = \{A \in SL(2, \mathbb{Z}) \mid A \equiv I_2 \bmod 2\}.$$

That is $t(\tau) = t(\tau')$ if and only if $\tau' = \gamma \cdot \tau = \frac{a\tau+b}{c\tau+d}$ for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$. Indeed $t(\tau)$ is the principal modular function of $\Gamma(2)$. Let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Then it is well known that $\mathbb{H}^*/\Gamma(2) \cong \mathbb{CP}^1$ with three cusp point $\infty, 0, 1$ which are mapped to $1, 0, \infty$ by $t(\tau)$ respectively. As the consequence of the isomorphism, we have

$$t'(\tau) = \frac{dt}{d\tau}(\tau) \neq 0, \ \ \forall \tau \in \mathbb{H},$$

namely the transformation $t(\tau)$ is locally one-to-one. Therefore, $t(\tau)$ maps any fundamental domain of $\Gamma(2)$ one-to-one onto $\mathbb{C}\setminus\{0,1\}$, and any solution $\lambda(t(\tau))$ is single-valued and meromorphic whenever τ is restricted in a fundamental domain of $\Gamma(2)$. As pointed out in §1, throughout this article, *a branch of a solution $\lambda(t)$* always means a solution $\lambda(t(\tau))$ defined for τ in a fundamental domain of $\Gamma(2)$.

The fundamental domain F_2 of $\Gamma(2)$ is

$$F_2 = \left\{\tau \mid 0 \leq \operatorname{Re}\tau < 2,\ |\tau - \tfrac{1}{2}| \geq \tfrac{1}{2},\ |\tau - \tfrac{3}{2}| > \tfrac{1}{2}\right\}. \tag{2.1}$$

When $\tau \in i\mathbb{R}^+$, $e_k(\tau)$ are real-valued and satisfies $e_2(\tau) < e_3(\tau) < e_1(\tau)$ (see e.g. [6]). From here, it is easy to see that $t(i\mathbb{R}^+) = (0,1)$, where $t(i\infty) = 1$ and $t(i0) = 0$. Here we used $\lim_{\tau\to i\infty} e_2(\tau) = \lim_{\tau\to i\infty} e_3(\tau) = -\frac{\pi^2}{3}$ (see §6). Furthermore, we could deduce from above that for any $\tau \in F_2$, $t(\tau) \in \mathbb{R}$ if and only if $\tau \in i\mathbb{R}^+ \cup \{\tau \in \mathbb{H} \mid |\tau - \frac{1}{2}| = \frac{1}{2}\} \cup \{\tau \in \mathbb{H} \mid \operatorname{Re}\tau = 1\}$.

By the formula (1.8), we see that $\wp(p(\tau)|\tau)$ is always a single-valued meromorphic function defined in $\mathbb{H}$. However, as a solution of (1.7), $p(\tau)$ has a branch point at those τ such that $p(\tau) \in E_\tau[2]$, where $E_\tau[2] := \{\frac{\omega_k}{2} | 0 \leq k \leq 3\}$ is the set of 2-torsion points in E_τ. The single-valueness of $\wp(p(\tau)|\tau)$ is one of the advantages of the elliptic form.

Recalling (1.21) and (2.1), F_0 is a half part of F_2. Then it is not difficult to prove that the transformation $t(\tau)$ maps the interior of F_0 onto the lower half plane $\mathbb{C}_-$, and $t(\tau)$ maps $F_2\backslash F_0$ onto $\mathbb{C}_+$; see §6. Hence it is convenient to use $\tau \in F_2$ when a branch of solution $\lambda(t)$ with $t \in \mathbb{C}\setminus\{0,1\}$ is discussed. Different branches of $\lambda(t)$ can be obtained from (1.8) by considering τ in another fundamental domain of $\Gamma(2)$.

2.2. **Isomonodromic deformation.** It is well known that Painlevé VI governs the isomonodromic deformation of some linear ODE. See [18] in this aspect. For the elliptic form (1.5), it was shown in [6] that it is convenient to use the so-called generalized Lamé equation (GLE):

$$y'' = \left[\begin{array}{l} \sum_{j=0}^3 n_j\left(n_j+1\right)\wp\left(z+\frac{\omega_j}{2}\right) + \frac{3}{4}\left(\wp\left(z+p\right)+\wp\left(z-p\right)\right) \\ +A\left(\zeta\left(z+p\right)-\zeta\left(z-p\right)\right)+B \end{array}\right] y. \tag{2.2}$$

Suppose $n_j \notin \frac{1}{2}+\mathbb{Z}$. Then *$p(\tau)$ is a solution of (1.5) if and only if there exist $A(\tau)$ and $B(\tau)$ such that GLE (2.2) preserves the monodromy as τ deforms*. The formula to connect parameters of (1.5) and (2.2) is:

$$\alpha_j = \tfrac{1}{2}\left(n_j + \tfrac{1}{2}\right)^2,\ j = 0,1,2,3. \tag{2.3}$$

See [6] for the proof. The advantage to employ GLE (2.2) is that for some cases, the monodromy representation is easier to describe. For example, let us consider $n_j = 0$ for all j. Then the elliptic form of PVI is (1.7), and GLE is

$$y'' = \left[\tfrac{3}{4}\left(\wp\left(z+p\right)+\wp\left(z-p\right)\right) + A\left(\zeta\left(z+p\right)-\zeta\left(z-p\right)\right)+B\right] y. \tag{2.4}$$

For any $p \notin E_\tau[2]$, $\pm p$ are the singular points of (2.4) with local exponents $-\frac{1}{2}$ and $\frac{3}{2}$. We always assume that $\pm p$ are *apparent singularities*. If $(r,s) \in$

$\mathbb{C}^2\backslash\frac{1}{2}\mathbb{Z}^2$ and $p(\tau) = p_{r,s}(\tau)$ is the solution given by (1.8), then we proved in [6] that the monodromy representation $\rho : \pi_1(E_\tau\backslash\{\pm p\}, q_0) \to SL(2, \mathbb{C})$ of GLE (2.4) is generated by

$$\rho(\gamma_\pm) = -I_2,\ \rho(\ell_1) = \begin{pmatrix} e^{-2\pi is} & 0 \\ 0 & e^{2\pi is} \end{pmatrix},\ \rho(\ell_2) = \begin{pmatrix} e^{2\pi ir} & 0 \\ 0 & e^{-2\pi ir} \end{pmatrix},$$

where q_0 is a base point, $\gamma_\pm \in \pi_1(E_\tau\backslash\{\pm p\}, q_0)$ encircles $\pm p$ once and $\ell_{1,2} \in \pi_1(E_\tau\backslash\{\pm p\}, q_0)$ are two fundamental circles of the torus E_τ such that $\gamma_+\gamma_- = \ell_2^{-1}\ell_1^{-1}\ell_2\ell_1$. In particular, the monodromy representation ρ is completely reducible.

2.3. **Bäcklund transformation.** In [28], Okamoto constructed the so-called Bäcklund transformations between solutions of Painlevé VI with different parameters. Indeed, this transformation is a birational transformation between the solutions of the corresponding Hamiltonian system, or equivalently, a birational transformation of $(\lambda(t), \lambda'(t))$ together. Since $\lambda(t)$ and $\lambda'(t)$ are algebraically independent generally (otherwise, Painlevé equation would be reduced to a first order ODE), the Bäcklund transformation is not a birational transformation of the solution $\lambda(t)$ only.

For example, it is known that a solution $\lambda(t)$ of $\mathrm{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ can be obtained from a solution $\hat{\lambda}(t)$ of $\mathrm{PVI}_{(0,0,0,\frac{1}{2})}$ by the following Bäcklund transformation (cf. [34, transformation s_2 in p.723]):

$$\lambda(t) = \hat{\lambda}(t) + \frac{1}{2\hat{\mu}(t)}, \quad \hat{\mu}(t) = \frac{t(t-1)\hat{\lambda}' - \hat{\lambda}(\hat{\lambda}-1)}{2\hat{\lambda}(\hat{\lambda}-1)(\hat{\lambda}-t)}. \tag{2.5}$$

As mentioned in the Introduction, for $\mathrm{PVI}_{(0,0,0,\frac{1}{2})}$, all its solutions are Picard solutions:

$$\hat{\lambda}(t) = \hat{\lambda}_{\nu_1,\nu_2}(t) = \wp(\nu_1\omega_1(t) + \nu_2\omega_2(t) \mid \omega_1(t), \omega_2(t)) + \frac{t+1}{3}, \tag{2.6}$$

where $(\nu_1, \nu_2) \in \mathbb{C}^2\backslash\frac{1}{2}\mathbb{Z}^2$ and $\omega_{1,2}(t)$ are given by (1.9). See [26, 12]. In principle, Hitchin's formula (1.8) could be obtained from Picard solution (2.6) via (2.5), as mentioned by [26] and some other references. However, the computation of $\hat{\lambda}'(t)$ via (2.6) is actually very difficult, and in practice, it is not easy at all to obtain Hitchin's formula from Picard solution (This is why we can not find a rigorous derivation of Hitchin's formula from Picard solution in the literature). In Appendix A, we will give a rigorous derivation from Hitchin's formula to Picard solution.

In the literature, researchers often restrict the study of Painlevé VI to special parameters via Bäcklund transformations. This leaves the impression that the theory for different parameters may be much the same. However, *this turns out not to be completely true* in general. For example, it is easy to see from the expression (A.1) that $\hat{\lambda}(t)$ is smooth for all $t \in \mathbb{C}\backslash\{0,1\}$ if and only if $(\nu_1, \nu_2) \in \mathbb{R}^2\backslash\frac{1}{2}\mathbb{Z}^2$. But this assertion is obviously false for $\mathrm{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$.

However, the Bäcklund transformation is useful to discuss branch points and essential singularities, which are preserved under the Bäcklund transformation. For example, if $t=0$ is a branch point for a solution, then after the Bäcklund transformation, $t=0$ is still a branch point for the new solution. Another example is that $t=1$ is an essential singularity for $\lambda_{r,s}(t)$ if $r\in\mathbb{R}$ and $s\in i\mathbb{R}$. Thus, $t=1$ is also an essential singularity for Picard solution $\hat{\lambda}_{\nu_1,\nu_2}(t)$ if $\nu_1\in\mathbb{R}$ and $\nu_2\in i\mathbb{R}$. For the discussion of branch points for real solutions, please see Appendix B.

3. RICCATI SOLUTIONS

First we review the classification theorem of solutions to the elliptic form (1.7) due to the associated monodromy representation of GLE (2.4). It was shown that solutions expressed in (1.8) does not contain all the solutions. Indeed, we have the following classification theorem proved in [6]. In this article, *when we talk about the monodromy representation, we always mean the one of GLE* (2.4).

Theorem C. ([6, Theorem 4.2]) *Suppose $p(\tau)$ is a solution to the elliptic form (1.7). Then the followings hold:*

(i) *The monodromy representation is completely reducible if and only if there exists $(r,s)\in\mathbb{C}^2\backslash\frac{1}{2}\mathbb{Z}^2$ such that $\wp(p(\tau)|\tau)$ is given by (1.8).*

(ii) *The monodromy representation is not completely reducible if and only if*

$$\lambda\left(t\right)=\frac{\wp\left(p\left(\tau\right)|\tau\right)-e_1\left(\tau\right)}{e_2\left(\tau\right)-e_1\left(\tau\right)},\ t=\frac{e_3\left(\tau\right)-e_1\left(\tau\right)}{e_2\left(\tau\right)-e_1\left(\tau\right)}\tag{3.1}$$

satisfies one of the following four Riccati equations:

$$\frac{d\lambda}{dt}=-\frac{1}{2t(t-1)}(\lambda^2-2t\lambda+t),\quad \mu\equiv 0,\tag{3.2}$$

$$\frac{d\lambda}{dt}=\frac{1}{2t(t-1)}(\lambda^2-2\lambda+t),\quad \mu\equiv\frac{1}{2\lambda},\tag{3.3}$$

$$\frac{d\lambda}{dt}=\frac{1}{2t(t-1)}(\lambda^2-t),\quad \mu\equiv\frac{1}{2(\lambda-1)},\tag{3.4}$$

$$\frac{d\lambda}{dt}=\frac{1}{2t(t-1)}(\lambda^2+2(t-1)\lambda-t),\quad \mu\equiv\frac{1}{2(\lambda-t)}.\tag{3.5}$$

Here $\mu(t)$ is defined by the second formula in (A.2), i.e., $(\lambda(t),\mu(t))$ satisfies the well-known Hamiltonian system of Painlevé VI.

It is known that Riccati equations can be transformed into second order linear equations (such as the Gauss hypergeometric equation). Hence, this classification shows that once the associated monodromy representation is not completely reducible, then solution $\lambda(t)$ can be expressed in terms of previously known functions, i.e., it does not define new transcendental functions.

Now we discuss the solutions of which the associated monodromy representation is not completely reducible, and the results in this section will be used to prove Theorem 1.1 in §5. For GLE (2.4) in E_τ with $p \notin E_\tau[2]$, we proved in [6] that there is always a solution which is expressed by:

$$y_{a_1}(z) := \exp\left(\tfrac{1}{2}z(\zeta(a_1+p)+\zeta(a_1-p))\right)\frac{\sigma(z-a_1)}{\sqrt{\sigma(z+p)\sigma(z-p)}}, \tag{3.6}$$

where the pair $\pm a_1 \in E_\tau$ is uniquely determined by

$$A = \tfrac{1}{2}\left[\zeta(p+a_1)+\zeta(p-a_1)-\zeta(2p)\right]. \tag{3.7}$$

If $a_1 \not\equiv -a_1 \bmod \Lambda_\tau$, then $y_{-a_1}(z)$ and $y_{a_1}(z)$ are linearly independent solutions to (2.4). In this case, the monodromy representation associated to (2.4) is completely reducible. In fact, we proved in [6, Lemma 2.3] that the monodromy representation for (2.4) is not completely reducible if and only if $a_1 \in E_\tau[2]$.

Now we assume $a_1 = \frac{\omega_k}{2} \in E_\tau[2]$. Let $y_1(z) = y_{a_1}(z)$ and $y_2(z) = \chi(z)y_1(z)$ be a linearly independent solution of (2.4) to $y_1(z)$. Clearly it is equivalent to $\chi(z) \not\equiv$const and

$$\frac{\chi''(z)}{\chi'(z)}+2\frac{y_1'(z)}{y_1(z)}=0, \text{ i.e., } \chi'(z)=\text{const}\cdot y_1(z)^{-2}. \tag{3.8}$$

On the other hand, by using $2\zeta(z)-\zeta(2z)=-\frac{1}{2}\frac{\wp''(z)}{\wp'(z)}$, (3.7) is equivalent to

$$A=-\frac{1}{4}\frac{\wp''(p-\frac{\omega_k}{2})}{\wp'(p-\frac{\omega_k}{2})}. \tag{3.9}$$

When $a_1=0$, we have

$$y_1(z)^{-2}=\frac{\sigma(z+p)\sigma(z-p)}{\sigma(z)^2}=c(\wp(z)-\wp(p)),$$

and then (3.8) yields $\chi(z)=c(\zeta(z)+\wp(p)z)$. So for any $c(\tau)\neq 0$, $(c(\tau)y_1, y_2)$ is a fundamental system of solutions to GLE (2.4), where $y_2(z)=(\zeta(z)+\wp(p)z)y_1(z)$. In particular,

$$\ell_j^*\begin{pmatrix} c(\tau)y_1 \\ y_2\end{pmatrix}=\begin{pmatrix} 1 & 0 \\ \frac{\eta_j+\wp(p)\omega_j}{c(\tau)} & 1\end{pmatrix}\begin{pmatrix} c(\tau)y_1 \\ y_2\end{pmatrix}. \tag{3.10}$$

Proposition 3.1. *The solutions of the Riccati equation (3.2) can be parametrized by $C \in \mathbb{CP}^1$:*

$$\lambda_C(t)=\frac{\wp(p_C(\tau)|\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}, \quad \wp(p_C(\tau)|\tau)=\frac{\eta_2(\tau)-C\eta_1(\tau)}{C-\tau}. \tag{3.11}$$

Proof. We separate the proof into two steps.

Step 1. We prove that for any constant $C \in \mathbb{CP}^1$, $\lambda_C(t)$ given by (3.11) solves the Riccati equation (3.2).

Fix any $C \in \mathbb{CP}^1$ and let $p(\tau) = p_C(\tau)$, $A(\tau) = -\frac{1}{4}\frac{\wp''(p(\tau))}{\wp'(p(\tau))}$ in the generalized Lamé equation (2.4).

If $C = \infty$, then $\wp(p(\tau)) = -\eta_1(\tau)$. Choose $c(\tau) = \eta_2(\tau) + \wp(p(\tau))\tau$. By the Legendre relation $\tau\eta_1(\tau) - \eta_2(\tau) = 2\pi i$, $c(\tau) = -2\pi i$. Thus by (3.10), we have

$$\ell_1^* \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix}, \ell_2^* \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix}.$$

That is, GLE (2.4) is monodromy preserving as τ deforms, so $p_\infty(\tau)$ is a solution of the elliptic form (1.7) (see Subsection 2.2).

If $C \neq \infty$, then (3.11) gives $\eta_1(\tau) + \wp(p(\tau)) \not\equiv 0$ and $C = \frac{\eta_2(\tau)+\wp(p(\tau))\tau}{\eta_1(\tau)+\wp(p(\tau))}$. Choose $c(\tau) = \eta_1(\tau) + \wp(p(\tau))$. Clearly except a set of discrete points in $\mathbb{H}$, $c(\tau) \neq 0$ and so

$$\ell_1^* \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix}, \ell_2^* \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix}. \tag{3.12}$$

As before, we conclude that $p_C(\tau)$ is a solution of the elliptic form (1.7).

We remark that the second formula in (3.11) was previously obtained in [15, 33], where there does not contain the relation between $\lambda_C(t)$ and Riccati equations. Here, together with our result in [6], we conclude that $\lambda_C(t)$ actually satisfies the Ricatti equation (3.2).

Step 2. Let $\lambda(t)$ be any solution of the Riccati equation (3.2). We prove the existence of $C \in \mathbb{CP}^1$ such that $\lambda(t) = \lambda_C(t)$.

Define $\pm p(\tau)$ by $\lambda(t)$ via (3.1) and $A(\tau) = -\frac{1}{4}\frac{\wp''(p(\tau))}{\wp'(p(\tau))}$. Then $p(\tau)$ is a solution of the elliptic form (1.7), which implies that (2.4) is monodromy preserving as τ deforms. Therefore, there exists a fundamental system of solutions $(\tilde{y}_1(z;\tau), \tilde{y}_2(z;\tau))$ to (2.4) such that the monodromy matrices M_1, M_2, which are defined by

$$\ell_j^* \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = M_j \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix}, \ j = 1, 2,$$

are independent of τ. We may assume $\wp(p(\tau)|\tau) \not\equiv \wp(p_\infty(\tau)|\tau)$, otherwise we are done. Then $c(\tau) := \eta_1(\tau) + \wp(p(\tau)) \not\equiv 0$. For any τ such that $c(\tau) \neq 0$, $(c(\tau)y_1, y_2)$ is also a fundamental system of solutions, so there is an invertible matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{pmatrix} = \gamma \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix}$. Clearly the monodromy matrices of $(c(\tau)y_1, y_2)$ is given by (3.12), where

$$C := \frac{\eta_2(\tau) + \wp(p(\tau)|\tau)\tau}{\eta_1(\tau) + \wp(p(\tau)|\tau)} \tag{3.13}$$

may depend on τ. Then

$$M_1 = \gamma \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \gamma^{-1} = \begin{pmatrix} 1 + \frac{bd}{ad-bc} & \frac{-b^2}{ad-bc} \\ \frac{d^2}{ad-bc} & 1 - \frac{bd}{ad-bc} \end{pmatrix},$$

$$M_2 = \gamma \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \gamma^{-1} = \begin{pmatrix} 1 + \frac{bd}{ad-bc}C & \frac{-b^2}{ad-bc}C \\ \frac{d^2}{ad-bc}C & 1 - \frac{bd}{ad-bc}C \end{pmatrix}.$$

Since M_1, M_2 are independent of τ and $bd \not\equiv 0$, we conclude that C is a constant independent of τ. Consequently, (3.13) implies $\wp(p(\tau)|\tau) = \wp(p_C(\tau)|\tau)$ and so $\lambda(t) = \lambda_C(t)$.

The proof is complete. □

Remark 3.2. It is easy to see that if $\operatorname{Im} C > 0$, then $\lambda_C(t)$ has singularities (at least a pole) in $\mathbb{C}\backslash\{0,1\}$. However, it is not so obvious to see whether $\lambda_C(t)$ has singularities or not if $\operatorname{Im} C \le 0$. In §6, we will exploit formulae (3.11) and (3.18) (below) to prove that any solution of the four Riccati equations has singularities in $\mathbb{C}\backslash\{0,1\}$.

