On Galois Theory of Several Variables

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Algebraic relations

We are interested in understanding transcendental invariants which arise naturally in mathematics. Satisfactory understanding means that we are able to determine all the algebraic relations among these very special values.

Let A be an abelian variety over $\overline{\mathbb{Q}}$ of dimension d, and let P be the period matrix of A. Grothendieck in 1960's made **conjecture** :

$$\mathrm{trdeg}_{\overline{\mathbb{Q}}}\overline{\mathbb{Q}}(P) = \dim \mathrm{MT}(A),$$

where $\mathrm{MT}(A)$ is the Mumford-Tate group of A and is an algebraic subgroup of $\mathrm{GL}_{2d} \times \mathbb{G}_m$. This Mumford-Tate group is the motivic Galois group of the motive $h_1(A) \oplus \mathbb{Q}(1)$.

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Example. Consider elliptic curve $E : y^2 = 4x^3 - g_2x - g_3$ over $\overline{\mathbb{Q}} \subset \mathbb{C}$. Its transcendental invariants are the periods and quasi-periods. Periods lattice $\Lambda_E = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ consisting of periods of dfk :

$$\int_c \frac{dx}{y}, \;\; c \in H_1(E(\mathbb{C}), \mathbb{Z}).$$

From differential of the 2nd kind xdx/y, one has quasi-periods $\eta(\omega)$, which is $\mathbb{Z}$-linear in $\omega \in \Lambda_E$. All the non-zero periods and quasi-periods are transcendental, by Siegel-Schneider 1930's. On the other hand Legendre's relation says

$$\det \begin{pmatrix} \omega_1 & \omega_2 \\ \eta(\omega_1) & \eta(\omega_2) \end{pmatrix} = \pm 2\pi\sqrt{-1}.$$

CM Elliptic curves

If E has no complex multiplications, one conjectures that $\omega_1, \omega_2, \eta(\omega_1), \eta(\omega_2)$ are algebraically independent, i.e.

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(\pi, \omega, \eta(\omega); \omega \in \Lambda_E) = 4.$$

Complex multiplications certainly give rise algebraic relations, thus in 1970's Chudnovsky showed that if E has CM, then the above finitely generated extension has transcendence degree only two.

In the non-CM case the motivic Galois group for E should be GL_2, in the CM case the Galois group is a 2-dim torus, which is just $\mathrm{Res}_{k/\mathbb{Q}} \mathbb{G}_m$, where k is the quadratic field of multiplications of E.

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Example. Consider the following values from arithmetic of $\mathbb{Q}$: $S = \{2\pi\sqrt{-1}, \zeta(2), \zeta(3), \cdots, \zeta(m) \cdots, \}$, where

$$\zeta(m) := \sum_{n=1}^{\infty} n^{-m}.$$

The value of Riemann zeta function at positive integer $m > 1$.

For m *even*, one knows from Euler the relations:

$$\zeta(m) = \frac{-(2\pi\sqrt{-1})^m B_m}{2m!}.$$

where B_m are the Bernoulli numbers:

$$\frac{Z}{e^Z - 1} = \sum_{m=0}^{\infty} B_m \frac{Z^m}{m!}.$$

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Mixed Tate motives

One **conjectures** that these relations generate all the algebraic relations among numbers from S over the field of algebraic numbers $\overline{\mathbb{Q}}$. In particular, all the zeta values $\zeta(m)$ for odd integer $m > 1$ should be transcendental, and algebraically independent from each other, as well as algebraically independent from π. (Presently the transcendence of $\zeta(3)$ is still unknown).

This follows from the period conjecture for Mixed Tate motives.

Arithmetic of positive characteristic.
$\mathbb{F}_q :=$ the finite field of q elements.
$k := \mathbb{F}_q(\theta) :=$ the rational function field in the variable θ over $\mathbb{F}_q$.
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World of positive characteristic

$k_\infty := \mathbb{F}_q((\frac{1}{\theta}))$, completion of k with respect to the infinite place.
$\overline{k_\infty}$:= a fixed algebraic closure of k_∞ containing $\bar{k}$.
$\mathbb{C}_\infty$:= completion of $\overline{k_\infty}$ with respect to the canonical extension of the infinite place.