Another observation is that $C = \infty$ gives that

$$\lambda_\infty(t) = -\frac{\eta_1(\tau) + e_1(\tau)}{e_2(\tau) - e_1(\tau)} \tag{3.14}$$

is a solution of $\mathrm{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$. Since $\lambda_\infty(t)$, $\lambda_\infty(t) - 1$ and $\lambda_\infty(t) - t$ can have only simple zeros (cf. [18, Proposition 1.4.1]), a direct consequence is

Theorem 3.3. *For fixed $k \in \{1,2,3\}$, the followings hold:*

(i) *Any zero of $\eta_1(\tau) + e_k(\tau)$ must be simple.*

(ii)

$$\frac{d}{d\tau}((\eta_1(\tau) + e_k(\tau))^{-1}) \neq \frac{1}{2\pi i} \text{ for any } \tau \in \mathbb{H}. \tag{3.15}$$

(iii) $\frac{\eta_2(\tau)+\tau e_k(\tau)}{\eta_1(\tau)+e_k(\tau)}$ *is a locally one-to-one map from $\mathbb{H}$ to $\mathbb{C} \cup \{\infty\}$.*

Proof. Recall

$$t = t(\tau) = \frac{e_3(\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)}.$$

Since $t'(\tau) \neq 0$ for all $\tau \in \mathbb{H}$, the assertion (i) follows readily from the fact that $\lambda_\infty(t)$ (for $k = 1$), $\lambda_\infty(t) - 1$ (for $k = 2$) and $\lambda_\infty(t) - t$ (for $k = 3$) can have only simple zeros.

For the assertion (ii), we note from the Legendre relation and (3.11) that

$$\lambda_C(t) = -\frac{\eta_1(\tau) + e_1(\tau) - \frac{2\pi i}{\tau - C}}{e_2(\tau) - e_1(\tau)}.$$

Fix any $\tau_0 \in \mathbb{H}$. If τ_0 is a zero of $\eta_1 + e_1$, then $\frac{d}{d\tau}((\eta_1 + e_1)^{-1})|_{\tau=\tau_0} = \infty$. So it suffices to consider the case $\eta_1(\tau_0) + e_1(\tau_0) \neq 0$. Then by letting

$$C = \tau_0 - \frac{2\pi i}{\eta_1(\tau_0) + e_1(\tau_0)},$$

we see that $t_0 = t(\tau_0)$ is a zero of $\lambda_C(t)$. Since $\lambda_C(t)$ has only simple zeros, we have

$$\frac{d}{d\tau}\left(\eta_1 + e_1 - \frac{2\pi i}{\tau - C}\right)|_{\tau=\tau_0} \neq 0.$$

This, together with $\eta_1(\tau_0)+e_1(\tau_0)-\frac{2\pi i}{\tau_0-C}=0$, easily implies $\frac{d}{d\tau}((\eta_1+e_1)^{-1})|_{\tau=\tau_0}\neq\frac{1}{2\pi i}$. This proves (3.15) for $k=1$. Similarly, by considering $\lambda_C(t)-1$ and $\lambda_C(t)-t$, we can prove (3.15) for $k=2,3$. This proves the assertion (ii).

Finally, using the Legendre relation leads to

$$\frac{\eta_2(\tau)+\tau e_k(\tau)}{\eta_1(\tau)+e_k(\tau)}=\tau-\frac{2\pi i}{\eta_1(\tau)+e_k(\tau)}.$$

Therefore, $\frac{\eta_2(\tau)+\tau e_k(\tau)}{\eta_1(\tau)+e_k(\tau)}$ is locally one-to-one. This completes the proof. □

Remark 3.4. In §6, we will see that the Hessian of the Green function $G(z|\tau)$ at $z=\frac{\omega_1}{2}=\frac{1}{2}$:

$$\det D^2G(\tfrac{1}{2}|\tau)=-C(\tau)\cdot\operatorname{Im}\left(\frac{\eta_2(\tau)+\tau e_1(\tau)}{\eta_1(\tau)+e_1(\tau)}\right)$$

for some $C(\tau)>0$, provided that $\eta_1(\tau)+e_1(\tau)\neq 0$. The local one-to-one of the map $\frac{\eta_2(\tau)+\tau e_1(\tau)}{\eta_1(\tau)+e_1(\tau)}$ is important for studying the curve in $\mathbb{H}$ where the half-period $\frac{\omega_1}{2}$ is a degenerate critical point of $G(z|\tau)$. See §6. Furthermore, we will prove a stronger result that $\eta_1(\tau)+e_1(\tau)$ has only one zero in any fundamental domain of $\Gamma(2)$; see Theorem 6.6.

Similarly, we can prove that all solutions of the other three Riccati equations can be parametrized by $\mathbb{CP}^1$. The calculation is as follows. Fix $k\in\{1,2,3\}$. When $a_1=\frac{\omega_k}{2}$, by (3.6) it is easy to see that

$$\chi(z)=-\frac{\wp(p)-e_k}{(e_k-e_i)(e_k-e_j)}\zeta(z-\tfrac{\omega_k}{2})-\left(1+e_k\frac{\wp(p)-e_k}{(e_k-e_i)(e_k-e_j)}\right)z$$

satisfies (3.8), where $\{i,j\}=\{1,2,3\}\backslash\{k\}$. As before, for any $c(\tau)\neq 0$, $(c(\tau)y_1(z),y_2(z))$ is a fundamental system of solutions to (2.4), where $y_2(z)=\chi(z)y_1(z)$. In particular,

$$\ell_j^*\begin{pmatrix}c(\tau)y_1\\ y_2\end{pmatrix}=\begin{pmatrix}1 & 0\\ -\frac{D\eta_j+\omega_j(1+De_k)}{c(\tau)} & 1\end{pmatrix}\begin{pmatrix}c(\tau)y_1\\ y_2\end{pmatrix}, \tag{3.16}$$

where

$$D:=\frac{\wp(p)-e_k}{(e_k-e_i)(e_k-e_j)}. \tag{3.17}$$

Proposition 3.5. *For $C\in\mathbb{CP}^1$, we let $\lambda_C(t)=\frac{\wp(p_C(\tau)|\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}$, where*

$$\wp(p_C(\tau)|\tau)=\frac{e_k(C\eta_1(\tau)-\eta_2(\tau))+(\frac{g_2}{4}-2e_k^2)(C-\tau)}{C\eta_1(\tau)-\eta_2(\tau)+e_k(C-\tau)}. \tag{3.18}$$

Then $\lambda_C(t)$ satisfies the Ricatti equation (3.3) if $k=1$, (3.4) if $k=2$, (3.5) if $k=3$. Furthermore, such $\lambda_C(t)$ give all the solutions of these three Riccati equations.

Proof. We sketch the proof for fixed $k \in \{1,2,3\}$. For any $C \in \mathbb{CP}^1$, we let $p(\tau) = p_C(\tau)$, $A(\tau) = -\frac{1}{4}\frac{\wp''(p(\tau)-\frac{\omega_k}{2})}{\wp'(p(\tau)-\frac{\omega_k}{2})}$ in (2.4). If $C = \infty$, i.e., $D\eta_1 + (1 + De_k) \equiv 0$, then we choose $c(\tau) = D\eta_2 + \tau(1 + De_k) = \frac{2\pi i}{\eta_1(\tau)+e_k(\tau)} \not\equiv 0$. By (3.16),

$$\ell_1^* \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix}, \ell_2^* \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix}.$$

If $C \neq \infty$, then (3.18) gives $D\eta_1 + (1 + De_k) \not\equiv 0$ and $C = \frac{D\eta_2+\tau(1+De_k)}{D\eta_1+(1+De_k)}$. Choose $c(\tau) = D\eta_1 + (1 + De_k)$. Then

$$\ell_1^* \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix}, \ell_2^* \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} c(\tau)y_1 \\ y_2 \end{pmatrix}.$$

Similarly as in Proposition 3.1, we see that $p_C(\tau)$ is a solution of the elliptic form (1.7). Again the formula in (3.18) was first obtained in [33]. Here, together with our result in [6], we conclude that $\lambda_C(t)$ actually satisfies the Ricatti equation (3.3) if $k = 1$, (3.4) if $k = 2$, (3.5) if $k = 3$. The rest of the proof is similar to that of Proposition 3.1. □

For solution $p_C(\tau)$ of the Riccati equations given in Propositions 3.1 and 3.5, we let $\tau' = \gamma \cdot \tau$ and $C' = \gamma \cdot C$ for $\gamma \in SL(2,\mathbb{Z})$. By using (4.2)-(4.4) (see §4) and the formula of $\wp\left(p_C(\tau)|\tau\right)$, it is easy to prove

$$\wp(p_{C'}(\tau')|\tau') = (c\tau + d)^2 \wp\left(p_C(\tau)|\tau\right). \tag{3.19}$$

Then we have the following result, which can be proved by the same argument of Proposition 4.4 in §4, so we omit the details of the proof here.

Proposition 3.6. *Let $\lambda_C(t)$ and $\lambda_{C'}(t)$ solve the same one of the four Riccati equations (3.2)-(3.5). Then they give the same solution to $PVI_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ if and only if $C' = \gamma \cdot C$ for some $\gamma \in \Gamma(2)$.*

We conclude this section by a remark. In [26], Mazzocco classified solutions of $\mathrm{PVI}_{((2\mu-1)^2/2,0,0,\frac{1}{2})}$ (write PVI_μ for convenience) for $\mu \in \frac{1}{2} + \mathbb{Z}$. Notice that $\mathrm{PVI}_{\frac{1}{2}}$ is precisely $\mathrm{PVI}_{(0,0,0,\frac{1}{2})}$ and PVI_μ can be transformed to $\mathrm{PVI}_{\frac{1}{2}}$ via Bäcklund transformations. Mazzocco proved for $\mu \in \frac{1}{2} + \mathbb{Z}$ and $\mu \neq \frac{1}{2}$, say $\mu = \frac{-1}{2}$ for instance, $\mathrm{PVI}_{\frac{-1}{2}}$ has two types of solutions: one is so-called *Picard type solutions*, which is obtained from Picard solutions (2.6) via Bäcklund transformations; the other one is so-called *Chazy solutions*, such as

$$\widetilde{\lambda}(t) = \frac{\frac{1}{8}\{[\omega_2 + \nu\omega_1 + 2t(\omega_2' + \nu\omega_1')]^2 - 4t(\omega_2' + \nu\omega_1')^2\}^2}{(\omega_2 + \nu\omega_1)(\omega_2' + \nu\omega_1')[2(t-1)(\omega_2' + \nu\omega_1') + \omega_2 + \nu\omega_1][\omega_2 + \nu\omega_1 + 2x(\omega_2' + \nu\omega_1')]}$$

(where $\nu \in \mathbb{C}$), which will turn to be the singular solutions $\lambda_0(t) \equiv 0, 1, t$ or ∞ of $\mathrm{PVI}_{\frac{1}{2}}$ via Bäcklund transformations. Here together with Theorem C and our argument in §2, in principle, solutions of the four Riccati equations could be obtained from Chazy solutions of $\mathrm{PVI}_{\frac{-1}{2}}$ via Bäcklund transformations, but the process would be too complicated to be computed.

4. Completely reducible solutions

4.1. **Simple zeros of Hecke form.** By Theorem C in §3, any solution $\lambda(t)$ of $\mathrm{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ with a completely reducible monodromy representation can be expressed by (3.1):

$$\lambda(t)=\frac{\wp(p(\tau)|\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)},\ t=\frac{e_3(\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)},$$

where $\wp(p(\tau)|\tau)$ is given by (1.8) with some $(r,s)\in\mathbb{C}^2\backslash\frac{1}{2}\mathbb{Z}^2$. From (1.8), we have the following application of the Painlevé property.

Theorem 4.1. *Suppose $(r,s)\in\mathbb{C}^2\backslash\frac{1}{2}\mathbb{Z}^2$ is a pair of complex constants. Then the Hecke form $Z_{r,s}(\tau)=\zeta(r+s\tau|\tau)-(r\eta_1(\tau)+s\eta_2(\tau))$ has only simple zeros.*

Proof. First, we note that the situations $r+s\tau\in E_\tau[2]$ and $Z_{r,s}(\tau)=0$ can not occur simultaneously. If not, then there are τ_0 and $m,n\in\mathbb{Z}$ such that $r+s\tau_0=m+n\tau_0+\frac{\omega}{2}$, where ω is any lattice points $\{0,\omega_1,\omega_2,\omega_3=\omega_1+\omega_2\}$, and also $\zeta(r+s\tau_0)=r\eta_1(\tau_0)+s\eta_2(\tau_0)$. Without loss of generality, we might assume $\omega=\omega_1$. The other cases can be proved similarly.

The second identity also implies

$$\begin{aligned}\tfrac{1}{2}\eta_1(\tau_0)=\zeta(\tfrac{\omega_1}{2})&=\zeta\left((r-m)+(s-n)\tau_0\right)\\&=\zeta\left(r+s\tau_0\right)-m\eta_1(\tau_0)-n\eta_2(\tau_0)\\&=(r-m)\eta_1(\tau_0)+(s-n)\eta_2(\tau_0).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&(r-m-\tfrac{1}{2})+(s-n)\tau_0=0,\\&(r-m-\tfrac{1}{2})\eta_1(\tau_0)+(s-n)\eta_2(\tau_0)=0,\end{aligned}$$

which implies $r-m-\frac{1}{2}=0$ and $s=n$ because the matrix $\begin{pmatrix}1&\tau\\\eta_1(\tau)&\eta_2(\tau)\end{pmatrix}$ is non-degenerate for any τ due to the Legendre relation. Obviously it contradicts to the assumption $(r,s)\notin\frac{1}{2}\mathbb{Z}^2$.

Now suppose $Z_{r,s}(\tau_0)=0$, which implies $\wp(p(\tau_0))=\infty$ by (1.8) because $\wp'(r+s\tau_0)\neq 0$. Consider the transformation $\tau_0\mapsto t_0$ via (3.1). Then by the Painlevé property, we know that $\lambda(t)$ has a pole at $t=t_0\notin\{0,1,\infty\}$. By substituting the local expansion of $\lambda(t)$ at $t=t_0$ into (1.2), it is easy to prove that the order of pole at $t=t_0$ is 1, which implies the zero of $Z_{r,s}$ at $\tau=\tau_0$ is simple. □

Remark 4.2. If (r,s) is an N-torison point, i.e., $(r,s)=(\frac{k_1}{N},\frac{k_2}{N})$ for positive integers $k_i,N\geq 3$ and $\gcd(k_1,k_2,N)=1$, then the function $Z_{r,s}(\tau)$ is a modular form of weight 1 with respect to the modular group $\Gamma(N)$. In this case, Theorem 4.1 was proved in [7], where the method of dessins d'enfants was used. For a real pair of (r,s), we will give an alternative proof in §5.

Since $\alpha_i = \frac{1}{8}$ for $0 \leq i \leq 3$, it is easy to see that for $1 \leq k \leq 3$, $p(\tau) + \frac{\omega_k}{2}$ is also a solution of the elliptic form (1.7) provided that $p(\tau)$ is a solution of (1.7). Then we have the following result, which will be used in §5.

Proposition 4.3. *Given $(r,s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, we define*

$$(r_k, s_k) = \begin{cases} (r - \frac{1}{2}, s) & \text{if } k = 1, \\ (r, s - \frac{1}{2}) & \text{if } k = 2, \\ (r - \frac{1}{2}, s - \frac{1}{2}) & \text{if } k = 3. \end{cases} \tag{4.1}$$

Then $p_{r,s}(\tau) + \frac{\omega_k}{2} = \pm p_{r_k,s_k}(\tau)$ in E_τ.

Proof. It was proved in [6] that (1.8) is equivalent to

$$\zeta(r + s\tau + p_{r,s}(\tau)) + \zeta(r + s\tau - p_{r,s}(\tau)) - 2(r\eta_1(\tau) + s\eta_2(\tau)) = 0.$$

Form here, we easily obtain

$$\begin{aligned}\zeta(r_k + s_k\tau + (p_{r,s}(\tau) + \tfrac{\omega_k}{2})) + \zeta(r_k + s_k\tau - (p_{r,s}(\tau) + \tfrac{\omega_k}{2}))& \\ - 2(r_k\eta_1(\tau) + s_k\eta_2(\tau)) = 0&,\end{aligned}$$

and so

$$\begin{aligned}&\wp(p_{r,s}(\tau) + \tfrac{\omega_k}{2}|\tau) \\ =&\wp\left(r_k + s_k\tau|\tau\right) + \frac{\wp'\left(r_k + s_k\tau|\tau\right)}{2\left(\zeta\left(r_k + s_k\tau|\tau\right) - (r_k\eta_1(\tau) + s_k\eta_2(\tau))\right)} \\ =&\wp(p_{r_k,s_k}(\tau)|\tau).\end{aligned}$$

This completes the proof. $\square$

We call a solution to (1.5) a *real solution* if the monodromy group of its associated GLE (2.2) is contained in $SU(2)$. For the case $\alpha_j = \frac{1}{8}$, $p(\tau)$ is a real solution if and only if it is given by (1.8) for some real pair $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$.

4.2. **Modularity.** In this subsection we study the modularity property of solutions to $\mathrm{PVI}_{(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})}$. Consider the pair $(z,\tau) \in \mathbb{C} \times \mathbb{H}$ and $z = r + s\tau$. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, conventionally γ can act on $\mathbb{C} \times \mathbb{H}$ by $\gamma(z,\tau) := (\frac{z}{c\tau+d}, \gamma \cdot \tau) = (\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d})$. Then

$$\frac{z}{c\tau + d} = \frac{r + s\tau}{c\tau + d} = r' + s'\tau', \text{ where } \tau' = \gamma \cdot \tau \text{ and } (s', r') = (s, r) \cdot \gamma^{-1}.$$

Using

$$\wp\left(\frac{z}{c\tau + d}\middle|\, \tau'\right) = (c\tau + d)^2 \wp\left(z|\tau\right), \;\; \tau' = \frac{a\tau + b}{c\tau + d}, \tag{4.2}$$

we derive

$$\zeta\left(\frac{z}{c\tau+d}\middle|\tau'\right)=(c\tau+d)\,\zeta\,(z|\tau)\,,$$

and so

$$\begin{pmatrix}\eta_2(\tau')\\ \eta_1(\tau')\end{pmatrix}=(c\tau+d)\gamma\cdot\begin{pmatrix}\eta_2(\tau)\\ \eta_1(\tau)\end{pmatrix}. \tag{4.3}$$

Set $(r,s)\cdot(\eta_1(\tau),\eta_2(\tau))^T=r\eta_1(\tau)+s\eta_2(\tau)$. Then $(r',s')\cdot(\eta_1(\tau'),\eta_2(\tau'))^T=(c\tau+d)(r,s)\cdot(\eta_1(\tau),\eta_2(\tau))^T$ and so

$$Z_{r',s'}(\tau')=(c\tau+d)Z_{r,s}(\tau). \tag{4.4}$$

Together (4.2) and (4.4), we obtain

$$\wp\left(p_{r',s'}(\tau')|\tau'\right)=(c\tau+d)^2\wp\left(p_{r,s}(\tau)|\tau\right)=\wp\left(\tfrac{p_{r,s}(\tau)}{c\tau+d}\middle|\tau'\right), \tag{4.5}$$

where $(r'+s'\tau',\tau')=\gamma(r+s\tau,\tau)$. Indeed, by a direct calculation, we could prove that $\frac{p_{r,s}(\tau)}{c\tau+d}$ as a function of τ' is a solution of the elliptic form (1.7) since $p_{r,s}(\tau)$ is a solution of (1.7). Particularly, $\frac{p_{r,s}(\tau)}{c\tau+d}=\pm p_{r',s'}(\tau')$ mod $\Lambda_{\tau'}$. Recall that $\lambda_{r,s}(t)$ is the corresponding solution of (1.2), namely

$$\lambda_{r,s}(t)=\frac{\wp(p_{r,s}(\tau)|\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}. \tag{4.6}$$

Then the above argument yields the following result.

Proposition 4.4. *$\lambda_{r,s}(t)$ and $\lambda_{r',s'}(t)$ belong to the same solution of $PVI_{(\frac18,\frac{-1}{8},\frac18,\frac38)}$ if and only if $(s,r)\equiv(s',r')\cdot\gamma \bmod \mathbb{Z}^2$ by some $\gamma\in\Gamma(2)$.*

Proof. For the sufficient part, assume $(s,r)\equiv(s',r')\cdot\gamma \bmod \mathbb{Z}^2$ by some $\gamma\in\Gamma(2)$. Recall from [6, Lemma 4.2] that

$$\wp(p_{r,s}(\tau)|\tau)=\wp\left(p_{\tilde r,\tilde s}(\tau)|\tau\right)\Longleftrightarrow(r,s)\equiv\pm(\tilde r,\tilde s)\quad(\text{mod }\mathbb{Z}^2), \tag{4.7}$$

which implies that all elements in $\pm(r,s)+\mathbb{Z}^2$ give precisely the same solution $\lambda_{r,s}(t)$. Hence we may assume $(s,r)=(s',r')\cdot\gamma$ by replacing (s,r) with some element in $(s,r)+\mathbb{Z}^2$ if necessary. Let $\ell_0\subset\mathbb{H}$ be a path starting from any fixed point τ_0 to $\tau_0'=\gamma\cdot\tau_0$. Then $\ell:=t(\ell_0)\in\pi_1(\mathbb{CP}^1\backslash\{0,1,\infty\},t_0)$, where $t(\tau)=\frac{e_3(\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}$ and $t_0=t(\tau_0)$. Let $U\subset\mathbb{H}$ be a small neighborhood of τ_0 and denote $V=t(U)$. Since

$$\lambda_{r',s'}(t)=\frac{\wp(p_{r',s'}(\tau)|\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)},\ \ \tau\in U,$$

so the analytic continuation $\ell^*\lambda_{r',s'}(t)$ of $\lambda_{r',s'}(t)$ along ℓ satisfies

$$\ell^*\lambda_{r',s'}(t)=\frac{\wp(p_{r',s'}(\gamma\cdot\tau)|\gamma\cdot\tau)-e_1(\gamma\cdot\tau)}{e_2(\gamma\cdot\tau)-e_1(\gamma\cdot\tau)},\ \ \tau\in U.$$

On the other hand, $(s,r) = (s',r')\cdot\gamma$ gives $(r'+s'\tau',\tau') = \gamma(r+s\tau,\tau)$, where $\tau' = \gamma\cdot\tau$. Moreover, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$ gives

$$e_j(\gamma\cdot\tau) = (c\tau+d)^2 e_j(\tau),\ \ j=1,2,3. \tag{4.8}$$

Then it follows from (4.5) and (4.8) that

$$\begin{aligned} \lambda_{r',s'}\left(t(\gamma\cdot\tau)\right) &= \frac{\wp(p_{r',s'}(\gamma\cdot\tau)|\gamma\cdot\tau) - e_1(\gamma\cdot\tau)}{e_2(\gamma\cdot\tau)-e_1(\gamma\cdot\tau)} \\ &= \frac{\wp(p_{r,s}(\tau)|\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)} = \lambda_{r,s}\left(t(\tau)\right),\ \ \tau\in U, \end{aligned} \tag{4.9}$$

namely

$$\lambda_{r,s}\left(t\right) = \ell^*\lambda_{r',s'}\left(t\right),\quad t\in V. \tag{4.10}$$

Conversely, assume that $\lambda_{r,s}\left(t\right)$ and $\lambda_{r',s'}\left(t\right)$ represent different branches of the same solution in a small neighborhood V of $t_0 \in \mathbb{CP}^1\backslash\{0,1,\infty\}$. Then there is $\ell\in\pi_1(\mathbb{CP}^1\backslash\{0,1,\infty\},t_0)$ such that (4.10) holds. Fix any $\tau_0\in\mathbb{H}$ such that $t_0 = t\left(\tau_0\right)$ and let $t^{-1}(\ell)\subset\mathbb{H}$ denote the lifting path of ℓ under the map $t\left(\tau\right) = \frac{e_3(\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}$ such that its starting point is τ_0. Denote its ending point by τ_0'. Then $t(\tau_0') = t_0 = t(\tau_0)$, which implies $\tau_0' = \gamma\cdot\tau_0$ for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$. Let U be a neighborhood of τ_0 such that $t(U)\subset V$. Then (4.6) and (4.10) give (4.9). Define $(\tilde{s},\tilde{r}) := (s',r')\cdot\gamma$, then $(r'+s'\tau',\tau') = \gamma(\tilde{r}+\tilde{s}\tau,\tau)$, where $\tau'=\gamma\cdot\tau$, and so (4.5) gives

$$\wp(p_{r',s'}(\gamma\cdot\tau)|\gamma\cdot\tau) = (c\tau+d)^2\wp(p_{\tilde{r},\tilde{s}}(\tau)|\tau). \tag{4.11}$$

Substituting (4.11) and (4.8) into (4.9) leads to

$$\wp(p_{r,s}(\tau)|\tau) = \wp\left(p_{\tilde{r},\tilde{s}}(\tau)|\tau\right),\ \tau\in U.$$

Again by (4.7) we obtain $(r,s)\equiv\pm(\tilde{r},\tilde{s}) \bmod \mathbb{Z}^2$, namely $(s,r)\equiv(s',r')\cdot(\pm\gamma) \bmod \mathbb{Z}^2$ where $\pm\gamma\in\Gamma(2)$. $\square$

Define for any N-torsion point $(r,s) = (\frac{k_1}{N},\frac{k_2}{N})\in Q_N$,

$$\Gamma_{(r,s)} := \left\{\gamma\in SL(2,\mathbb{Z})\mid(s,r)\cdot\gamma\equiv\pm(s,r) \bmod \mathbb{Z}^2\right\}.$$

Then $\wp(p_{r,s}(\tau)|\tau)$ is a modular form of weight 2 with respect to $\Gamma_{(r,s)}$ in the sense

$$\wp(p_{r,s}(\tau')|\tau') = (c\tau+d)^2\wp(p_{r,s}(\tau)|\tau),\ \forall\gamma\in\Gamma_{(r,s)}.$$

For example, if $r=0$, then

$$\Gamma_{(r,s)} = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z})\,\middle|\, b\equiv 0, a\equiv\pm 1 \bmod N\right\}.$$

5. Geometry of Ω_5

In this and the next sections, our main purpose is to prove Theorems 1.1-1.4. In these two sections, we mainly consider $\tau \in F_0$, where $F_0 \subset \mathbb{H}$ is the fundamental domain for $\Gamma_0(2)$ defined by

$$F_0 := \{\tau \in \mathbb{H} \mid 0 \le \operatorname{Re}\tau \le 1, \, |\tau - \tfrac{1}{2}| \ge \tfrac{1}{2}\}. \tag{5.1}$$

Remark 5.1. Recall that

$$\Gamma_0(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \mid c \equiv 0 \pmod N\}.$$

It is well known that the modular curve $X_0(N) = \mathbb{H}/\Gamma_0(N)$ parametrizes the pair (E, C) of an elliptic curve E together with a cyclic subgroup $C \subset E$ with $|C| = N$. For $N = p$ being a prime, $[\mathrm{SL}(2,\mathbb{Z}) : \Gamma_0(p)] = p + 1$ and a fundamental domain for $\Gamma_0(p)$ is given by

$$\tilde{F} = F \cup S(F) \cup ST(F) \cup \cdots \cup ST^{p-1}(F),$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and F is any fundamental domain for $\mathrm{SL}(2,\mathbb{Z})$.

For $N = p = 2$, $X_0(2)$ parametrizes (E, q) with q a half period. An alternative choice of fundamental domain is $F_0 = F \cup TS(F) \cup (TS)^2(F)$ (notice that $(TS)^3 = -\mathrm{Id}$ and TS fixes $\rho = e^{\pi i/3}$).

Recall the Hecke form

$$Z_{r,s}(\tau) := \zeta(r + s\tau|\tau) - (r\eta_1(\tau) + s\eta_2(\tau)),$$

which is doubly periodic in $(r, s) \in \mathbb{R}^2$. It is related to the Green function on E_τ via $Z_{r,s}(\tau) = -4\pi\partial_z G(r + s\tau|\tau)$.

Recall also the q expansion for ζ with $q := e^{2\pi i\tau}$:

$$\begin{aligned}\zeta(z|\tau) = {} & \eta_1(\tau)z - \pi i\frac{1 + e^{2\pi iz}}{1 - e^{2\pi iz}} \\ & - 2\pi i\sum_{n=1}^{\infty}\left(\frac{e^{2\pi iz}q^n}{1 - e^{2\pi iz}q^n} - \frac{e^{-2\pi iz}q^n}{1 - e^{-2\pi iz}q^n}\right).\end{aligned} \tag{5.2}$$

(This can be deduced form the Jacobi triple product formula for theta function ϑ and the relation between ϑ and σ, see e.g. [36].)

We use the Legendre relation $\eta_1\tau - \eta_2 = 2\pi i$ and the above q expansion to compute the q expansion for Z:

$$\begin{aligned}Z_{r,s}(\tau) = {} & 2\pi is - \pi i\frac{1 + e^{2\pi iz}}{1 - e^{2\pi iz}} \\ & - 2\pi i\sum_{n=1}^{\infty}\left(\frac{e^{2\pi iz}q^n}{1 - e^{2\pi iz}q^n} - \frac{e^{-2\pi iz}q^n}{1 - e^{-2\pi iz}q^n}\right),\end{aligned} \tag{5.3}$$

where $z = r + s\tau$. See also [9, 14].

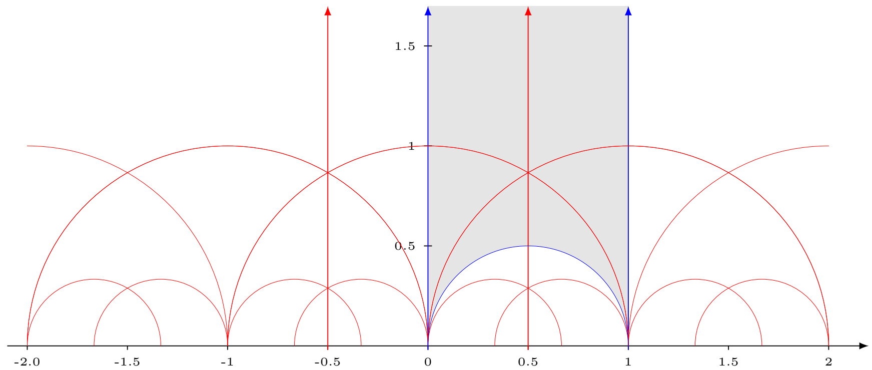

FIGURE 2. $F_0 = F \cup TS(F) \cup (TS)^2(F)$.

For fixed $s \in [0,1)$, (5.3) then implies that

$$\lim_{\tau\to\infty} Z_{r,s}(\tau) = \begin{cases} 2\pi i(s-\frac{1}{2}) & \text{if} \quad s \neq 0, \\ \pi \cot \pi r & \text{if} \quad s = 0. \end{cases} \tag{5.4}$$

By the periodicity, the limit is a discontinuous linear function with discontinuity at $s \in \mathbb{Z}$.

To compute the limit as $\tau \to 0$, we use the transformation $\tau \mapsto S\cdot\tau = -1/\tau$, and (4.4) yields

$$Z_{r,s}(-1/\tau) = \tau Z_{-s,r}(\tau), \tag{5.5}$$

and for $r \in (0,1)$,

$$Z_{r,s}(\tau) = \frac{-1}{\tau} Z_{-s,r}(-1/\tau) = \frac{2\pi i}{\tau}\big(\tfrac{1}{2} - r + o(1)\big) \tag{5.6}$$

as $\tau \to 0$. For $r = 0$, a contribution $\pi \cot \pi s/\tau$ appears as the dominant term instead. For other r, the value is determined by periodicity.

It is also easy to see that under the translation $\tau \mapsto T\cdot\tau = \tau + 1$, (4.4) yields

$$Z_{r,s}(\tau+1) = Z_{r+s,s}(\tau), \tag{5.7}$$

and for $r + s \in (0,1)$,

$$Z_{r,s}(\tau) = Z_{r+s,s}(\tau - 1) = \frac{2\pi i}{\tau - 1}\big(\tfrac{1}{2} - (r+s) + o(1)\big) \tag{5.8}$$

as $\tau \to 1$. For $r + s = 0$, the dominant term is replaced by $\pi \cot \pi s/(\tau - 1)$. For general $r + s$, the value is again determined by periodicity.

We will analyze the structure of the solutions $\tau \in F_0$ for $Z_{r,s}(\tau) = 0$ by varying (r,s). Since half periods are trivial solutions for all τ, we exclude those cases by assuming that r, s are not half integers in our discussion.

For the proof of Theorems 1.2 and 1.3, we need the following result about the critical points of the Green function $G(z|\tau)$ if $\tau \in \partial F_0$. Recall that for $\tau \in \partial F_0 \cap \mathbb{H}$, E_τ is conformally equivalent to rectangular tori.

Theorem D. [22] *If E_τ is a rectangular torus, then $G(z|\tau)$ has only three critical points, i.e, the three half-periods $\frac{\omega_k}{2}$, $k = 1, 2, 3$.*

Using our language, Theorem D just says that for any $(r,s) \in \mathbb{R}^2 \backslash \frac{1}{2}\mathbb{Z}^2$, $Z_{r,s}(\tau) \neq 0$ for $\tau \in \partial F_0 \cap \mathbb{H}$. Based on this, the idea of our analysis is to make use of the argument principle along the curve ∂F_0 to analyze the number of zeros of $Z_{r,s}$ in F_0.

We start with a simple yet important observation:

Lemma 5.2. *For any $\tau \in \mathbb{H}$,*

(i) $\zeta(\frac{3}{4}\omega_1 + \frac{1}{4}\omega_2) \neq \frac{3}{4}\eta_1 + \frac{1}{4}\eta_2$.

(ii) $\zeta(\frac{1}{6}\omega_1 + \frac{1}{6}\omega_2) \neq \frac{1}{6}\eta_1 + \frac{1}{6}\eta_2$.

(iii) $\zeta(\frac{2}{6}\omega_1 + \frac{3}{6}\omega_2) \neq \frac{2}{6}\eta_1 + \frac{3}{6}\eta_2$.

In particular, solution $\lambda_{r,s}(t)$ of $PVI_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ has no poles provided that $(r,s) \in \{(\frac{3}{4},\frac{1}{4}), (\frac{1}{6},\frac{1}{6}), (\frac{2}{6},\frac{3}{6})\}$.

Proof. We will use the addition formula

$$\frac{\wp'(z)}{\wp(z) - \wp(u)} = \zeta(z+u) + \zeta(z-u) - 2\zeta(z).$$

For (i), we choose $z = \frac{1}{4}(3\omega_1 + \omega_2) = \frac{1}{2}\omega_1 + \frac{1}{4}\omega_3$ and $u = \frac{1}{4}\omega_3$. Then $\zeta(z-u) = \zeta(\frac{1}{2}\omega_1) = \frac{1}{2}\eta_1$ and $\zeta(z+u) = \zeta(\omega_1 + \frac{1}{2}\omega_2) = \eta_1 + \frac{1}{2}\eta_2$. Hence

$$\begin{aligned}\zeta(\tfrac{3}{4}\omega_1 + \tfrac{1}{4}\omega_2) - (\tfrac{3}{4}\eta_1 + \tfrac{1}{4}\eta_2) &= \zeta(z) - \tfrac{1}{2}(\zeta(z+u) + \zeta(z-u)) \\ &= -\frac{1}{2}\frac{\wp'(\frac{3}{4}\omega_1 + \frac{1}{4}\omega_2)}{\wp(\frac{3}{4}\omega_1 + \frac{1}{4}\omega_2) - \wp(\frac{1}{4}\omega_3)} \neq 0.\end{aligned}$$

This proves (i).

For (ii), we choose $z = \frac{1}{6}(\omega_1 + \omega_2) = \frac{1}{6}\omega_3$ and $u = \frac{1}{3}\omega_3$. Then

$$\begin{aligned}0 \neq \frac{\wp'(z)}{\wp(z) - \wp(u)} &= \zeta(\tfrac{1}{2}\omega_3) + \zeta(-\tfrac{1}{6}\omega_3) - 2\zeta(\tfrac{1}{6}\omega_3) \\ &= -3(\zeta(\tfrac{1}{6}\omega_1 + \tfrac{1}{6}\omega_2) - \tfrac{1}{6}\eta_1 - \tfrac{1}{6}\eta_2).\end{aligned}$$

This proves (ii).

For (iii), we choose $z = \frac{1}{3}\omega_1 + \frac{1}{2}\omega_2$ and $u = \frac{1}{3}\omega_1$. Then $\wp'(z) \neq 0$ and

$$\begin{aligned}0 &\neq \zeta(\tfrac{2}{3}\omega_1 + \tfrac{1}{2}\omega_2) + \zeta(\tfrac{1}{2}\omega_2) - 2\zeta(\tfrac{1}{3}\omega_1 + \tfrac{1}{2}\omega_2) \\ &= \zeta(-\tfrac{1}{3}\omega_1 - \tfrac{1}{2}\omega_2) + (\eta_1 + \eta_2) + \tfrac{1}{2}\eta_2 - 2\zeta(\tfrac{1}{3}\omega_1 + \tfrac{1}{2}\omega_2) \\ &= -3(\zeta(\tfrac{1}{3}\omega_1 + \tfrac{1}{2}\omega_2) - \tfrac{1}{3}\eta_1 - \tfrac{1}{2}\eta_2).\end{aligned}$$

This proves (iii). □

Now we are in the position to prove Theorem 1.3.

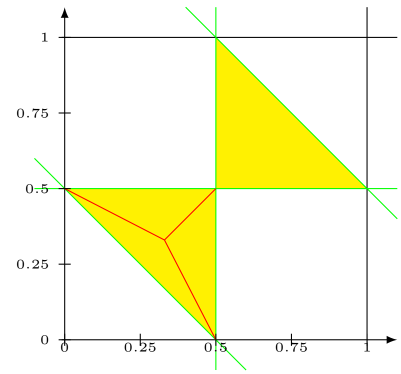

FIGURE 3. Triangle region $\triangle_0$ describing all r, s coordinates of $p(\tau)$. The upper one third is bijective to $\Omega_5 \subset \mathcal{M}_1$.

Proof of Theorem 1.3. We separate the proof into three steps.

Step 1. We will show that $Z_{r,s}(\tau)$ has no solutions if $(r,s) \notin \overline{\triangle_0}$.
Indeed, if $s, r, r+s \neq \frac{1}{2}$, then (5.4), (5.6) and (5.8) imply that

$$Z_{r,s}(\tau) \not\to 0 \quad \text{as } \tau \to \infty, 0, 1$$

respectively. Furthermore, the pole order at $\tau = 0, 1$ is unchanged among such (r,s)'s.

Thus an extended version of the argument principle shows that the number of zero of $Z_{r,s}(\tau)$ is constant in the region

$$\triangle_3 := \{(r,s) \mid r > 0, s > 0, r+s < \tfrac{1}{2}\}.$$

By Lemma (5.2) (ii), $Z_{1/6,1/6}(\tau)$ has no solutions. Since $(\frac{1}{6}, \frac{1}{6}) \in \triangle_3$, this implies that $Z_{r,s}(\tau)$ has no solutions for any $(r,s) \in \triangle_3$.

Similarly $Z_{r,s}(\tau)$ has no solutions for $(r,s) \in \square$, where

$$\square := \{(t,s) \mid \tfrac{1}{2} < t < 1 \text{ and } 0 < s < \tfrac{1}{2}\}.$$

This follows from Lemma (5.2) (i) and the fact that $(\frac{3}{4}, \frac{1}{4}) \in \square$.

Step 2. $Z_{r,s}(\tau)$ has no solutions if $(r,s) \notin \triangle_0$.

Indeed, it follows easily form the *argument principle* in complex analysis that the points (r,s) such that $Z_{r,s}(\tau)$ has only finite solutions form an open set. In particular, by Step 1, for $(r,s) \in \overline{\square \cup \triangle_3}$, the function $Z_{r,s}(\tau)$ either has no solutions or has infinite solutions (which corresponds to the trivial case $r, s \in \frac{1}{2}\mathbb{Z}$ and $Z_{r,s} \equiv 0$).

Step 3. In order to conclude the proof of the theorem, by the same reasoning as in Step 1 we only need to establish the existence and uniqueness of solution $Z_{r,s}(\tau) = 0$ in $\tau \in F_0$ for one special point $(r,s) \in \triangle_0$. For this purpose we take $(r,s) = (\frac{1}{3}, \frac{1}{3}) \in \triangle_0$.

By an easy symmetry argument (c.f. [22]), $Z_{\frac{1}{3},\frac{1}{3}}(\tau) = 0$ for $\tau = \rho := e^{\pi i/3}$. Conversely we will prove that $\rho \in F_0$ is the unique zero of $Z_{\frac{1}{3},\frac{1}{3}}$ and it is a simple zero. The following argument motivated by [14, 2] is the only place where the theory of modular forms is used.

Recall

$$Z_{(3)}(\tau) = \prod{}' Z_{\frac{k_1}{3},\frac{k_2}{3}}(\tau),$$

where the product is over all pairs (k_1, k_2) with $0 \le k_1, k_2 \le 2$ and with $\gcd(k_1, k_2, 3) = 1$. In this case it simply means $(k_1, k_2) \ne (0, 0)$. There are 8 factors in the product and in fact $Z_{(3)}$ is a modular function of weight 8 with respect to the full modular group $SL(2, \mathbb{Z})$. The counting formula for the zeros of $Z_{(3)}$ then reads as

$$\nu_\infty(Z_{(3)}) + \frac{1}{2}\nu_i(Z_{(3)}) + \frac{1}{3}\nu_\rho(Z_{(3)}) + \sum_{p \ne \infty, i, \rho} \nu_p(Z_{(3)}) = \frac{8}{12}.$$

Since $Z_{\frac{1}{3},\frac{1}{3}}(\rho) = Z_{\frac{2}{3},\frac{2}{3}}(\rho) = 0$, we have $\nu_\rho(Z_{(3)}) \ge 2$. The counting formula then implies that $\nu_\rho(Z_{(3)}) = 2$ and all the other terms vanish. Hence $\tau = \rho$ is a simple (and unique) zero for $Z_{\frac{1}{3},\frac{1}{3}}(\tau)$ (as well as for $Z_{\frac{2}{3},\frac{2}{3}}(\tau)$).