Natural transcendental "numbers" from function field arithmetic : Carlitz zeta values (1935), $m \geq 1$,

$$\zeta_C(m) = \sum_{a \in \mathbb{F}_q[\theta]_+} \frac{1}{a^m} \in \mathbb{F}_q((\frac{1}{\theta})),$$

where $\mathbb{F}_q[\theta]_+$ consists of monic polynomials in $\mathbf{A} := \mathbb{F}_q[\theta]$.

Development of transcendence theory in positive characteristic: 1st stage : Transporting classical (characteristic zero) theory (e.g. methods of Siegel, Schneider, Lang, Baker, and Wüstholz) to positive characteristic world, from 1980's to 1990's.

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Euler-Carlitz relations

One has the obvious Frobenius relations among special zeta values in characteristic p :

$$\zeta_C(m)^p = \zeta_C(mp).$$

If m is *even, i.e.* $m \equiv 0 \pmod{q-1}$ because the ring $\mathbb{F}_q[\theta]$ has $q-1$ signs, one also has the Euler-Carlitz relation.

$$\zeta_C(m) = \frac{\tilde{\pi}^m \widetilde{B}_m}{\Gamma_{m+1}},$$

where $\tilde{\pi}$ is a *fundamental period* of the Carlitz module for $\mathbb{F}_q$:

$$\tilde{\pi} = \theta(-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1},$$

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Bernoulli-Carlitz

The Γ_m are Carlitz factorials :

- setting $D_0 = 1$, and $D_i = (\theta^{q^i} - \theta^{q^{i-1}}) \cdots (\theta^{q^i} - \theta)$, for $i \geq 1$,
- writing down the q-adic expansion $\sum_{i=0}^{\infty} n_i q^i$ of n, and let

$$\Gamma_{n+1} = \prod_{i=0}^{\infty} D_i^{n_i} \ .$$

The $\widetilde{B_m} \in \mathbb{F}_q(\theta)$ are the Bernoulli-Carlitz "numbers" given by

$$\frac{z}{\exp_C(z)} = \sum_{m=0}^{\infty} \widetilde{B_m} \frac{z^m}{\Gamma_{m+1}}.$$

Here Carlitz exponential is the series

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Carlitz period $\tilde{\pi}$ fits in exact sequence of $\mathbb{F}_q$-linear maps :

$$0 \to \mathbb{F}_q[\theta]\,\tilde{\pi} \to \mathbb{C}_\infty \xrightarrow{\exp_C} \mathbb{C}_\infty \to 0.$$

Carlitz exponential linearizes the $\mathbb{F}_q[t]$-action (Carlitz module) given by

$$\phi_C(t) : x \longmapsto \theta x + x^q,$$

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\mathbb{C}_\infty \arrow[r, "\exp_C"] \arrow[d, "\theta(\cdot)"'] & \mathbb{G}_a(\mathbb{C}_\infty) = \mathbb{C}_\infty \arrow[d, "\phi_C(t)"] \\
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Toward algebraic independence

For arbitrary q, interested in following set of special zeta values :

$$S_q = \{\tilde{\pi}, \zeta_C(1), \zeta_C(2), \cdots, \zeta_C(m), \cdots.\}$$

Yu 1991 proves that all these are transcendental over $\bar{k} = \overline{\mathbb{F}_q(\theta)}$, and in 1997 it is shown that all linear relations among them come from the Euler-Carlitz relations.

2nd stage of positive characteristic transcendence theory in last decade: Go beyond its classical counterparts, i.e. from linear independence to algebraic independence. In particular an analogue of Grothendieck's motivic design actually works (Anderson, Brownawell, C.-Y. Chang, Papanikolas, and J. Yu).

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Let t, σ be variables independent of θ.
Let $\bar{k}(t)[\sigma, \sigma^{-1}]$ be **noncommutative** ring of Laurent polynomials in σ with coefficients in $\bar{k}(t)$, subject to the relation

$$\sigma f := f^{(-1)}\sigma \text{ for all } f \in \bar{k}(t).$$

Here $f^{(-1)}$ is the rational function obtained from $f \in \bar{k}(t)$ by twisting all its coefficients $a \in \bar{k}$ to $a^{\frac{1}{q}}$.

A pre-t-motive M over $\mathbb{F}_q$ is a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module which is finite-dimensional over $\bar{k}(t)$.
Let $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(M)$ be a $\bar{k}(t)$-basis of M.
Multiplying by σ on M is represented by $\sigma(\mathbf{m}) = \Phi\mathbf{m}$ for some matrix $\Phi \in \mathrm{GL}_r(\bar{k}(t))$.