The proof of the theorem is complete. □

Corollary 5.3. *The set $\Omega_5 \subset \mathcal{M}_1$ is an "unbounded" simply connected domain.*

Proof. Let $\tilde{\Omega}_5$ be the lifting of Ω_5 in F_0, i.e.,

$$\tilde{\Omega}_5 = \{\tau \in F_0 \mid G(z|\tau) \text{ has five critical points}\}.$$

Theorem 1.3 establishes a continuous map $\phi : (r, s) \mapsto \tau$ from $\triangle_0$ onto $\tilde{\Omega}_5$. The map ϕ is one to one due to the uniqueness theorem of extra pair of nontrivial critical points of Green function G; see Theorem A in §1. Being the continuous image of a simply connected domain $\triangle_0$ under a one to one continuous function ϕ on $\mathbb{R}^2$, $\tilde{\Omega}_5$ must also be a simply connected domain. (This is the classic result on "Invariance of Domain" proved in algebraic topology. In the current case it follows easily from the inverse function theorem since ϕ is differentiable.)

It is also proven in [22] that the domain $\tilde{\Omega}_5$ contains the vertical line $\frac{1}{2} + ib$ for $b \ge b_1$ where $b_1 \in (1/2, \sqrt{3}/2)$, hence it is unbounded.

The corresponding statement for Ω_5 follows from the obvious $\mathbb{Z}_3$ identification. □

Define

$$\triangle_1 := \{(r, s) \mid \tfrac{1}{2} < r < 1, 0 < s < \tfrac{1}{2}, r + s > 1\},$$
$$\triangle_2 := \{(r, s) \mid \tfrac{1}{2} < r < 1, 0 < s < \tfrac{1}{2}, r + s < 1\}.$$

Corollary 5.4. *Let $(r, s) \in [0, 1] \times [0, \frac{1}{2}] \backslash \frac{1}{2}\mathbb{Z}^2$. Then*

(i) *$\lambda_{r,s}(t) = \infty$ (equivalently, $p_{r,s}(\tau) = 0$) has a solution $t = t(\tau)$ with $\tau \in F_0$ if and only if $(r, s) \in \triangle_0$.*

(ii) *$\lambda_{r,s}(t) = 0$ (equivalently, $p_{r,s}(\tau) = \frac{\omega_1}{2}$) has a solution $t = t(\tau)$ with $\tau \in F_0$ if and only if $(r,s) \in \triangle_1$.*
(iii) *$\lambda_{r,s}(t) = 1$ (equivalently, $p_{r,s}(\tau) = \frac{\omega_2}{2}$) has a solution $t = t(\tau)$ with $\tau \in F_0$ if and only if $(r,s) \in \triangle_2$.*
(iv) *$\lambda_{r,s}(t) = t$ (equivalently, $p_{r,s}(\tau) = \frac{\omega_3}{2}$) has a solution $t = t(\tau)$ with $\tau \in F_0$ if and only if $(r,s) \in \triangle_3$.*

Proof. Noting from (1.8) that $p_{r,s}(\tau) = 0$ in E_τ if and only if $Z_{r,s}(\tau) = 0$, this corollary follows readily from Theorem 1.3, Proposition 4.3 and (4.7). □

Now we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose $\lambda(t)$ is smooth for all $t \in \mathbb{C}\backslash\{0,1\}$. We will prove in Corollary 6.7 (see §6) that any Riccati solution has singularities in $\mathbb{C}\backslash\{0,1\}$. Therefore, $\lambda(t) = \lambda_{r,s}(t)$ for some $(r,s) \in [0,1) \times [0, \frac{1}{2}]\backslash\frac{1}{2}\mathbb{Z}^2$.

First we claim $(r,s) \in \mathbb{Q}^2$. By Corollary 5.4, we must have $(r,s) \in \cup_{k=0}^{3}\partial\triangle_k$. Recalling from (4.9) that for any $\gamma \in \Gamma(2)$,

$$\lambda_{r',s'}(t(\tau)) = \lambda_{r,s}(t(\gamma \cdot \tau)), \text{ whenever } (s',r') = (s,r) \cdot \gamma, \tag{5.9}$$

we see that $\lambda_{r',s'}(t)$ is also smooth for all $t \in \mathbb{C}\backslash\{0,1\}$, namely

$$\pm(r',s') \in \cup_{k=0}^{3}\partial\triangle_k + \mathbb{Z}^2 \text{ for any } \gamma \in \Gamma(2) \text{ and } (s',r') = (s,r) \cdot \gamma.$$

Taking $\gamma = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$, we conclude that $\{r, s, r+s\} \cap \mathbb{Q} \neq \emptyset$ and $\{4r + 3s, 3r + 2s, 7r + 5s\} \cap \mathbb{Q} \neq \emptyset$, which implies $(r,s) \in \mathbb{Q}^2$.

Once $(r,s) \in \mathbb{Q}^2$, it is straightforward to check that if for some $N \geq 3$ there are no N-torsion points contained in $\cup_{k=0}^{3}\triangle_k$, then $N = 4$. Thus (r,s) must be a 4-torsion point. By Proposition 4.4, it is easy to check that $\lambda_{\frac{1}{4},0}$ and $\lambda_{\frac{1}{4},\frac{2}{4}}$ give the same solution; $\lambda_{0,\frac{1}{4}}$ and $\lambda_{\frac{2}{4},\frac{1}{4}}$ give the same solution; $\lambda_{\frac{1}{4},\frac{1}{4}}$ and $\lambda_{\frac{3}{4},\frac{1}{4}}$ give the same solution. Therefore, $\{\lambda_{\frac{1}{4},0}, \lambda_{0,\frac{1}{4}}, \lambda_{\frac{1}{4},\frac{1}{4}}\}$ gives all the solutions that are smooth in $\mathbb{C} \setminus \{0,1\}$. □

6. Geometry of $\partial\Omega_5$

Even though $\tilde{\Omega}_5$, the lifting of Ω_5 in F_0, is a simply connected domain, its boundary may still possibly be ill-behaved. The purpose in this section is to show that this is not the case.

For $i = 1,2,3$ we put

$$C_i(F_0) := \{\tau \in F_0 \mid \tfrac{1}{2}\omega_i \text{ is a degenerate critical point of } G(z|\tau)\}. \tag{6.1}$$

It is known that all the half period points $\frac{1}{2}\omega_i$'s are non-degenerate critical points of $G(z|\tau)$ if $\tau \in \partial F_0$. Hence $C_i(F_0) \cap \partial F_0 = \emptyset$ for all i. When no confusion may possibly arise, we will drop the dependence on F_0 and simply write C_i.

The first main result of this section is

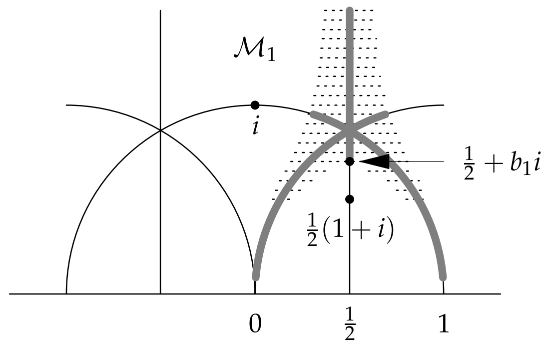

FIGURE 4. The dotted region is the lifted domain $\tilde{\Omega}_5 \subset F_0$. The lower boundary curve $C_1 \ni \frac{1}{2}+b_1 i$ consists of τ with $\frac{1}{2}\omega_1$ being a degenerate critical point of G. The upper left (resp. right) boundary is C_3 (resp. C_2) respectively.

Theorem 6.1. *(1) For each i, C_i is a smooth connected curve.*
(2)

$$\partial\tilde{\Omega}_5 = \bigcup_{i=1}^{3} C_i.$$

We first derive the equation for C_i, and then extend the discussion in [22] for rhombus tori to the general cases. To compute the Hessian of $G(z|\tau)$ at $\frac{1}{2}\omega_i$, we recall that for $\tau = a+bi$, $z = x+iy$,

$$4\pi G_z = -(\log\vartheta)_z - 2\pi i\frac{y}{b}, \tag{6.2}$$

where ϑ denotes the theta function ϑ_1. Then

$$\begin{aligned} 2\pi G_{xx} &= -\mathrm{Re}\,(\log\vartheta)_{zz}, \\ 2\pi G_{xy} &= +\mathrm{Im}\,(\log\vartheta)_{zz}, \\ 2\pi G_{yy} &= +\mathrm{Re}\,(\log\vartheta)_{zz} + \frac{2\pi}{b}, \end{aligned} \tag{6.3}$$

and the Hessian H is given by

$$\begin{aligned} H &= \det D^2 G \\ &= \frac{-1}{4\pi^2}\left(|(\log\vartheta)_{zz}|^2 + \frac{2\pi}{b}\mathrm{Re}\,(\log\vartheta)_{zz}\right) \\ &= \frac{-1}{4\pi^2}\left(\left|(\log\vartheta)_{zz} + \frac{\pi}{b}\right|^2 - \frac{\pi^2}{b^2}\right). \end{aligned} \tag{6.4}$$

The relation to the Weierstrass elliptic functions is linked by

$$(\log\vartheta)_z(z) = \zeta(z) - \eta_1 z. \tag{6.5}$$

For $z = \frac{1}{2}\omega_i$ we have then

$$(\log\vartheta)_{zz}(\tfrac{1}{2}\omega_i) = -\wp(\tfrac{1}{2}\omega_i) - \eta_1 = -(e_i+\eta_1). \tag{6.6}$$

For our discussion, using the $\mathrm{SL}(2,\mathbb{Z})$ action (see (6.13) below) we only need to work on the case $i = 1$. At the critical point $z = \frac{1}{2}$, we have clearly (by (6.2) and (6.5))

$$(\log\vartheta)_z(\tfrac{1}{2};\tau) = 0. \tag{6.7}$$

Recall the heat equation for theta function

$$\vartheta_{zz} = 4\pi i\vartheta_\tau.$$

It allows us to transform the Hessian into deformations in τ. Then

$$(\log\vartheta)_{zz} = 4\pi i(\log\vartheta)_\tau - (\log\vartheta)_z^2. \tag{6.8}$$

At $z = \frac{1}{2}$ we get $(\log\vartheta)_{zz} = 4\pi i(\log\vartheta)_\tau$, and (6.4) becomes

$$\begin{aligned} H(\tfrac{1}{2};\tau) &= -4|(\log\vartheta)_\tau|^2 + \frac{2}{b}\mathrm{Im}\,(\log\vartheta)_\tau \\ &= \frac{-1}{4b^2}\big(|-4bi(\log\vartheta)_\tau - 1|^2 - 1\big). \end{aligned} \tag{6.9}$$

That is, the curve C_i is the inverse image of the unit circle centered at $\xi = 1$ under the analytic (but not holomorphic) map $F_0 \to \mathbb{C}$:

$$\tau \mapsto \xi := -4bi(\log\vartheta)_\tau(\tfrac{1}{2};\tau) = \frac{b}{\pi}(e_1 + \eta_1). \tag{6.10}$$

To proceed, we need to calculate $(\log\vartheta)_{\tau\tau}$ at $z = \frac{1}{2}$. By (6.8), (6.7) and (6.5),

$$\begin{aligned} 4\pi i(\log\vartheta)_{\tau\tau} &= (\log\vartheta)_{zz\tau} + 2(\log\vartheta)_z(\log\vartheta)_{z\tau} \\ &= (4\pi i)^{-1}\big((\log\vartheta)_{zz} + (\log\vartheta)_z^2\big)_{zz} \\ &= (4\pi i)^{-1}\big(-\wp''(\tfrac{1}{2}) + 2(\log\vartheta)_{zz}^2\big), \end{aligned}$$

which implies that

$$(\log\vartheta)_{\tau\tau} - 2(\log\vartheta)_\tau^2 = \frac{\wp''(\frac{1}{2})}{16\pi^2} \neq 0 \tag{6.11}$$

since $\wp''(\frac{1}{2}) = (e_1 - e_2)(e_1 - e_3)$.

If $(\log\vartheta)_\tau(\frac{1}{2};\tau) = 0$, then $(\log\vartheta)_{\tau\tau} \neq 0$ and $\nabla_\tau H(\frac{1}{2};\tau) \neq 0$ since

$$\begin{aligned} \partial H/\partial a &= \frac{2}{b}\mathrm{Im}\,(\log\vartheta)_{\tau\tau}, \\ \partial H/\partial b &= \frac{2}{b}\mathrm{Re}\,(\log\vartheta)_{\tau\tau}. \end{aligned}$$

In particular C_1 is smooth near such τ.

If $(\log\vartheta)_\tau(\frac{1}{2};\tau) \neq 0$, we may write

$$H = -\frac{2}{b}|(\log\vartheta)_\tau|^2\mathrm{Im}\left(2\tau + \frac{1}{(\log\vartheta)_\tau}\right)$$

and C_1 is defined by $\operatorname{Im} f = 0$ where

$$f(\tau) := 2\tau + \frac{1}{(\log \vartheta)_\tau(\frac{1}{2};\tau)}. \tag{6.12}$$

We compute

$$f' = 2 - \frac{(\log \vartheta)_{\tau\tau}}{(\log \vartheta)_\tau^2} = -\frac{(\log \vartheta)_{\tau\tau} - 2(\log \vartheta)_\tau^2}{(\log \vartheta)_\tau^2} \neq 0.$$

Since

$$\frac{\partial \operatorname{Im} f}{\partial a} = \operatorname{Im} f', \qquad \frac{\partial \operatorname{Im} f}{\partial b} = \operatorname{Re} f',$$

we conclude again that C_1 is smooth near such τ.

Hence C_i are smooth curves for $i = 1, 2, 3$.

To characterize $\partial\tilde{\Omega}_5$, we first show that $C_i \cap \tilde{\Omega}_5 = \emptyset$. If not, say $C_i \cap \tilde{\Omega}_5$ is a (not necessarily connected) smooth curve in the open set $\tilde{\Omega}_5$. Let $\tau_0 \in C_1 \cap \tilde{\Omega}_5$. Either $(\log \vartheta)_\tau(\frac{1}{2};\tau_0) = 0$ or $\operatorname{Im} f(\tau_0) = 0$. Since $(\log \vartheta)_\tau(\frac{1}{2};\tau)$ has only discrete zeros (it is non-constant since $(\log \vartheta)_{\tau\tau} \neq 0$ over the zeros), we may choose τ_0 so that $(\log \vartheta)_\tau(\frac{1}{2};\tau_0) \neq 0$. Since $\tilde{\Omega}_5$ is open, there is a neighborhood U of τ_0 inside $\tilde{\Omega}_5$ such that $(\log \vartheta)_\tau(\frac{1}{2};\tau) \neq 0$ for all $\tau \in U$. Thus, $f(\frac{1}{2};\tau)$ is a holomorphic function in U.

By Theorem B, $z = \frac{1}{2}$ is a saddle point of $G(z|\tau)$ for all $\tau \in U$. Thus $H(\tau) \leq 0$ for all $\tau \in U$; this is equivalent to that $\operatorname{Im} f \geq 0$ over U. But $\operatorname{Im} f$ is a harmonic function on U and $\operatorname{Im} f(\tau_0) = 0$, the maximal principle implies that $\operatorname{Im} f(\tau) \equiv 0$ on U and $f(\tau)$ is a constant, which leads to a contradiction. Thus $C_i \cap \tilde{\Omega}_5 = \emptyset$ for all i.

Similar argument applies to the open set $\tilde{\Omega}_3^\circ$, the interior of $\tilde{\Omega}_3$, where $z = \frac{1}{2}$ is known to be a minimal point and $H \geq 0$ (c.f. [22]). Again the maximum principle implies $C_i \cap \tilde{\Omega}_3^\circ = \emptyset$ for all i.

Hence we have proved the following result:

Proposition 6.2. $\partial\tilde{\Omega}_5 = \partial\tilde{\Omega}_3 = \bigcup_{i=1}^3 C_i$. *In particular, for $\tau \in \tilde{\Omega}_5 \cup \tilde{\Omega}_3^\circ$, all the half period points are non-degenerate critical points.*

Proof of Theorem 6.1. It remains to show that C_i is connected for each i. Since $\partial\tilde{\Omega}_5 = \bigcup_{i=1}^3 C_i$ and $\tilde{\Omega}_5$ is simply connected, C_i can not bound any bounded domain. (We note that this can not be proved by the maximal principle as we have done above since f might have singularities on the boundary of this bounded domain. Instead, the contradiction is draw from the unboundedness and simply connectedness of $\tilde{\Omega}_5$.)

It thus suffices to show that at each cusp (i.e. $0, 1, \infty$), C_1 has at most one component near a neighborhood of them.

It is known that as $\operatorname{Im}\tau \to +\infty$, $2\eta_1(\tau) - e_1(\tau) \to 0$ and $e_1(\tau) \to \frac{2}{3}\pi^2$. Thus (6.9)-(6.10) yield that

$$C_1 \cap \{\tau \in F_0 \mid \operatorname{Im}\tau \geq R\} = \emptyset$$

for large R. Since C_1 is symmetric with respect to the line $\operatorname{Re}\tau = \frac{1}{2}$, it suffices to show that $C_1 \cap \{\tau \in F_0 \mid |\tau| < \delta_0\}$ is a smooth curve for small $\delta_0 > 0$.

It is readily checked that the Hessian of G satisfies

$$H((c\tau+d)z;\tau) = |c\tau+d|^4 H(z;\tau'), \tag{6.13}$$

where

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), \qquad \tau' = \gamma\cdot\tau = \frac{a\tau+b}{c\tau+d}.$$

Let $\gamma = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, i.e. $\tau' = (\tau-1)/\tau$. Then γ maps F_0 onto F_0 with $\gamma(\infty) = 0$. By (6.13) we have

$$H(\tfrac{1}{2};\tau) = |\tau|^4 H(\tfrac{1}{2}(1-\tau');\tau') = |\tau|^4 H(\tfrac{1}{2}(1+\tau');\tau'). \tag{6.14}$$

Therefore the degeneracy curve C_1 is mapped to the degeneracy curve C_3 and it suffices to show that $C_3 \cap \{\tau \in F_0 \mid \operatorname{Im}\tau \geq R\}$ is a smooth curve for large R.

In doing so, we use the following $q = e^{2\pi i\tau}$ expansion for $\wp(z|\tau)$:

Proposition 6.3. [21, p.46] *For $|q| < |e^{2\pi iz}| < 1/|q|$, we have*

$$\frac{\wp(z|\tau)}{(2\pi i)^2} = \frac{1}{12} + \frac{e^{2\pi iz}}{(1-e^{2\pi iz})^2} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} nq^{mn}(e^{2\pi inz} + e^{-2\pi inz} - 2).$$

By substituting $z = \frac{1}{2} + \frac{1}{2}\tau$, we have $e^{2\pi inz} = (-1)^n q^{n/2}$. After rearranging terms and simplifications, we get

$$e_3(\tau) = -\frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty}\Big((-1)^n \sum_{d\in\mathbb{N},\, n/d \text{ odd}} d - \sum_{d\in\mathbb{N},\, n/d \text{ even}} d\Big) q^{n/2}. \tag{6.15}$$

By integrating the q expansion in Proposition 6.3, we get a second q expansion for $\zeta(z|\tau)$ which does not involve $\eta_1(\tau)$ (c.f. (5.2)):

Corollary 6.4.

$$\begin{aligned}\frac{\zeta(z|\tau)}{2\pi i} &= -\frac{2\pi iz}{12} - \frac{1}{1-e^{2\pi iz}} + \frac{1}{2} \\ &\quad + 4\pi iz\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \sum_{n=1}^{\infty}\frac{e^{2\pi inz} - e^{-2\pi inz}}{1-q^n}.\end{aligned} \tag{6.16}$$

By substituting $z = \frac{1}{2}$, $e^{2\pi inz} = (-1)^n$, we get

$$\eta_1(\tau) = \frac{\pi^2}{3} - 8\pi^2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}. \tag{6.17}$$

Thus for $\tau = a+ib$,

$$e_3(\tau) + \eta_1(\tau) = 8\pi^2 e^{\pi i\tau}(1 + O(e^{-\pi b})).$$

By (6.4) and (6.6), it is easy to see that $H(\frac{1}{2}\omega_3;\tau)=0$ if and only if (a,b) satisfies

$$\cos\pi a = 4\pi b e^{-\pi b}(1+O(b^{-1})).$$

This implies that near ∞ the curve C_3 is (smooth and) connected.

The proof is complete. $\square$

Remark 6.5. Similarly, for $z=\frac{1}{2}$, $e^{2\pi i z}=-1$, Proposition 6.3 leads to

$$e_1(\tau)=\frac{2\pi^2}{3}+16\pi^2\sum_{n=1}^{\infty}\Big(\sum_{0<d|n,\, d\ \text{odd}} d\Big)q^n.$$

It had been shown in [22] that along the line $\tau=\frac{1}{2}+ib$, $e_1\nearrow\frac{2}{3}\pi^2$, $\frac{1}{2}e_1-\eta_1\nearrow 0$ and $e_1+\eta_1\nearrow\pi^2$ as $b\to+\infty$.

Recalling (6.4) and (6.6), we have

$$H(\tfrac{\omega_k}{2};\tau)=-\frac{1}{4\pi^2 b}|e_k(\tau)+\eta_1(\tau)|^2\operatorname{Im}\Big(\tau-\frac{2\pi i}{e_k(\tau)+\eta_1(\tau)}\Big).$$

In the following, we use $H(\frac{\omega_k}{2};\tau)$ to determine the location of zeros of $e_k(\tau)+\eta_1(\tau)$. Note that if $e_k(\tau)+\eta_1(\tau)\neq 0$, then $H(\frac{\omega_k}{2};\tau)=0$ if and only if $\operatorname{Im}(\tau-\frac{2\pi i}{e_k(\tau)+\eta_1(\tau)})=0$. Theorem 3.3 says that $e_k(\tau)+\eta_1(\tau)$ has only simple zeros, which can also be obtained by (6.11) as well.

Clearly $e_k(\tau)+\eta_1(\tau)$ is not a modular form. However, any zero of $e_k(\tau)+\eta_1(\tau)$ lies on the curve $H(\frac{\omega_k}{2};\tau)=0$. Recall that $H(\frac{\omega_k}{2};\tau)=0$ is the degenerate curve of $\frac{\omega_k}{2}$ as a critical point of $G(z|\tau)$. Since $E_{\tau'}$ is conformally equivalent to E_τ if $\tau'=\gamma\cdot\tau$ for some $\gamma\in SL(2,\mathbb{Z})$, but transforms $H(\frac{\omega_k}{2};\tau)=0$ to another degenerate curve $H(\frac{\omega_j}{2};\tau')=0$ (by (6.13)). Therefore, without loss of generality, we may assume $k=1$. Then by (6.10), it is equivalent to determine the location of zeros of $(\log\vartheta)_\tau(\frac{1}{2};\tau)$.

From (6.13), if $\gamma\in\Gamma_0(2)=\{\gamma\in\mathrm{SL}(2,\mathbb{Z})\mid c\equiv 0 \pmod 2\}$, the image of $C_1(F_0)$ is mapped to $C_1(F_0')$ for another fundamental domain $F_0':=\gamma(F_0)$. For example, if $\gamma=TS^{-1}T^2S^{-1}=\begin{pmatrix}1&-1\\2&-1\end{pmatrix}$, i.e.

$$\tau'=\gamma\cdot\tau=\frac{\tau-1}{2\tau-1},$$

then F_0' is the domain bounded by 3 half circles:

$$F_0'=\{\tau\in\mathbb{H}\mid |\tau-\tfrac{1}{2}|\le\tfrac{1}{2},\ |\tau-\tfrac{1}{4}|\ge\tfrac{1}{4},\ |\tau-\tfrac{3}{4}|\ge\tfrac{1}{4}\}.$$

Noting that the curve $\{\tau\mid|\tau-\frac{1}{2}|=\frac{1}{2}\}$ is invariant under γ, the curves $\overline{C_1(F_0)}$, $\overline{C_1(F_0')}$ bound a *simply connected domain* $\mathcal{D}$ in

$$F_0\cup F_0'=\{\tau\in\mathbb{H}\mid 0\le\operatorname{Re}\tau\le 1,\ |\tau-\tfrac{1}{4}|\ge\tfrac{1}{4},\ |\tau-\tfrac{3}{4}|\ge\tfrac{1}{4}\},$$

where $\overline{\mathcal{D}}\cap\mathbb{R}=\{0,1\}$. Since $\Gamma_0(2)=\Gamma(2)\cup\gamma\Gamma(2)$, $F_0\cup F_0'$ is also a fundamental domain of $\Gamma(2)$ (different from (2.1)). Note that for any $\tau\in\mathcal{D}$, the half period $\frac{1}{2}$ is a minimum point of $G(z|\tau)$ in E_τ, and Theorem 6.1 yields that $\frac{1}{2}$ is actually a non-degenerate critical point of $G(z|\tau)$.

Thus, the map $\kappa = f(\tau) = 2\tau + (\log\vartheta)_\tau^{-1}$ maps $\overline{C_1(F_0) \cup C_1(F_0')}$ to the real axis. By [22, Theorem 1.6],

$$C_1(F_0) \cap \{\tau \mid \operatorname{Re}\tau = \tfrac{1}{2}\} = \{\tfrac{1}{2} + ib_1\}, \tag{6.18}$$

where $b_1 \in (\frac{1}{2}, \frac{\sqrt{3}}{2})$ is the unique zero of the increasing function in b

$$e_1 + \eta_1 - \frac{2\pi}{b} \tag{6.19}$$

along the vertical line $\frac{1}{2} + ib$. Similarly, $C_1(F_0') \cap \{\tau \mid \operatorname{Re}\tau = \frac{1}{2}\} = \{\frac{1}{2} + ib_0\}$ where $b_0 \in (0, \frac{1}{2})$ is the unique zero of the increasing function $e_1 + \eta_1$ along $\frac{1}{2} + ib$. Then

$$f(\tfrac{1}{2} + ib) = 2(\tfrac{1}{2} + ib) - \frac{4\pi i}{e_1 + \eta_1} = 1 + \frac{2bi}{e_1 + \eta_1}\Big(e_1 + \eta_1 - \frac{2\pi}{b}\Big).$$

In particular, $f(\frac{1}{2} + ib_1) = 1$, $f(\frac{1}{2} + ib_0) = 1 - i\infty$ and f maps $\mathcal{D}$ to the lower half plane $\mathbb{C}_- = \{\kappa \mid \operatorname{Im}\kappa < 0\}$ in a locally one-to-one manner, because for any $\tau \in \mathcal{D}$, the half period $\frac{1}{2}$ is a non-degenerate minimum point. The local one-to-one is due to Theorem 3.3. Then f is actually one to one over $\overline{\mathcal{D}}$ onto $\mathbb{C}_- \cup \mathbb{R} \cup \{\infty\}$.

Since $(\log\vartheta)_\tau(\frac{1}{2};\tau) \to \infty$ when $\tau \in C_1(F_0) \cup C_1(F_0')$ tends to the boundary point 0 (resp. 1), we have by (6.12) that $f(\tau) \to 0$ (resp. 2). Therefore f maps $\overline{C_1(F_0)}$ and $\overline{C_1(F_0')}$ onto $[0, 2]$ and $\mathbb{R} \cup \{\infty\} \backslash (0, 2)$ respectively. Then $f(\tau) = \infty$ has only one solution $\tau = \frac{1}{2} + ib_0$.

Therefore we have proved the following theorem:

Theorem 6.6. *The function $(\log\vartheta)_\tau(\frac{1}{2};\tau)$ has a unique zero $\tau_0 \in F_0 \cup F_0'$. It takes the form $\tau_0 = \frac{1}{2} + ib_0$ where $0 < b_0 < \frac{1}{2}$ is the unique zero for $e_1 + \eta_1$ along the vertical line $\operatorname{Re}\tau = \frac{1}{2}$. Therefore, in any fundamental domain of $\Gamma(2)$, $(\log\vartheta)_\tau(\frac{1}{2};\tau)$ has a unique zero.*

Remark that although $(\log\vartheta)_\tau(\frac{1}{2};\tau)$ is not a modular form, the curve $C_1(F_0) \cup C_1(F_0')$, the degenerate curve of $\frac{1}{2}\omega_1$ in $F_0 \cup F_0'$, is invariant under $\Gamma(2)$, a fact coming from the invariance of the Green function under $SL(2, \mathbb{Z})$ action and that $\frac{1}{2}\omega_1$ is preserved under $\Gamma(2)$. The last statement of Theorem 6.6 follows from this and the fact that zeros of $(\log\vartheta)_\tau(\frac{1}{2};\tau)$ lie on this curve.

As a consequence, we are in a position to prove the following result about Riccati solutions.

Corollary 6.7. *Any solution of the four Riccati equations (3.2)-(3.5) has singularities in $\mathbb{C} \setminus \{0, 1\}$.*

Proof. For $k \in \{1,2,3\}$, we define

$$f_k(\tau) := \frac{\tau e_k(\tau) + \eta_2(\tau)}{e_k(\tau) + \eta_1(\tau)} = \tau - \frac{2\pi i}{e_k(\tau) + \eta_1(\tau)}.$$

Clearly $f_1(\tau) = \frac{1}{2} f(\tau)$, where f is defined in (6.12). Then the proof of Theorem 6.6 shows that f_1 *is one-to-one from* $\overline{\mathcal{D}}$ *onto* $\mathbb{C}_- \cup \mathbb{R} \cup \{\infty\}$ and $f_1(0) = 0$, $f_1(1) = 1$. Remark that $\overline{\mathcal{D}} \backslash \{0,1\} \subset \mathbb{H}$, so

$$f_1\left(\overline{\mathcal{D}} \backslash \{0,1\}\right) = \mathbb{C}_- \cup \mathbb{R} \cup \{\infty\} \backslash \{0,1\} \subset f_1(\mathbb{H}). \tag{6.20}$$

Step 1. Let $\lambda_C(t)$ be any solution of the Riccati equation (3.2). We show that $\lambda_C(t)$ has singularities in $\mathbb{C} \backslash \{0,1\}$.

If $C \in \mathbb{H}$, we let $\tau_0 = C$. By the Legendre relation $\tau_0 \eta_1(\tau_0) - \eta_2(\tau_0) = 2\pi i$, we easily deduce from (3.11) that $t(\tau_0)$ is a pole of λ_C.

It suffices to consider $C \in \mathbb{C}_- \cup \mathbb{R} \cup \{\infty\}$. First we assume $C \notin \{0,1\}$. Then (6.20) shows the existence of $\tau_0 \in \overline{\mathcal{D}} \backslash \{0,1\} \subset \mathbb{H}$ such that $f_1(\tau_0) = C$, which is equivalent to

$$\wp(p_C(\tau_0)|\tau_0) = \frac{\eta_2(\tau_0) - C\eta_1(\tau_0)}{C - \tau_0} = e_1(\tau_0).$$

Therefore, $\lambda_C(t(\tau_0)) = 0$, i.e., $t(\tau_0)$ is a type 1 singularity of λ_C.

Before we consider the final case $C \in \{0,1\}$, we prove that

$$\mathbb{C}_- \cup \mathbb{R} \backslash \{-1\} \subset f_2(\mathbb{H}), \tag{6.21}$$

$$\mathbb{C}_- \cup \mathbb{R} \cup \{\infty\} \backslash \{0, \tfrac{1}{2}\} \subset f_3(\mathbb{H}). \tag{6.22}$$

Let $\tau' = \gamma \cdot \tau = \frac{-\tau}{\tau - 1}$ and $C' = \gamma \cdot C = \frac{-C}{C-1}$, where $\gamma = STS = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. Using (4.2) it is easy to see $t(\tau') = \frac{t(\tau)}{t(\tau)-1}$. Let $\lambda_C(t)$ and $\lambda_{C'}(t)$ be solutions of the Riccati equation (3.2).Then by (3.19), we easily obtain

$$\lambda_{C'}\left(\tfrac{t(\tau)}{t(\tau)-1}\right) = \lambda_{C'}(t(\tau')) = \frac{\lambda_C(t(\tau)) - t(\tau)}{1 - t(\tau)}.$$

For any $C \in \mathbb{C}_- \cup \mathbb{R} \cup \{\infty\} \backslash \{0, \frac{1}{2}\}$, we have $C' \in \mathbb{C}_- \cup \mathbb{R} \cup \{\infty\} \backslash \{0,1\}$, which implies the existence of $\tau_0 \in \mathbb{H}$ such that $\lambda_{C'}\left(\frac{t(\tau_0)}{t(\tau_0)-1}\right) = 0$. Consequently, $\lambda_C(t(\tau_0)) = t(\tau_0)$, i.e., $\wp(p_C(\tau_0)|\tau_0) = e_3(\tau_0)$. This, together with (3.11), gives $f_3(\tau_0) = C$. This proves (6.22). To prove (6.21), we let $\tau' = S \cdot \tau = \frac{-1}{\tau}$ and $C' = S \cdot C = \frac{-1}{C}$. Then $t(\tau') = 1 - t(\tau)$ and

$$\lambda_{C'}(1 - t(\tau)) = \lambda_{C'}(t(\tau')) = 1 - \lambda_C(t(\tau)).$$

From here, we can prove (6.21) similarly.

Now for $C \in \{0,1\}$, (6.21) shows the existence of $\tau_0 \in \mathbb{H}$ such that $f_2(\tau_0) = C$, which is equivalent to $\lambda_C(t(\tau_0)) = 1$. Thus, λ_C has a type 2 singularity at $t(\tau_0)$. This completes the proof of Step 1.

Step 2. Let $\lambda(t)$ be any solution of the three Riccati equations (3.3)-(3.5). We show that $\lambda(t)$ has singularities in $\mathbb{C} \backslash \{0,1\}$.

For $\lambda(t)$ satisfying (3.3), we define

$$\tilde{\lambda}(t) := \frac{t}{\lambda(t)}.$$

Then a straightforward computation shows that $\tilde{\lambda}(t)$ solves (3.2). Since Step 1 shows that $\tilde{\lambda}(t)$ has singularities in $\mathbb{C}\backslash\{0,1\}$, so does $\lambda(t)$.

For $\lambda(t)$ satisfying (3.4), we define

$$\tilde{\lambda}(t) := \frac{\lambda(t)-t}{\lambda(t)-1}.$$

Again $\tilde{\lambda}(t)$ solves (3.2), which implies that $\lambda(t)$ has singularities in $\mathbb{C}\backslash\{0,1\}$.

For $\lambda(t)$ satisfying (3.5), we define

$$\tilde{\lambda}(t) := t\frac{\lambda(t)-1}{\lambda(t)-t}.$$

Again $\tilde{\lambda}(t)$ solves (3.2), so $\lambda(t)$ has singularities in $\mathbb{C}\backslash\{0,1\}$.

The proof is complete. □

Remark 6.8. There is another way to prove Step 2 of Corollary 6.7. That is, we can exploit the formula (3.18) and (6.20)-(6.22) to show that $\lambda_C(t)$ has singularities just as done in Step 1. We leave the details to the reader.

We conclude this section by giving the proof of Theorem 1.4. Recall the fundamental domain F_2 of $\Gamma(2)$ defined in (2.1) and F_0 of $\Gamma_0(2)$ defined in (5.1). As mentioned in Subsection 2.1, first we prove the following

Lemma 6.9. *The map $t(\tau) = \frac{e_3(\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}$ maps the interior of F_0 onto the lower half plane $\mathbb{C}_-$, and maps $F_2\backslash F_0$ onto $\mathbb{C}_+$.*

Proof. Note that $t(i\mathbb{R}^+) = (0,1)$ (see e.g. [6]). Using $t(T\cdot\tau) = \frac{1}{t(\tau)}$ and $t(ST^{-1}\cdot\tau) = 1-\frac{1}{t(\tau)}$ (see e.g. Propositions B.2 and B.3 in Appendix B), we obtain $t(1+i\mathbb{R}^+) = (1,+\infty)$ and $t(\{\tau\in\mathbb{H} \mid |\tau-\frac{1}{2}| = \frac{1}{2}\}) = (-\infty,0)$. That is, $t(\tau)$ maps $\partial F_0\cap\mathbb{H}$ onto $\mathbb{R}\backslash\{0,1\}$.

Recalling $\rho = e^{\frac{\pi i}{3}} \in F_0$, we claim $t(\rho)\in\mathbb{C}_-$. Indeed, By [22, (2.10)] we have

$$\wp(z|\rho) = \rho^2\wp(\rho z|\rho),$$

by which, it is easy to see that $e_3(\rho) = \rho^2 e_1(\rho)$ and $e_2(\rho) = \rho^{-2}e_1(\rho) = \rho^4 e_1(\rho)$. Hence,

$$t(\rho) = \frac{\rho^2-1}{\rho^4-1} = \bar{\rho} = \frac{1}{2}(1-\sqrt{3}) \in \mathbb{C}_-.$$

Now this lemma follows readily from the fact that $t(\tau)$ is one-to-one from F_2 onto $\mathbb{C}\backslash\{0,1\}$. □

Proof of Theorem 1.4. Suppose $\lambda(t)$ is a real solution. Then there exists $(r,s) \in [0,1] \times [0,\frac{1}{2}] \setminus \frac{1}{2}\mathbb{Z}^2$ such that $\lambda(t) = \lambda_{r,s}(t)$. The goal is to prove that any branch of $\lambda_{r,s}(t)$ has at most one singular point in both $\overline{\mathbb{C}_-} \setminus \{0,1\}$ and $\mathbb{C}_+$. For this purpose, it suffices to consider the F_2 branch (i.e., the branch corresponding to $\tau \in F_2$) when a branch of $\lambda_{r,s}(t)$ in $\mathbb{C} \setminus \{0,1\}$ is discussed. By (5.9) (or (4.9)), *for any other branch of the same real solution $\lambda_{r,s}(t)$ in $\mathbb{C} \setminus \{0,1\}$*, which can be obtained from (1.8) by considering τ in another fundamental domain of $\Gamma(2)$, *its restriction in $\overline{\mathbb{C}_-} \setminus \{0,1\}$ (resp. in $\mathbb{C}_+$) is just the restriction in $\overline{\mathbb{C}_-} \setminus \{0,1\}$ (resp. in $\mathbb{C}_+$) of the F_2 branch of a "new" real solution $\lambda_{r',s'}(t)$.* Therefore, we only need to prove this theorem for the F_2 branch.

Step 1. We consider $\tau \in F_0$. Applying Corollary 5.4 and Lemma 6.9, we see that the F_2 branch of $\lambda_{r,s}(t)$ has at most one singular point in $\overline{\mathbb{C}_-} \setminus \{0,1\} = t(F_0)$. More precisely, this F_2 branch of $\lambda_{r,s}(t)$ has no singularities in $\mathbb{R} \setminus \{0,1\} = t(\partial F_0 \cap \mathbb{H})$ (see also [6]); if $(r,s) \in \cup_{k=0}^{3} \partial \triangle_k$, then it has no singularities in $\mathbb{C}_-$ either; while for $k \in \{0,1,2,3\}$, it has only a type k singularity in $\mathbb{C}_-$ if and only if $(r,s) \in \triangle_k$. Recalling that $\tilde{\Omega}_5$ is the lifting of Ω_5 in F_0, we have

$$\begin{aligned}\tilde{\Omega}_5 &= \{\tau \in F_0 \mid G(z|\tau) \text{ has five critical points}\} \\ &= \left\{\tau \in F_0 \mid Z_{r,s}(\tau) = 0 \text{ for some } (r,s) \in [0,1] \times [0,\tfrac{1}{2}] \setminus \tfrac{1}{2}\mathbb{Z}^2\right\} \\ &= \{\tau \in F_0 \mid Z_{r,s}(\tau) = 0 \text{ for some } (r,s) \in \triangle_0\}.\end{aligned}$$

This, together with the definition (1.22) of $\Omega_-^{(0)}$, easily implies

$$\Omega_-^{(0)} = t\left(\{\tau \in F_0 \mid Z_{r,s}(\tau) = 0 \text{ for some } (r,s) \in \triangle_0\}\right) = t(\tilde{\Omega}_5).$$

Therefore, we conclude from Theorem 6.1 that $\Omega_-^{(0)}$ is open and simply connected and $\partial\Omega_-^{(0)}$ consists of three smooth curves connecting $0, 1, \infty$ respectively; they are precisely $t(C_i(F_0))$ for $i = 1,2,3$, where $C_i(F_0)$ is defined in (6.1).

Now we recall $\Omega_-^{(k)}$ defined in (1.23) and fix $k \in \{1,2,3\}$. It follows from Proposition 4.3, (4.7) and Corollary 5.4 that

$$\begin{aligned}\Omega_-^{(k)} &= t\left(\left\{\tau \in F_0 \mid p_{r,s}(\tau) = \tfrac{\omega_k}{2} \text{ for some } (r,s) \in [0,1] \times [0,\tfrac{1}{2}] \setminus \tfrac{1}{2}\mathbb{Z}^2\right\}\right) \\ &= t\left(\left\{\tau \in F_0 \mid p_{r_k,s_k}(\tau) = 0 \text{ for some } (r,s) \in [0,1] \times [0,\tfrac{1}{2}] \setminus \tfrac{1}{2}\mathbb{Z}^2\right\}\right) \\ &= t\left(\left\{\tau \in F_0 \mid p_{r,s}(\tau) = 0 \text{ for some } (r,s) \in [0,1] \times [0,\tfrac{1}{2}] \setminus \tfrac{1}{2}\mathbb{Z}^2\right\}\right) \\ &= t\left(\{\tau \in F_0 \mid Z_{r,s}(\tau) = 0 \text{ for some } (r,s) \in \triangle_0\}\right) \\ &= \Omega_-^{(0)}.\end{aligned}$$

Step 2. We consider $\tau \in F_2 \setminus F_0$. Then $\tau' = T^{-1} \cdot \tau = \tau - 1 \in F_0$. By (B.11), we have $t(\tau) = 1/t(\tau')$ and

$$\lambda_{r,s}(t(\tau)) = \frac{\lambda_{r_1,s_1}(t(\tau'))}{t(\tau')},$$

where $(r_1, s_1) \in [0,1] \times [0, \frac{1}{2}] \backslash \frac{1}{2}\mathbb{Z}^2$ is given by (B.10):

$$(r_1, s_1) := \begin{cases} (r+s, s) & \text{if } r+s<1, \\ (r+s-1, s) & \text{if } r+s \geq 1. \end{cases} \tag{6.23}$$

Therefore, by the result of Step 1, we conclude that the F_2 branch of $\lambda_{r,s}(t)$ has at most one singular point in $\mathbb{C}_+ = t(F_2 \setminus F_0)$ and $\Omega_+^{(k)} = \Omega_+^{(0)} = \{t \in \mathbb{C}_+ \mid t^{-1} \in \Omega_-^{(0)}\}$ (see (1.23) for the definition of $\Omega_+^{(k)}$).