The category of pre-t-motives over $\mathbb{F}_q$ forms an abelian $\mathbb{F}_q(t)$-linear tensor category.

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Frobenius difference equations

From a pre-t-motive, one associates a "system of Frobenius difference equation" which has solutions in series of t.

Consider the operator on $\mathbb{C}_\infty$ by $x \mapsto x^{\frac{1}{q}}$.
Then extend this operator to $\mathbb{C}_\infty((t))$ as follows,
for $f = \sum_i a_i t^i \in \mathbb{C}_\infty((t))$ define $f^{(-1)} := \sum_i a_i^{q^{-1}} t^i$.

More generally, for matrix B with entries in $\mathbb{C}_\infty((t))$ define twisting $B^{(-1)}$ by the rule $B^{(-1)}{}_{ij} = B_{ij}{}^{(-1)}$.

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Let $t = \theta$

We view Ψ as giving a "fundamental" solution of the system of **Frobenius** difference equations described by the algebraic matrix Φ coming from M.

Note that if $\Psi' \in \mathrm{Mat}_r(\mathbb{C}_\infty((t)))$ is also a solution of the Frobenius system from Φ, then $\Psi'^{-1}\Psi \in \mathrm{GL}_r(\mathbb{F}_q(t))$.

A power series $f = \sum_{i=0}^{\infty} a_i t^i \in \mathbb{C}_\infty[[t]]$ that converges everywhere and satisfies

$$[k_\infty(a_0, a_1, a_2, \dots) : k_\infty] < \infty$$

is called an entire power series. As a function of t it takes values in $\overline{k_\infty}$, when restricted to $\overline{k_\infty}$. The ring of the entire power series is denoted by $\mathbb{E}$.

If all entries of a solution Ψ of the Frobenius system in question are in $\mathbb{E}$, one can **specializ** Ψ to $\Psi(\theta)$.

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Rigid analytic trivialization

$|\cdot|_\infty :=$ a fixed absolute value for the completed field $\mathbb{C}_\infty$.
$\mathbb{T} := \{f \in \mathbb{C}_\infty[[t]] \mid f$ converges on $|t|_\infty \leq 1\}$.
$\mathbb{L} :=$ the fraction field of $\mathbb{T}$.

Pre t-motive M is called rigid analytically trivial if there exists $\Psi \in \mathsf{GL}_r(\mathbb{L})$ such that

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Such matrix Ψ is called a rigid analytic trivialization of the pre t-motive in question.

The category $\mathcal{R}$ of rigid analytically trivial pre-t-motives over $\mathbb{F}_q$ forms a neutral Tannakian category over $\mathbb{F}_q(t)$.

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Given object M in $\mathcal{R}$ and let $\mathcal{T}_M$ be the strictly full Tannakian subcategory of $\mathcal{R}$ generated by M. That is, $\mathcal{T}_M$ consists of all objects of $\mathcal{R}$ isomorphic to subquotients of finite direct sums of

$$M^{\otimes u} \otimes (M^\vee)^{\otimes v} \text{ for various } u, v,$$

where $M^\vee$ is the dual of M. By Tannakian **duality**, $\mathcal{T}_M$ is representable by an affine algebraic group scheme Γ_M over $\mathbb{F}_q(t)$. Such Γ_M is called the **motivic Galois group** of M.

Given rigid analytically trivial pre-t-motive M, the motivic Galois group Γ_M is isomorphic over $\mathbb{F}_q(t)$ to the linear algebraic Galois group Γ_Ψ of the associated Frobenius difference equation.

Tannakian duality

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where $M^{\vee}$ is the dual of M. By Tannakian **duality**, $\mathcal{T}_M$ is representable by an affine algebraic group scheme Γ_M over $\mathbb{F}_q(t)$. Such Γ_M is called the **motivic Galois group** of M.