In conclusion, the F_2 branch of $\lambda_{r,s}(t)$ has at most two singular points in $\mathbb{C} \setminus \{0,1\}$. If it has two singular points, then one is in $\mathbb{C}_+$ and the other one is in $\mathbb{C}_-$. Furthermore, they are the same type 0 (resp. type 1) singular points if and only if both (r,s) and (r_1, s_1) given by (6.23) belong to $\triangle_0$ (resp. $\triangle_1$); while they are the same type 2 (resp. type 3) if and only if $(r,s) \in \triangle_2$ and $(r_1, s_1) \in \triangle_3$ (resp. $(r,s) \in \triangle_3$ and $(r_1, s_1) \in \triangle_2$). Therefore, it is easy to see from the definition of $\triangle_k$ and (6.23) that these two singular points can not be the same type. Finally, any real solution is smooth in $\mathbb{C} \setminus (\Omega_-^{(0)} \cup \Omega_+^{(0)} \cup \{0,1\})$.

The proof is complete. □

7. Algebraic solutions

In this section, we study the monic polynomial $\ell_N(j)$ defined in (1.28) and prove Theorems 1.6 and 1.7. In the following we always assume $N \geq 5$.

Lemma 7.1.

$$\deg \ell_N = \begin{cases} \frac{|Q_N|}{24} & \textit{if } N \textit{ is odd,} \\ \frac{|Q_N|}{24} - \frac{1}{2}\varphi(\frac{N}{2}) & \textit{if } N \textit{ is even.} \end{cases}$$

Proof. Recalling $q = e^{2\pi i \tau}$, we use the q-expansions (cf. [16, p.193]):

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{+\infty} (1-q^n)^{24}, \tag{7.1}$$

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots. \tag{7.2}$$

Let $\tau = ib$ with $b \uparrow +\infty$, then $q = e^{-2\pi b} \downarrow 0$. By (B.3), (B.6), (B.7) in Appendix B and (1.27), we have for $(r,s) \in Q_N$ that, as $b \uparrow +\infty$,

$$Z_{r,s}(\tau) = \begin{cases} 2\pi i(s - \frac{1}{2}) + O(q^s) & \text{if } s \notin \{0, \frac{1}{2}\}, \\ \pi \cot \pi r + O(q) & \text{if } s = 0, \\ 4\pi \sin(2\pi r) q^{\frac{1}{2}} + O(q) & \text{if } s = \frac{1}{2}. \end{cases} \tag{7.3}$$

First we assume that N is odd. Then $r, s \neq \frac{1}{2}$, which implies that $Z_{(N)}(\tau)$ converges to a nonzero constant as $b \uparrow +\infty$. Substituting (7.1) and (7.2) into (1.28) and computing the leading term, we obtain $\deg \ell_N = m = \frac{|Q_N|}{24}$.

Now we consider that N is even. Then the number of $(r, \frac{1}{2})$ in Q_N is $2\varphi(\frac{N}{2})$, which implies from (7.3) that $Z_{(N)}(\tau) \sim q^{\varphi(\frac{N}{2})}$ as $b \uparrow +\infty$. Again by (7.1), (7.2) and (1.28), we obtain $\deg \ell_N = m - \frac{1}{2}\varphi(\frac{N}{2})$. $\square$

Lemma 7.2. *The constant C_{2m} and all the coefficients of $\ell_N(j)$ are rational numbers. In particular, all zeros of $\ell_N(j)$ are algebraic numbers.*

Proof. Denote $a = e^{2\pi i/N}$. Then for any $(r, s) = (\frac{k_1}{N}, \frac{k_2}{N}) \in Q_N$, $e^{2\pi i(r+s\tau)} = a^{k_1}q^s = a^{k_1}q^{\frac{k_2}{N}}$. Recalling the q-expansion (5.3) of $Z_{r,s}$, we have

$$\begin{aligned}\frac{Z_{r,s}(\tau)}{\pi i} =& 2s - (1 + a^{k_1}q^s)\sum_{l=0}^{\infty}(a^{k_1}q^s)^l \\ & - 2\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\left((a^{k_1}q^{n+s})^l - (a^{-k_1}q^{n-s})^l\right), \; if \; s \neq 0, \\ \frac{Z_{r,0}(\tau)}{\pi i} =& -\frac{1 + a^{k_1}}{1 - a^{k_1}} - 2\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\left((a^{k_1}q^n)^l - (a^{-k_1}q^n)^l\right).\end{aligned} \tag{7.4}$$

Therefore,

$$\frac{Z_{(N)}(\tau)}{(\pi i)^{|Q_N|}} = \prod_{(r,s)\in Q_N}\frac{Z_{r,s}(\tau)}{\pi i} = R_0(a) + \sum_{n=1}^{\infty}R_n(a)q^n, \tag{7.5}$$

where $R_j(a)$ are rational functions of a with integer coefficients. Here by (1.28) and (7.1)-(7.2) we know that there are no terms of q with fractional powers in (7.5). Define

$$P_N := \{k \in \mathbb{N} \mid 1 \leq k \leq N-1, \; \gcd(k, N) = 1\}.$$

Fix any $k \in P_N$. For any $(r, s) = (\frac{k_1}{N}, \frac{k_2}{N}) \in Q_N$, we also have $\gcd(kk_1, k_2, N) = 1$. Denote $r' = kr - [kr] \in [0, 1)$, then $(r', s) \in Q_N$ and $a^N = 1$ gives $e^{2\pi i(r'+s\tau)} = (a^k)^{k_1}q^s$. Thus, repeating the argument of (7.4)-(7.5) leads to

$$\prod_{(r,s)\in Q_N}\frac{Z_{r',s}(\tau)}{\pi i} = R_0(a^k) + \sum_{n=1}^{\infty}R_n(a^k)q^n.$$

Since (r', s) takes over all elements in Q_N whenever (r, s) does, we conclude that

$$\frac{Z_{(N)}(\tau)}{(\pi i)^{|Q_N|}} = \prod_{(r,s)\in Q_N}\frac{Z_{r',s}(\tau)}{\pi i} = R_0(a^k) + \sum_{n=1}^{\infty}R_n(a^k)q^n, \quad \forall k \in P_N.$$

Comparing this with (7.5), we have for any $j \geq 0$ that $R_j(a) = R_j(a^k), \forall k \in P_N$, which implies that $R_j(a)$ are rational numbers.

Recall $|Q_N| = 24m$ and $i^{|Q_N|} = 1$. It follows from (1.28), (7.1) and (7.5) that all the coefficients of the q-expansion of $C_{2m}(\ell_N(j))^2 = \frac{Z_{(N)}(\tau)}{\Delta(\tau)^{2m}}$ are rational numbers. This, together with (7.2), easily implies that C_{2m} and all the coefficients of $\ell_N(j)$ are rational numbers. This completes the proof. $\square$

Motivated by (1.27), we define

$$Q'_N = \left\{ (r,s) \in Q_N \;\middle|\; \begin{array}{c} r < \frac{1}{2} \text{ if } s = 0;\ s < \frac{1}{2} \text{ if } r = 0; \\ s \leq \frac{1}{2} \text{ if } r \neq 0, s \neq 0 \end{array} \right\}. \tag{7.6}$$

Clearly $|Q'_N| = |Q_N|/2$. We are now in the position to prove Theorem 1.6. Recall the fundamental domain F of $SL(2,\mathbb{Z})$ defined in (1.34).

Proof of Theorem 1.6. The assertion (i) follows readily from the fact that for $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, $\lambda_{r,s}(t_0) = \infty$ for some $t_0 = t(\tau_0)$ if and only if $Z_{r,s}(\tau_0) = 0$. So it suffices to prove (ii) and (iii).

(ii) Assume by contradiction that $j_0 = j(\tau_0)$, $\tau_0 \in F$, is a multiple zero of $\ell_N(j)$. Then by (1.28) and (1.27), there exist at least two $(r_i, s_i) \in Q'_N$ such that $Z_{r_i,s_i}(\tau_0) = 0$ for $i = 1,2$. The definition (7.6) of Q'_N implies that $r_1 + s_1\tau_0 \neq \pm(r_2 + s_2\tau_0)$ in the torus E_{τ_0}. Thus, $G(z|\tau_0)$ has two pairs of nontrivial critical points $\pm(r_1 + s_1\tau_0)$ and $\pm(r_2 + s_2\tau_0)$, a contradiction with Theorem A.

(iii) Suppose that for some $N_1 \neq N_2$, $\ell_{N_1}(j)$ and $\ell_{N_2}(j)$ has a common zero $j_0 = j(\tau_0)$, $\tau_0 \in F$. Then there exists $(r_i, s_i) \in Q'_{N_i}$ such that $Z_{r_i,s_i}(\tau_0) = 0$ for $i = 1,2$. Clearly $r_1 + s_1\tau_0 \neq \pm(r_2 + s_2\tau_0)$ in the torus E_{τ_0}, again we obtain a contradiction.

The proof is complete. $\square$

To give the proof of Theorem 1.7, we exploit the following result in [22]. Recall from (6.19) that $b_1 \in (\frac{1}{2}, \frac{\sqrt{3}}{2})$ is the unique zero of the increasing function $e_1 + \eta_1 - \frac{2\pi}{b}$ in b where τ is along the vertical line $\frac{1}{2} + ib$.

Theorem E. [22, Lemma 6.4 and Theorem 6.7] *For any $b > b_1$, $G(z|\tau)$ with $\tau = \frac{1}{2} + ib$ has a critical point of the form $\frac{1}{2} + iy(b)$ with $y(b) \in (0, \frac{b}{2})$.*

Recalling from (6.18) that $\tau_1 := \frac{1}{2} + ib_1 \in \partial\tilde{\Omega}_5$, where $\tilde{\Omega}_5$ is the lifting of Ω_5 in F_0, so $G(z|\tau_1)$ has only three critical points $\frac{1}{2}$, $\frac{\tau_1}{2}$ and $\frac{1+\tau_1}{2}$. Therefore, $y(b) \downarrow 0$ as $b \downarrow b_1$. Write the critical point $\frac{1}{2} + iy(b) = r(b) + s(b)\tau$, then

$$2r(b) + s(b) = 1, \;\; s(b) = \frac{y(b)}{b} \in \left(0, \tfrac{1}{2}\right), \quad \lim_{b \to b_1} s(b) = 0. \tag{7.7}$$

Lemma 7.3. *As a function of $b \in (b_1, +\infty)$, $s(b)$ is strictly increasing. Furthermore, $s(\frac{\sqrt{3}}{2}) = \frac{1}{3}$ and $\lim_{b\to\infty} s(b) = \frac{1}{2}$.*

Proof. It was shown in [22] that if $\tau = \rho = e^{\frac{\pi i}{3}}$, then $G(z|\tau)$ has a critical point at $\frac{1+\tau}{3}$, which gives $s(\frac{\sqrt{3}}{2}) = \frac{1}{3}$.

If $s(b)$ is not strictly increasing, then there exist $b_3 > b_2 > b_1$ such that $s(b_2) = s(b_3)$. Clearly (7.7) gives $r(b_2) = r(b_3)$ and $(r,s) := (r(b_2), s(b_2)) \in \triangle_0$. Write $\tau_k = \frac{1}{2} + ib_k$ for $k = 2,3$, then $\tau_k \in F_0$ by (5.1). Since $G(z|\tau_k)$ has a critical point at $r + s\tau_k$, so $Z_{r,s}(\tau)$ has two zeros $\tau_2, \tau_3 \in F_0$, which contradicts to Theorem 1.3.

Finally, we prove $\lim_{b\to\infty} s(b) = \frac{1}{2}$. Suppose $\lim_{b\to\infty} s(b) = \bar{s} < \frac{1}{2}$. Define a function $K : (0, \frac{1}{2}) \times F_0 \to \mathbb{C}$ by $K(s,\tau) := Z_{\frac{1-s}{2},s}(\tau)$. Since $(\frac{1-\bar{s}}{2}, \bar{s}) \in \triangle_0$, Theorem 1.3 shows that there is a unique $\bar{\tau} \in F_0$ such that $K(\bar{s},\bar{\tau}) = 0$. Furthermore, Theorem 4.1 gives $\frac{\partial K}{\partial \tau}(\bar{s},\bar{\tau}) \neq 0$. Then by the implicit function theorem, there exists a function $\tau(s)$ for $s \in (\bar{s}-\epsilon, \bar{s}+\epsilon)$ such that $\tau(\bar{s}) = \bar{\tau}$ and $K(s,\tau) = 0$ for $s \in (\bar{s}-\epsilon, \bar{s}+\epsilon)$ if and only if $\tau = \tau(s)$, where $\epsilon > 0$ is small. Thus Theorem E implies $\tau(s) = \frac{1}{2} + ib$ for $s \in (\bar{s}-\epsilon, \bar{s})$ and $b \uparrow +\infty$ as $s \uparrow \bar{s}$, which is a contradiction with $\tau(s) \to \bar{\tau}$. This completes the proof. □

Now we can give the proof of Theorem 1.7.

Proof of Theorem 1.7. Let $j = j(\tau_0) \in J(N)$, $\tau_0 \in F$, be a real zero of $\ell_N(j)$. Since $J(3) = \{0\}$, the same proof as Theorem 1.6-(iii) shows that $j(\tau_0) \neq 0$. Recall that in F, $j(\tau)$ maps $\{ib \mid b \geq 1\}$ onto $[1728, +\infty)$; maps $\{\tau \in F \mid |\tau| = 1\}$ onto $[0, 1728]$; maps $\{\frac{1}{2} + ib | b > \frac{\sqrt{3}}{2}\}$ onto $(-\infty, 0)$. Since Theorem D says that $Z_{r,s}(\tau) \neq 0$ for any $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and $\tau \in i\mathbb{R}^+$, we deduce from Theorem 1.6-(i) that $\ell_N(j)$ has no zeros in $[1728, +\infty)$. Thus, $j(\tau_0) \in (-\infty, 0) \cup (0, 1728)$.

Step 1. We prove that $\ell_N(j)$ has $\#J_N^-$ zeros in $(-\infty, 0)$.

Assume that $j(\tau_0) \in (-\infty, 0)$, $\tau_0 \in F$, is a zero of $\ell_N(j)$. Then $\tau_0 \in \{\frac{1}{2} + ib \mid b > \frac{\sqrt{3}}{2}\}$ and there exists $(r,s) \in Q_N'$ such that $Z_{r,s}(\tau_0) = 0$. Write $\tau_0 = \frac{1}{2} + i\hat{b}$ with $\hat{b} > \frac{\sqrt{3}}{2}$. Then Theorem E, Lemma 7.3 and (7.7) imply that $Z_{r(\hat{b}),s(\hat{b})}(\tau_0) = 0$ and $\frac{1}{3} < s(\hat{b}) < \frac{1}{2}$. Since (r,s), $(r(\hat{b}), s(\hat{b})) \in [0,1) \times [0, \frac{1}{2}]$ and $G(z|\tau_0)$ has at most one pair of nontrivial critical points $\pm(r + s\tau_0)$, we conclude that $(r,s) = (r(\hat{b}), s(\hat{b}))$, i.e., $(r,s) \in J_N^-$.

Conversely, given $(r,s) \in J_N^-$, by Theorem E, Lemma 7.3 and (7.7), there exists $\bar{b} \in (\frac{\sqrt{3}}{2}, +\infty)$ such that $s = s(\bar{b})$, namely $G(z|\bar{\tau})$ with $\bar{\tau} = \frac{1}{2} + i\bar{b}$ has a critical point at $r + s\bar{\tau}$. Thus $Z_{r,s}(\bar{\tau}) = 0$ and then $j(\bar{\tau}) \in (-\infty, 0)$ is a zero of $\ell_N(j)$.

For any two different $(r_2, s_2), (r_3, s_3) \in J_N^-$, we have $s_2 \neq s_3$, say $s_2 < s_3$. Then there exist $b_3 > b_2 > b_1$ such that $s_k = s(b_k)$ for $k = 2, 3$. Write $\tau_k = \frac{1}{2} + ib_k$. Then $b_3 > b_2$ implies $j(\tau_2) > j(\tau_3)$, so $j(\tau_2) \neq j(\tau_3)$ are two different zeros of $\ell_N(j)$ in $(-\infty, 0)$. This proves the one-to-one correspondence between elements of J_N^- and negative zeros of $\ell_N(j)$. Therefore, $\ell_N(j)$ has exactly $\#J_N^-$ zeros in $(-\infty, 0)$.

Step 2. We prove that $\ell_N(j)$ has $\#J_N^+$ zeros in $(0, 1728)$.

Assume that $j(\tau_0) \in (0, 1728)$, $\tau_0 \in F$, is a zero of $\ell_N(j)$. Then $\tau_0 \in \{\tau \in F \mid |\tau| = 1\}$. Let $\tau' = \gamma \cdot \tau = \frac{1}{1-\tau}$, where $\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Then $\tau' = \frac{1+i}{2}$ if

$\tau = i$, $\tau' = \tau$ if $\tau = e^{\frac{\pi i}{3}}$, and

$$\tau \in \{\tau \in F \mid |\tau| = 1\} \Longleftrightarrow \tau' \in \{\tfrac{1}{2} + ib \mid \tfrac{1}{2} \leq b \leq \tfrac{\sqrt{3}}{2}\}.$$

Furthermore, $j(\tau') = j(\tau)$, which gives that j maps $\{\frac{1}{2} + ib \mid \frac{1}{2} \leq b \leq \frac{\sqrt{3}}{2}\}$ onto $[0, 1728]$ with $j(\frac{1+i}{2}) = 1728$ and $j(e^{\frac{\pi i}{3}}) = 0$. Therefore, $\tau_0' \in \{\frac{1}{2} + ib \mid \frac{1}{2} < b < \frac{\sqrt{3}}{2}\}$ and there exists $(r, s) \in Q_N'$ such that $Z_{r,s}(\tau_0') = 0$. Since it was shown in [22] that $G(z|\tau)$ has only three critical points $\frac{1}{2}$, $\frac{\tau}{2}$ and $\frac{1+\tau}{2}$ if $\tau = \frac{1}{2} + ib$ with $\frac{1}{2} \leq b \leq b_1$, we see that $\tau_0' = \frac{1}{2} + i\hat{b}$ with $b_1 < \hat{b} < \frac{\sqrt{3}}{2}$. Then Theorem E, Lemma 7.3 and (7.7) imply that $Z_{r(\hat{b}),s(\hat{b})}(\tau_0') = 0$ and $0 < s(\hat{b}) < \frac{1}{3}$. Similarly as in Step 1, we conclude that $(r, s) = (r(\hat{b}), s(\hat{b}))$, i.e., $(r, s) \in J_N^+$.

Conversely, given $(r, s) \in J_N^+$, there exists $\bar{b} \in (b_1, \frac{\sqrt{3}}{2})$ such that $s = s(\bar{b})$, namely $G(z|\bar{\tau})$ with $\bar{\tau} = \frac{1}{2} + i\bar{b}$ has a critical point at $r + s\bar{\tau}$. Thus $Z_{r,s}(\bar{\tau}) = 0$ and then $j(\bar{\tau}) \in (0, 1728)$ is a zero of $\ell_N(j)$.

Finally, we can prove that any two different points in J_N^+ correspond to two different positive zeros of $\ell_N(j)$ as in Step 1. Therefore, $\ell_N(j)$ has exactly $\#J_N^+$ zeros in $(0, 1728)$.

The proof is complete. □

In the rest of this section, we give an generic approach to compute $\ell_N(j)$ for small N. Fix $N \geq 5$ with $N \neq 6$. Recalling that $J(N)$ is the zero set of $\ell_N(j)$, we denote

$$J(N) = \{j_k \mid 1 \leq k \leq \deg \ell_N\}.$$

Instead of considering the product like $Z_{(N)}(\tau)$, we consider the summation of $\lambda_{r,s}(t)$ with $(r, s) \in Q_N'$ (see (7.6)), because (4.7) implies that $\lambda_{r,s}(t) = \lambda_{1-r,1-s}(t)$ if $r, s \neq 0$, $\lambda_{0,s}(t) = \lambda_{0,1-s}(t)$ and $\lambda_{r,0}(t) = \lambda_{1-r,0}(t)$. Define

$$y_N(t) := \sum_{(r,s) \in Q_N'} \lambda_{r,s}(t) = \frac{1}{2} \sum_{(r,s) \in Q_N} \lambda_{r,s}(t). \tag{7.8}$$

Clearly Proposition 4.4 implies that $y_N(t)$ is meromorphic and single-valued in $\mathbb{C} \cup \{\infty\}$. Furthermore, Propositions B.1 and B.2 yield that $y_N(t)$ is holomorphic at $t = 0, 1$ (i.e., neither 0 nor 1 is a pole), and Proposition B.3 shows that $y_N(t)$ is at most linear growth at $t = \infty$. Therefore, $y_N(t)$ is a rational function. Note from (B.9) that

$$\lambda_{r,s}(t) = 1 - \lambda_{\tilde{r},\tilde{s}}(1 - t),$$

where $(\tilde{r}, \tilde{s}) \in Q_N'$ is determined by (r, s) via (B.8). Since $(\tilde{r}, \tilde{s})$ take over all elements of Q_N' whenever (r, s) does, we obtain

$$y_N(t) = |Q_N'| - y_N(1 - t). \tag{7.9}$$

Similarly, by (B.10) and (B.11), we have

$$y_N(t) = t y_N(\tfrac{1}{t}). \tag{7.10}$$

On the other hand, it is known (see [18, Proposition 1.4.1] or the proof of Theorem 4.1) that poles of any solution of $\mathrm{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ must be simple poles. Moreover, similarly as the proof of Theorem 1.6-(ii), we see that for any two different $(r_k,s_k)\in Q_N'$, $k=1,2$, $\lambda_{r_1,s_1}(t(\tau))$ and $\lambda_{r_2,s_2}(t(\tau))$ have no common poles as functions of τ. Therefore, $t_0=t(\tau_0)$ *is a pole of* $y_N(t)$ *if and only if there exists a* $(r,s)\in Q_N'$ *such that* t_0 *is a pole of* $\lambda_{r,s}(t)$ ((r,s) is uniquely determined by τ_0, i.e., if $t_0=t(\tau_1)$ with $\tau_1\neq\tau_0$, then (r,s) might be different, because by Proposition 4.4, (r,s) will permute in Q_N' after the analytic continuation along a path connecting τ_0 and τ_1 although $y_N(t)$ remains invariant). Furthermore, t_0 is a simple pole with

$$\operatorname*{Res}_{t=t_0} y_N(t)=\operatorname*{Res}_{t=t_0}\lambda_{r,s}(t)=-2t_0(t_0-1). \tag{7.11}$$

Here the second equality in (7.11) was proved in [6].

From (7.9)-(7.10), we see that if t_0 is pole of $y_N(t)$, then any of

$$\Xi(t_0):=\left\{t_0,\ 1-t_0,\ \tfrac{1}{t_0},\ 1-\tfrac{1}{t_0},\ \tfrac{1}{1-t_0},\ \tfrac{t_0}{t_0-1}\right\} \tag{7.12}$$

is also a pole of $y_N(t)$. This, together with Theorem 1.6-(i), implies that all elements in $\Xi(t_0)$ give the same j-value $j(t_0)\in J(N)$ via (1.26). For $j_k\in J(N)$, since $j_k\notin\{0,1728\}$ by Theorem 1.7, there are exactly six different t's which satisfies $j(t)=j_k$. We fix a $t_k\in\mathbb{C}$ such that $j(t_k)=j_k$, then $\Xi(t_k)$ gives precisely these six different t's. Therefore, we conclude that

$$\bigcup_{k=1}^{\deg\ell_N}\Xi(t_k)$$

gives precisely all the poles of $y_N(t)$.

From the above argument, we have

$$y_N(t)=-\sum_{k=1}^{\deg\ell_N}\sum_{a\in\Xi(t_k)}\frac{2a(a-1)}{t-a}+Ct+D, \tag{7.13}$$

where C,D are two constants that can be easily determined. Indeed, by (7.12), $\sum_{a\in\Xi(t_k)}a=3$, which implies

$$y_N(0)=2\sum_{k=1}^{\deg\ell_N}\sum_{a\in\Xi(t_k)}(a-1)+D=D-6\deg\ell_N.$$

First we assume that N is odd. Then $s<\frac{1}{2}$ for any $(r,s)\in Q_N'$. By Proposition B.1, we have $y_N(1)=|Q_N'|$. This, together with (7.9)-(7.10), gives $y_N(0)=0$ and $y_N(t)=o(t)$ as $t\to\infty$. Therefore, $C=0$ and $D=6\deg\ell_N=\frac{|Q_N|}{4}$, namely

$$y_N(t)=-\sum_{k=1}^{\deg\ell_N}\sum_{a\in\Xi(t_k)}\frac{2a(a-1)}{t-a}+\frac{|Q_N|}{4},\ \text{ if } N \text{ odd}.$$

Now we consider that N is even. Then the number of $(r, \frac{1}{2})$ in Q'_N is $\varphi(\frac{N}{2})$, and Proposition B.1 gives $y_N(1) = |Q'_N| - 2\varphi(\frac{N}{2})$, $y_N(0) = 2\varphi(\frac{N}{2})$ and $y_N(t) = 2\varphi(\frac{N}{2})t + O(1)$ as $t \to \infty$. Therefore, $C = 2\varphi(\frac{N}{2})$ and $D = 6\deg \ell_N + 2\varphi(\frac{N}{2}) = \frac{|Q_N|}{4} - \varphi(\frac{N}{2})$, namely

$$y_N(t) = -\sum_{k=1}^{\deg \ell_N} \sum_{a \in \Xi(t_k)} \frac{2a(a-1)}{t-a} + 2\varphi(\tfrac{N}{2})t + \frac{|Q_N|}{4} - 2\varphi(\tfrac{N}{2}), \text{ if } N \text{ even.}$$

We turn back to the problem of computing $J(N)$. The key observation is that *the coefficients of the Taylor expression of $y_N(t)$ at $t = 0$ are expressed in terms of $j_k \in J(N)$*. For example, we use $\sum_{a \in \Xi(t_k)} \frac{a-1}{a} = 3$ to obtain

$$y'_N(0) = 2\sum_{k=1}^{\deg \ell_N} \sum_{a \in \Xi(t_k)} \frac{a-1}{a} + C = 6\deg \ell_N + C;$$

we use the following formula, which is obtained from (7.12) and (1.26):

$$\frac{1}{2}\sum_{a \in \Xi(t_k)} \frac{a-1}{a^2} = 3 - \frac{(t_k^2 - t_k + 1)^3}{t_k^2(t_k-1)^2} = 3 - \frac{j_k}{256},$$

to obtain

$$y''_N(0) = 4\sum_{k=1}^{\deg \ell_N} \sum_{a \in \Xi(t_k)} \frac{a-1}{a^2} = 8\sum_{k=1}^{\deg \ell_N} \left(3 - \frac{j_k}{256}\right).$$

Similarly, a direct computation gives

$$y'''_N(0) = 12\sum_{k=1}^{\deg \ell_N} \left(3 - \frac{j_k}{256}\right);$$

$$y''''_N(0) = 48\sum_{k=1}^{\deg \ell_N} \left[\left(3 - \frac{j_k}{256}\right) - 2\left(3 - \frac{j_k}{256}\right)^2 + 12\right];$$

and so on. Thus, if $\deg \ell_N = 1$ (such as $N = 5, 8$), then $J(N)$ can be computed from $y''_N(0)$. If $\deg \ell_N = 2$ (such as $N = 7$), then $J(N)$ can be computed from $y''_N(0)$ and $y''''_N(0)$. In general, $J(N)$ should be determined by $y_N^{(2l)}(0)$ with $1 \leq l \leq \deg \ell_N$. On the other hand, by exploiting the same argument as Proposition B.1 in Appendix B, we can compute the Taylor expansion of $y_N(t)$ at $t = 1$ up to the term $(t-1)^{2\deg \ell_N}$ (which can be done by using *Mathematica*). Consequently, by using (7.9) we obtain the Taylor expansion of $y_N(t)$ at $t = 0$ up to the term $t^{2\deg \ell_N}$, from which we can compute $J(N)$ as explained above. Once $J(N)$ is determined, all poles of $y_N(t)$ (or equivalently, poles of $\lambda_{r,s}(t)$ with $(r,s) \in Q'_N$) can be computed via (1.26).

By exploiting the above approach, we computed for the cases $N = 5, 7, 8, 9$ and obtained (1.31)-(1.33). We take $N = 7$ as an example.

Example 7.4. Let $N = 7$, then $\deg \ell_7 = 2$. By using *Mathematica*, the Taylor expansion of $y_7(t)$ at $t = 0$ is

$$y_7(t) = 12t + \frac{19243064}{703125}t^2 + \frac{9621532}{703125}t^3 - \frac{536777924542148}{27 \times 703125^2}t^4 + O(t^5).$$

Hence,

$$4\sum_{k=1}^{2}\left(3 - \frac{j_k}{256}\right) = \frac{19243064}{703125},$$

$$2\sum_{k=1}^{2}\left[\left(3 - \frac{j_k}{256}\right) - 2\left(3 - \frac{j_k}{256}\right)^2 + 12\right] = -\frac{536777924542148}{27 \times 703125^2}.$$

From here, a straightforward computation gives

$$\ell_7(j) = j^2 - (j_1 + j_2)j + j_1 j_2 = j^2 + \frac{2^{12} \cdot 37001}{3^2 \cdot 5^7}j - \frac{2^{24} \cdot 571787}{3^7 \cdot 5^7},$$

and so $J(7)$ is given by (1.32).

8. Further discussion

In this final section, we make some further remarks about the results proved in this paper.

First we turn back to Step 2 in the proof of Theorem 1.7 in §7. Denote

$$j_0 := j(\tfrac{1}{2} + ib_1) \in (0, 1728),$$

because $\frac{1}{2} < b_1 < \frac{\sqrt{3}}{2}$ and j maps $\{\frac{1}{2} + ib \mid \frac{1}{2} \leq b \leq \frac{\sqrt{3}}{2}\}$ one-to-one onto $[0, 1728]$ with $j(\frac{1+i}{2}) = 1728$ and $j(e^{\frac{\pi i}{3}}) = 0$. Since $G(z|\tau)$ has only three critical points $\frac{1}{2}$, $\frac{\tau}{2}$ and $\frac{1+\tau}{2}$ if $\tau = \frac{1}{2} + ib$ with $\frac{1}{2} \leq b \leq b_1$, we see that $Z_{r,s}(\frac{1}{2} + ib) \neq 0$ for any $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and $\frac{1}{2} \leq b \leq b_1$, which implies that $\ell_N(j)$ has no zeros in $[j_0, 1728]$. Therefore, Theorem 1.7 can be restated in a sharper form: *$\ell_N(j)$ has no zeros in $\{0\} \cup [j_0, +\infty)$ and has exactly $\#J_N^+$ zeros in $(0, j_0)$.*

For each prime $N \geq 5$, we define $(r_N, s_N) := (\frac{N-1}{2N}, \frac{1}{N})$. Clearly $(r_N, s_N) \in J_N^+$. The proof of Theorem 1.7 shows that $Z_{r_N,s_N}(\tau)$ has a zero $\frac{1}{2} + ib_N$ with $b_N \downarrow b_1$ as $N \uparrow +\infty$. Therefore, $\ell_N(j)$ has a positive zero $j_N := j(\frac{1}{2} + ib_N)$ which satisfies $j_N \uparrow j_0$ as $N \uparrow +\infty$. Even though j_N is an algebraic number for each prime N, we still do not know *whether j_0 is an algebraic number or not.* This question seems very difficult and remains open.

We conjecture that *the polynomial $\ell_N(j)$ is irreducible in $\mathbb{Q}[j]$ and moreover $\mathbb{Q}[j]/(\ell_N(j))$ is a Galois extension of $\mathbb{Q}$.* Once this conjecture can be proved, all the zeros of $\ell_N(j)$ should not be algebraic integers provided $N \geq 5$, which implies that all the corresponding τ are transcendental.

Now let us turn to Theorem 1.4. Recall from Lemma 6.9 that $t(i\mathbb{R}^+) = (0, 1)$, $t(1 + i\mathbb{R}^+) = (1, +\infty)$ and $t(\{\tau \in \mathbb{H} \,|\, |\tau - \frac{1}{2}| = \frac{1}{2}\}) = (-\infty, 0)$.

Clearly

$$\mathbb{C}_- \setminus \Omega_-^{(0)} = U_1 \cup U_2 \cup U_3 \cup t(C_1) \cup t(C_2) \cup t(C_3),$$

where U_1 (resp. U_2, U_3) is the domain bounded by $(-\infty, 0]$ and the curve $t(C_1)$ (resp. by $[1, +\infty)$ and $t(C_2)$, by $[0,1]$ and $t(C_3)$) (these can be seen from Figures 2 and 4). Let $\mathcal{D}_1 \subset F_0$ be the domain bounded by C_1 and $\{\tau \in \mathbb{H} \,|\, |\tau - \frac{1}{2}| = \frac{1}{2}\}$, then $t(\mathcal{D}_1) = U_1$. Recalling the domain $\mathcal{D}$ defined in the proof of Theorem 6.6, clearly $\mathcal{D}_1 \cup C_1 \subset \overline{\mathcal{D}} \setminus \{0,1\}$. Then Step 1 of the proof of Corollary 6.7 shows that for any $t_0 = t(\tau_0) \in U_1 \cup t(C_1)$ with $\tau_0 \in \mathcal{D}_1 \cup C_1$, we have $\lambda_C(t_0) = 0$, i.e., t_0 is a type 1 singularity of λ_C, where $C = f_1(\tau_0)$ and λ_C is a solution of the Riccati equation (3.2). Therefore, *each element in $U_1 \cup t(C_1)$ is a type 1 singularity of some solution of the Riccati equation (3.2).*

We can prove analogous results for $U_2 \cup t(C_2)$ and $U_3 \cup t(C_3)$. Recalling (6.13)-(6.14), we let $\gamma = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, i.e. $\tau' = (\tau - 1)/\tau$. Then γ maps F_0 onto F_0 and

$$\tau \in \{\tau \in \mathbb{H} \,|\, |\tau - \tfrac{1}{2}| = \tfrac{1}{2}\} \Longleftrightarrow \tau' \in i\mathbb{R}^+.$$

Furthermore, (6.14) implies that γ maps C_1 onto C_3. Denote $\mathcal{D}_3 \subset F_0$ to be the domain bounded by C_3 and $i\mathbb{R}^+$ (see Figure 4). Then it is easy to see that $t(\mathcal{D}_3) = U_3$ and

$$\tau \in \mathcal{D}_1 \cup C_1 \Longleftrightarrow \tau' \in \mathcal{D}_3 \cup C_3. \tag{8.1}$$

Let $C' = \gamma \cdot C = (C-1)/C$. Similarly as Step 1 of the proof of Corollary 5.4, we can prove $t(\tau') = 1/(1 - t(\tau))$ and

$$\lambda_{C'}(t(\tau')) = \frac{\lambda_C(t(\tau)) - 1}{t(\tau) - 1} = \frac{\lambda_C(t(\tau))}{t(\tau) - 1} + t(\tau'),$$

namely $t(\tau') \in U_3 \cup t(C_3)$ is a type 3 singular point of $\lambda_{C'}$ provided that $t(\tau) \in U_1 \cup t(C_1)$ is a type 1 singular point of λ_C. Therefore, we have proved that *each element in $U_3 \cup t(C_3)$ is a type 3 singularity of some solution of the Riccati equation (3.2).* Similarly, by letting $\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, i.e. $\tau' = 1/(1-\tau)$, which maps $\{\tau \in \mathbb{H} \,|\, |\tau - \frac{1}{2}| = \frac{1}{2}\}$ onto $1 + i\mathbb{R}^+$, we can prove that *each element in $U_2 \cup t(C_2)$ is a type 2 singularity of some solution of the Riccati equation (3.2).*

Finally, we make a remark about Theorem C in §3. Let $\mathcal{M}_E$ denote the solution space of the elliptic form (1.7), and $\mathcal{M}_{\mathbb{C}}$ denote the solution space of $\text{PVI}_{(\frac{1}{8}, \frac{-1}{8}, \frac{1}{8}, \frac{3}{8})}$. Define $(r,s) \sim (\tilde{r}, \tilde{s})$ if $(r,s) \equiv \pm(\tilde{r}, \tilde{s}) \bmod \mathbb{Z}^2$. Then by Theorem C and Propositions 3.6, 4.4 and (4.7), we have

$$\mathcal{M}_E \cong \left((\mathbb{C}^2 \setminus \tfrac{1}{2}\mathbb{Z}^2) \,/\, \sim \right) \cup \text{ four copies of } \mathbb{CP}^1,$$

and

$$\mathcal{M}_{\mathbb{C}} \cong \left((\mathbb{C}^2 \setminus \tfrac{1}{2}\mathbb{Z}^2) / (\sim \cup \Gamma(2)) \right) \cup \text{ four copies of } \mathbb{CP}^1/\Gamma(2).$$

APPENDIX A. PICARD SOLUTION AND HITCHIN'S SOLUTION

In this Appendix, as an application of GLE (2.4), we give a rigorous derivation of Picard solution from Hitchin's formula. First, we note that Picard solution (2.6) can be also written as

$$\hat{\lambda}(t)=\hat{\lambda}_{\nu_1,\nu_2}(t)=\frac{\wp(\nu_1+\nu_2\tau|\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)},\ \ t=\frac{e_3(\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}. \tag{A.1}$$

This can be seen from (1.3) and the fact that the elliptic form (1.5) of $\mathrm{PVI}_{(0,0,0,\frac{1}{2})}$ is $\frac{d^2p(\tau)}{d\tau^2}=0$ (i.e., $\nu_1+\nu_2\tau$ is the general solution). Here we give a proof of (A.1) if $\hat{\lambda}(t)$ is given by (2.6). Clearly $\frac{t+1}{3}=\frac{-e_1(\tau)}{e_2(\tau)-e_1(\tau)}$. Recalling (1.9), it is well known that locally the inverse function of $t(\tau)=\frac{e_3(\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}$ can be expressed as

$$\tau=\tau(t)=i\frac{F(\frac{1}{2},\frac{1}{2},1;t)}{F(\frac{1}{2},\frac{1}{2},1;1-t)}=\frac{\omega_2(t)}{\omega_1(t)}.$$

Furthermore,

$$e_2(\tau)-e_1(\tau)=-\pi^2F(\tfrac{1}{2},\tfrac{1}{2},1;1-t)^2=\omega_1(t)^2.$$

Therefore, we easily deduce from (2.6) that

$$\hat{\lambda}(t)=\frac{\wp(\nu_1+\nu_2\tau\mid 1,\tau)}{\omega_1(t)^2}-\frac{e_1(\tau)}{e_2(\tau)-e_1(\tau)}=\frac{\wp(\nu_1+\nu_2\tau|\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)},$$

which proves (A.1).

On the other hand, $\hat{\lambda}(t)$ can be obtained from solution $\lambda(t)$ of $\mathrm{PVI}_{(\frac{1}{8},\frac{-1}{8},\frac{1}{8},\frac{3}{8})}$ by the following Bäcklund transformation (cf. [34, transformation s_2 in p.723]):

$$\hat{\lambda}(t)=\lambda(t)-\frac{1}{2\mu(t)},\ \mu(t)=\frac{t(t-1)\lambda'+\frac{1}{2}\lambda^2-t(\lambda-\frac{1}{2})}{2\lambda(\lambda-1)(\lambda-t)}. \tag{A.2}$$

Now let $\lambda(t)=\lambda_{r,s}(t)$ given by Hitchin's formula (1.8), namely

$$\lambda(t)=\frac{\wp(p_{r,s}(\tau)|\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}=\frac{\wp(r+s\tau|\tau)-e_1(\tau)+\frac{\wp'(r+s\tau|\tau)}{2Z_{r,s}(\tau)}}{e_2(\tau)-e_1(\tau)}, \tag{A.3}$$

where $Z_{r,s}$ is the Hecke form (1.16). The important thing is that, by studying the isomonodromic deformation of GLE (2.4), we proved in [6, Theorem 4.2] (without computing $\lambda'(t)$) that

$$\begin{aligned}\mu(t)&=\frac{e_2(\tau)-e_1(\tau)}{2(\wp(p_{r,s}(\tau)|\tau)-\wp(r+s\tau|\tau))}\\&=\frac{(e_2(\tau)-e_1(\tau))Z_{r,s}(\tau)}{\wp'(r+s\tau|\tau)}.\end{aligned} \tag{A.4}$$

We do not think that it is easy to obtain (A.4) via (A.3) and the second formula of (A.2). Substituting (A.3) and (A.4) into the first formula of

(A.2), we immediately obtain the formula (A.1) of Picard solution $\hat{\lambda}(t)$ with $(\nu_1, \nu_2) = (r, s)$.

Appendix B. Asymptotics of real solutions at $\{0, 1, \infty\}$

In this appendix, we prove the following asymptotic behaviors for real solutions at branch points, which are needed in §7. See [12] for asymptotics of solutions to Painlevé VI with generic parameters. As before, we may assume $\lambda(t) = \lambda_{r,s}(t)$ for some $(r, s) \in [0, 1) \times [0, \frac{1}{2}] \setminus \frac{1}{2}\mathbb{Z}^2$. First we consider $t \to 1$. The covering map $t = t(\tau)$ has infinitely many branches over $(0, 1)$. For our purpose we only need to consider $\tau \in i\mathbb{R}^+$.

Proposition B.1. *Suppose that $t = t(\tau)$, $\tau \in i\mathbb{R}^+$ and $\lambda(t) = \lambda_{r,s}(t)$ is a real solution with $(r, s) \in [0, 1) \times [0, \frac{1}{2}] \setminus \frac{1}{2}\mathbb{Z}^2$, then the followings hold:*

(i) *if $s \in (0, \frac{1}{2})$, then*

$$\lambda(t) = 1 + \frac{8se^{2\pi ir}}{2s-1}\left(\frac{1-t}{16}\right)^{2s} + O((1-t) + (1-t)^{4s}) \text{ as } t \uparrow 1. \tag{B.1}$$

(ii) *if $s = 0$, then*

$$\lambda(t) = 1 + \frac{t-1}{2} + O((t-1)^2) \text{ as } t \uparrow 1.$$

(iii) *if $s = \frac{1}{2}$, then*

$$\lambda(t) = -1 + \frac{1}{4}\left(\cos(2\pi r) - 2\right)(t-1) + O((t-1)^2) \text{ as } t \uparrow 1.$$

Proof. By $t(\tau) = \frac{e_3(\tau)-e_1(\tau)}{e_2(\tau)-e_1(\tau)}$, we have that if $\tau = ib$ with $b \in \mathbb{R}^+$, then $t \in (0, 1)$ and $t \uparrow 1$ as $b \to +\infty$. To compute the limit, we recall the formula for $\wp(z|\tau)$ in Proposition 6.3: if $q = e^{2\pi i\tau}$ and $|q| < |e^{2\pi iz}| < |q|^{-1}$, then

$$\wp(z|\tau) = -\frac{\pi^2}{3} - 4\pi^2\left[\frac{e^{2\pi iz}}{(e^{2\pi iz}-1)^2} + \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\left(e^{2\pi inz} + e^{-2\pi inz} - 2\right)\right]. \tag{B.2}$$

Now we put $z = r + s\tau = r + ibs$ with $(r, s) \in [0, 1) \times [0, \frac{1}{2}]$. Then $e^{-2\pi b} = |q| < |e^{2\pi iz}| = e^{-2\pi sb} < |q|^{-1}$. We consider three cases separately.

Case 1. $0 < s < \frac{1}{2}$.

In this case, by (B.2) and (5.3) we have

$$\wp(r + s\tau|\tau) = -\frac{\pi^2}{3} - 4\pi^2 e^{2\pi ir}e^{-2\pi bs} + O\left(e^{2\pi(s-1)b}\right),$$

$$\wp'(r + s\tau|\tau) = -8\pi^3 ie^{2\pi ir}e^{-2\pi bs} + O(e^{2\pi(s-1)b} + e^{-4\pi bs}),$$

$$Z_{r,s}(ib) = \pi i\,(2s-1) + O(e^{-2\pi bs}), \tag{B.3}$$

as $b \to +\infty$. Since

$$\wp(p_{r,s}(\tau)|\tau) = \wp(r + s\tau|\tau) + \frac{\wp'(r + s\tau|\tau)}{2Z_{r,s}(\tau)},$$

we have

$$\wp(p_{r,s}(\tau)|\tau) = -\frac{\pi^2}{3} - \frac{8s}{2s-1}\pi^2 e^{2\pi i r} e^{-2\pi b s} + O\left(e^{2\pi(s-1)b} + e^{-4\pi b s}\right) \text{ as } b \to +\infty.$$

On the other hand, by letting $z = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$ in (B.2) respectively, we easily obtain the following expansions for $e_i(\tau)$ (see (6.15) and Remark 6.5):

$$e_1(\tau) = \frac{2\pi^2}{3} + 16\pi^2 \sum_{k=1}^{\infty} a_k q^k, \quad e_2(\tau) = -\frac{\pi^2}{3} - 8\pi^2 \sum_{k=1}^{\infty} a_k q^{\frac{k}{2}},$$

$$e_3(\tau) = -\frac{\pi^2}{3} - 8\pi^2 \sum_{k=1}^{\infty} (-1)^k a_k q^{\frac{k}{2}},$$

where $a_k = \sum_{0<d|k,\, d \text{ odd}} d$. From here, we easily deduce

$$t - 1 = \frac{e_3(\tau) - e_2(\tau)}{e_2(\tau) - e_1(\tau)} = -16 e^{-\pi b} + O\left(e^{-2\pi b}\right), \tag{B.4}$$

and

$$\lambda(t) - 1 = \frac{\wp(p_{r,s}(\tau)|\tau) - e_2(\tau)}{e_2(\tau) - e_1(\tau)}$$
$$= \frac{8s}{2s-1} e^{2\pi i r} e^{-2\pi b s} + O\left(e^{-\pi b} + e^{-4\pi b s}\right) \tag{B.5}$$

as $b \to +\infty$, which implies (B.1) by using (B.4).

Case 2. $s = 0$.

In this case, as $b \to +\infty$, we have

$$\wp(r|\tau) = -\frac{\pi^2}{3} + \frac{\pi^2}{\sin^2(\pi r)} + 16\pi^2 \sin^2(\pi r) e^{-2\pi b} + O(e^{-4\pi b}),$$

$$\wp'(r|\tau) = \frac{-2\pi^3}{\sin^2(\pi r)} \cot(\pi r) + 16\pi^3 \sin(2\pi r) e^{-2\pi b} + O(e^{-4\pi b}),$$

$$Z_{r,0}(ib) = \pi \cot(\pi r) + 4\pi \sin(2\pi r) e^{-2\pi b} + O(e^{-4\pi b}), \tag{B.6}$$

$$\wp(p_{r,0}(\tau)|\tau) = -\frac{\pi^2}{3} + 8\pi^2 \left(1 + 4\sin^2(\pi r)\right) e^{-2\pi b} + O(e^{-4\pi b}),$$

and so, as $t \uparrow 1$,

$$\lambda(t) - 1 = -8e^{-\pi b} + 16(3 - 2\sin^2(\pi r)) e^{-2\pi b} + O(e^{-3\pi b})$$
$$= \frac{t-1}{2} + O((t-1)^2).$$

Case 3. $s = \frac{1}{2}$.

In this case, we note that $|e^{2\pi iz}| = e^{-\pi b} = |qe^{-2\pi iz}|$. As $b \to +\infty$, a straightforward computation gives

$$\begin{aligned}
\wp(r+\tfrac{\tau}{2}|\tau) &= -\frac{\pi^2}{3} - 8\pi^2\cos(2\pi r)e^{-\pi b} \\
&\qquad + 8\pi^2\left(1-2\cos(4\pi r)\right)e^{-2\pi b} + O(e^{-3\pi b}), \\
\wp'(r+\tfrac{\tau}{2}|\tau) &= 16\pi^3\sin(2\pi r)e^{-\pi b} + 64\pi^3\sin(4\pi r)e^{-2\pi b} + O(e^{-3\pi b}), \\
Z_{r,\frac{1}{2}}(ib) &= 4\pi\sin(2\pi r)e^{-\pi b} + 4\pi\sin(4\pi r)e^{-2\pi b} + O(e^{-3\pi b}), \qquad \text{(B.7)} \\
\wp(p_{r,\frac{1}{2}}(\tau)|\tau) &= \frac{5\pi^2}{3} + 4\pi^2\cos(2\pi r)e^{-\pi b} + O(e^{-2\pi b}),
\end{aligned}$$

and so, as $t \uparrow 1$,

$$\begin{aligned}
\lambda(t)+1 &= \left(8-4\cos(2\pi r)\right)e^{-\pi b} + O(e^{-2\pi b}) \\
&= \frac{1}{4}\left(\cos(2\pi r)-2\right)(t-1) + O((t-1)^2).
\end{aligned}$$

This completes the proof. □

Proposition B.2. *Suppose that $t = t(\tau)$, $\tau \in i\mathbb{R}^+$ and $\lambda(t) = \lambda_{r,s}(t)$ is a real solution with $(r,s) \in [0,1) \times [0,\frac{1}{2}] \setminus \frac{1}{2}\mathbb{Z}^2$, then the followings hold:*

(i) *if $r \notin \{0, \frac{1}{2}\}$, then*

$$\lambda(t) = -\frac{8\tilde{s}e^{2\pi i\tilde{r}}}{2\tilde{s}-1}\left(\frac{t}{16}\right)^{2\tilde{s}} + O(t + t^{4\tilde{s}}) \text{ as } t \downarrow 0,$$

where

$$(\tilde{r},\tilde{s}) = \begin{cases} (s, 1-r) & \text{if } r \in [\frac{1}{2}, 1), \\ (1-s, r) & \text{if } r \in [0,\frac{1}{2}), s > 0, \\ (0, r) & \text{if } r \in [0, \frac{1}{2}), s = 0. \end{cases} \qquad \text{(B.8)}$$

(ii) *if $r = 0$, then*

$$\lambda(t) = \frac{t}{2} + O(t^2) \text{ as } t \downarrow 0.$$

(iii) *if $r = \frac{1}{2}$, then*

$$\lambda(t) = 2 + \frac{1}{4}\left(\cos(2\pi s)-2\right)t + O(t^2) \text{ as } t \downarrow 0.$$

Proof. Let $\tau' = S\cdot\tau = -\frac{1}{\tau}$ and $(s',r') = (s,r)\cdot S^{-1} = (-r,s)$. By using (4.5), $e_1(\tau') = \tau^2 e_2(\tau)$, $e_2(\tau') = \tau^2 e_1(\tau)$ and $e_3(\tau') = \tau^2 e_3(\tau)$, we obtain

$$\begin{aligned}
\lambda_{r,s}(t(\tau)) &= \frac{\wp(p_{r,s}(\tau)|\tau) - e_1(\tau)}{e_2(\tau)-e_1(\tau)} = -\frac{\wp(p_{r',s'}(\tau')|\tau') - e_2(\tau')}{e_2(\tau')-e_1(\tau')} \qquad \text{(B.9)} \\
&= -\lambda_{r',s'}(t(\tau')) + 1 = -\lambda_{r',s'}(1-t(\tau)) + 1 \\
&= -\lambda_{\tilde{r},\tilde{s}}(1-t(\tau)) + 1,
\end{aligned}$$

where $(\tilde{r},\tilde{s}) \in \pm(r',s') + \mathbb{Z}^2$ is given by (B.8), namely $\tilde{s} = \min\{r, 1-r\} \in [0,\frac{1}{2}]$. Now our assertion follows readily from Proposition B.1. □

To give the asymptotic behavior as $t \uparrow +\infty$, we remark that $t(\tau) \in (1, +\infty)$ provided $\tau \in 1 + i\mathbb{R}^+$, see the proof of the following result.

Proposition B.3. *Suppose that $t = t(\tau)$, $\tau \in 1 + i\mathbb{R}^+$ and $\lambda(t) = \lambda_{r,s}(t)$ is a real solution with $(r, s) \in [0, 1) \times [0, \frac{1}{2}] \setminus \frac{1}{2}\mathbb{Z}^2$. Define*

$$(r_1, s_1) := \begin{cases} (r + s, s) \ \ \text{if} \ \ r + s < 1, \\ (r + s - 1, s) \ \ \text{if} \ \ r + s \geq 1. \end{cases} \tag{B.10}$$

Then the followings hold:

(i) *if $r_1 \notin \{0, \frac{1}{2}\}$, then*

$$\lambda(t) = -\frac{\tilde{s}e^{2\pi i \tilde{r}}}{2(2\tilde{s} - 1)} (16t)^{1-2\tilde{s}} + O(1 + t^{1-4\tilde{s}}) \text{ as } t \uparrow +\infty,$$

where

$$(\tilde{r}, \tilde{s}) = \begin{cases} (s_1, 1 - r_1) \ \ \text{if} \ \ r_1 \in (\frac{1}{2}, 1), \\ (1 - s_1, r_1) \ \ \text{if} \ \ r_1 \in (0, \frac{1}{2}), s_1 > 0, \\ (0, r_1) \ \ \text{if} \ \ r_1 \in (0, \frac{1}{2}), s_1 = 0. \end{cases}$$

(ii) *if $r_1 = 0$, then*

$$\lambda(t) = \frac{1}{2} + O(t^{-1}) \ \text{ as } t \uparrow +\infty.$$

(iii) *if $r_1 = \frac{1}{2}$, then*

$$\lambda(t) = 2t + \frac{1}{4} (\cos(2\pi s_1) - 2) + O(t^{-1}) \ \text{ as } t \uparrow +\infty.$$

Proof. Let $\tau' = T^{-1} \cdot \tau = \tau - 1 \in i\mathbb{R}^+$. Then $e_1(\tau') = e_1(\tau)$, $e_2(\tau') = e_3(\tau)$ and $e_3(\tau') = e_2(\tau)$, which implies $t(\tau) = \frac{1}{t(\tau')} \in (1, +\infty)$. Define $(s', r') = (s, r) \cdot \gamma^{-1} = (s, r + s)$. By using (4.5) we have

$$\begin{aligned} \lambda_{r,s}(t(\tau)) &= \frac{\wp(p_{r,s}(\tau)|\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)} = \frac{\wp(p_{r',s'}(\tau')|\tau') - e_1(\tau')}{e_3(\tau') - e_1(\tau')} \\ &= \frac{\lambda_{r',s'}(t(\tau'))}{t(\tau')} = t(\tau)\lambda_{r',s'}(t(\tau)^{-1}) \\ &= t(\tau)\lambda_{r_1,s_1}\left(\tfrac{1}{t(\tau)}\right), \end{aligned} \tag{B.11}$$

where $(r_1, s_1) \in (r', s') + \mathbb{Z}^2$ is given by (B.10). Consequently, this proposition follows readily from Proposition B.2. □

As pointed out in §1, no solution is real-valued along the real-axis. To see it, we first classify all solutions $\lambda_{r,s}(t)$ which are real-valued along $t(\tau) \in (0, 1)$ with $\tau \in i\mathbb{R}^+$.

Proposition B.4. *Let $t = t(\tau)$, $\tau \in i\mathbb{R}^+$ and $(r, s) \in \mathbb{C}^2 \setminus \frac{1}{2}\mathbb{Z}^2$. Then $\lambda_{r,s}(t(\tau))$ is real-valued along $\tau \in i\mathbb{R}^+$ if and only if either $r \in \mathbb{R}$, $s \in \frac{1}{2}\mathbb{Z} + i\mathbb{R}$ or $s \in \mathbb{R}$, $r \in \frac{1}{2}\mathbb{Z} + i\mathbb{R}$. In particular, for such a solution $\lambda_{r,s}(t(\tau))$, it is smooth for $t(\tau) \in (0, 1)$ if and only if it is also a real solution.*

Proof. Since $\tau \in i\mathbb{R}^+$, it is easy to see from the definition (1.4) of $\wp(z)$ that $\overline{\wp(z)} = \wp(\bar{z})$, $\overline{\wp'(z)} = \wp'(\bar{z})$ and $\overline{\zeta(z)} = \zeta(\bar{z})$. In particular, $e_j(\tau) \in \mathbb{R}$, $\eta_1(\tau) \in \mathbb{R}$ and $\eta_2(\tau) \in i\mathbb{R}$. Clearly $\lambda_{r,s}(t(\tau))$ is real-valued for $\tau \in i\mathbb{R}^+$ if and only if $\wp(p_{r,s}(\tau))$ is real-valued for $\tau \in i\mathbb{R}^+$, which is equivalent to

$$\begin{aligned}\wp(p_{r,s}(\tau)) &= \overline{\wp(p_{r,s}(\tau))} \\ &= \overline{\wp(r+s\tau)} + \overline{\frac{\wp'(r+s\tau)}{2(\zeta(r+s\tau) - r\eta_1(\tau) - s\eta_2(\tau))}} \\ &= \wp(\bar{r}-\bar{s}\tau) + \frac{\wp'(\bar{r}-\bar{s}\tau)}{2(\zeta(\bar{r}-\bar{s}\tau) - \bar{r}\eta_1(\tau) + \bar{s}\eta_2(\tau))} \\ &= \wp(p_{\bar{r},-\bar{s}}(\tau)) \ \text{ for all } \tau \in i\mathbb{R}^+. \end{aligned} \tag{B.12}$$

Together with (4.7), we conclude that $\lambda_{r,s}(t(\tau))$ is real-valued for $\tau \in i\mathbb{R}^+$ if and only if $(r,s) \equiv \pm(\bar{r}, -\bar{s}) \bmod \mathbb{Z}^2$. This proves the first assertion.

For the second assertion, we recall [6, Theorem 1.7] where we proved that any real solution $\lambda_{r,s}(t)$ has no singularities in $\mathbb{R}\backslash\{0,1\}$, i.e., $\lambda_{r,s}(t) \notin \{0,1,t,\infty\}$ for all $t \in \mathbb{R}\backslash\{0,1\}$, so the sufficient part holds. For the necessary part, it suffices to prove $r,s \in \mathbb{R}$. If not, without loss of generality, we may assume $\operatorname{Im} s \neq 0$. Then $r \in \mathbb{R}$ and $s \in \frac{1}{2}\mathbb{Z} + i\mathbb{R}$. Clearly there exists $\tau_0 \in i\mathbb{R}^+$ such that $r + s\tau_0 \in \{0, \frac{1}{2}\tau_0\} + \Lambda_{\tau_0}$, by which we have $\lambda_{r,s}(t(\tau_0)) \in \{1,\infty\}$, namely $\lambda_{r,s}(t(\tau))$ has a singularity $t(\tau_0) \in (0,1)$, a contradiction with the assumption that $\lambda_{r,s}(t(\tau))$ is smooth in $(0,1)$. □

Remark B.5. From Propositions B.1,B.2 and B.4, we see that for any real solution $\lambda_{r,s}(t(\tau))$ which is real-valued along $\tau \in i\mathbb{R}^+$, its analytic continuation to the line $1 + i\mathbb{R}^+$ (i.e. $t(\tau) \in (1,+\infty)$) or to the arc $\{\tau \in \mathbb{H} \,|\, |\tau - \frac{1}{2}| = \frac{1}{2}\}$ (i.e. $t(\tau) \in (-\infty, 0)$) turns out not real-valued. It is easy to see that any other solution can not be real-valued along the real-axis either, because it has at least a branch point at one of $\{0,1,\infty\}$ by Propositions 3.6 and 4.4.

The following result seems an interesting consequence of our smoothness result.

Proposition B.6. *Let $t = t(\tau)$, $\tau \in i\mathbb{R}^+$ and $(r,s) \in [0,1) \times [0,\frac{1}{2}] \setminus \frac{1}{2}\mathbb{Z}^2$. Then*

$$\begin{aligned} &0 < t < \lambda_{r,0}(t) < 1, \quad 0 < \lambda_{0,s}(t) < t < 1, \\ &\lambda_{r,\frac{1}{2}}(t) < 0, \quad \lambda_{\frac{1}{2},s}(t) > 1. \end{aligned} \tag{B.13}$$

In particular, $\lambda_{r,0}(t)$ and $\lambda_{0,s}(t)$ are both one-to-one from $(0,1)$ onto $(0,1)$.

Proof. By Proposition B.4 and the assumption, $\lambda_{r,0}$, $\lambda_{0,s}$, $\lambda_{r,\frac{1}{2}}$ and $\lambda_{\frac{1}{2},s}$ are all real-valued for $\tau \in i\mathbb{R}^+$. To prove (B.13), we use again that any real solution $\lambda_{r,s}(t)$ satisfies $\lambda_{r,s}(t) \notin \{0,1,t,\infty\}$ for all $t \in \mathbb{R}\backslash\{0,1\}$. Together this with Propositions B.1 and B.2, (B.13) follows readily.

It suffices to prove the one-to-one for $\lambda_{r,0}(t)$. The proof for $\lambda_{0,s}(t)$ is similar and we omit the details. Recall from Propositions B.1, B.2 and (B.13) that

$$\lim_{t\downarrow 0}\lambda_{r,0}(t)=0,\ \ \lim_{t\uparrow 1}\lambda_{r,0}(t)=1,\ \ t<\lambda_{r,0}(t)<1.$$

Suppose $\lambda_{r,0}(t)$ is not one-to-one, then there is a critical point $t_0\in(0,1)$ such that $\lambda_{r,0}'(t_0)=0$ and $\lambda_{r,0}''(t_0)\leq 0$, which implies from Painlevé VI (1.2) that

$$\frac{1}{8}-\frac{1}{8}\frac{t_0}{\lambda(t_0)^2}-\frac{1}{8}\frac{1-t_0}{(\lambda(t_0)-1)^2}-\frac{3}{8}\frac{t_0(1-t_0)}{(\lambda(t_0)-t_0)^2}\geq 0.$$

Thus, $t_0<\lambda(t_0)^2$ and $1-t_0<(\lambda(t_0)-1)^2$, which imply $2\lambda(t_0)(\lambda(t_0)-1)>0$, a contradiction to $0<\lambda(t_0)<1$. This completes the proof. □

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