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This algebraic Galois group Γ_Ψ from solution Ψ has the key property

$$\dim \Gamma_\Psi = \mathrm{tr.deg}_{\overline{k}(t)}\ \overline{k}(t)(\Psi).$$

If furthermore $\Psi \in \mathrm{Mat}_r(\mathbb{E})$ and satisfies

$$\mathrm{tr.deg}_{\overline{k}(t)} \overline{k}(t)(\Psi) = \mathrm{tr.deg}_{\bar{k}} \overline{k}(\Psi(\theta)),$$

then we say that M has the **GP** property. It follows that

$$\dim \Gamma_M = \mathrm{tr.deg}_{\bar{k}}\ \overline{k}(\Psi(\theta)).$$

Pre-t-motives having the **GP** property first come from Anderson-Brownawell-Papanikolas 2004, through reformulating the submodule theorem of Yu 1997 which plays the role of Wüstholz subgroup theorem (1989).

Papanikolas theory 2008

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Galois theory

We are interested in finitely generated extension of $\bar{k} = \overline{\mathbb{F}_q(\theta)}$ generated by a set S of special values, denoted by K_S. In particular we want to determine all algebraic relations among elements of S. From known algebraic relations, we can guess the transcendence degree of K_S over $\bar{k}$, and the goal is to prove that specific degree.

We construct a t-motive M_S for this purpose, so that it has the **GP** property and its "periods" $\Psi_S(\theta)$ from rigid analytic trivialization generate also the field K_S, then computing the dimension of the motivic Galois group Γ_{M_S}.

Following transcendental arithmetic values have been tackled:

- Periods and quasi-periods of Drinfeld $\mathbb{F}_q[t]$-module defined over $\bar{k}$ (arbitrary rank), Chang-Papanikolas 2009.
- Logarithms at algebraic points of Drinfeld $\mathbb{F}_q[t]$-module over $\bar{k}$, rank 1 Papanikolas 2008, rank 2 Chang-Papanikolas 2009.

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Construction of motives

- The Carlitz zeta values $\zeta_C(m), m \geq 1$, Chang-Yu 2007.
- Geometric Gamma values $\Gamma(\alpha), \alpha \in \mathbb{F}_q(\theta) - \mathbb{F}_q(\theta)_+$, Anderson-Brownawell-Papanikolas 2004 (analogue of Lang-Rohrlich conjecture).
- Arithmnetic Gamma values $r!, r \in \mathbb{Q} \cap (\mathbb{Z}_p - \mathbb{Z})$ (p is the characteristic), Chang-Thakur-Papanikolas-Yu 2008.

Construction of the t-motives in question rely on the arithmetic-geometric structures in question:

- Canonical t-motive associated to Drinfeld module over $\bar{k}$ of rank r following Anderson. In case the Drinfelds module has full CM, the Galois group is a torus of dim r. In the generic case, the Galois group is GL_r.
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Galois groups

- By a formula of Anderson-Thakur, these special zeta values are linear combinations of polylogarithms at algebraic points. The Galois group is an extension of $\mathbb{G}_m$ by a vector group.
- The motive construction for these special geometric Gamma values is by way of geometric cyclotomy, or "solitons". The Galois groups for these values come from tori which are obtained from $\mathbb{G}_m$ via restriction of scalars from the geometric CM field of the motive in question.
- The motive here is the one associated to the Carlitz module with CM from a constant field extension. The Galois groups for special arithmetic Gamma values are tori obtained from $\mathbb{G}_m$ via restriction of scalars from the constant field extension in question.

The Carlitz motive C. Let $C = \bar{k}(t)$ with σ-action:

$$\sigma f = (t-\theta) f^{(-1)}, \quad f \in C.$$

Here $\Phi = (t-\theta)$. Analytic solution Ψ of the equation $\Psi^{(-1)} = (t-\theta)\Psi$ is given by

$$\Psi_C(t) = (-\theta)^{-q/(q-1)} \prod_{i=1}^{\infty} (1 - t/\theta^{q^i}).$$

Note Galois group here is $\Gamma_C = \mathbb{G}_m$ which has dimension 1. Therefore $\Psi_C(\theta) = \frac{-1}{\tilde{\pi}}$ is transcendental over $\bar{k}$,

Drinfeld modules

Let $\tau : x \mapsto x^q$ be the Frobenius endomorphism of $\mathbb{G}_a/\mathbb{F}_q$.

Let $\bar{k}[\tau]$ be the twisted polynomial ring :

$$\tau c = c^q \tau, \ \text{ for all } \ c \in \bar{k}.$$

A Drinfeld $\mathbb{F}_q[t]$-module ρ of rank r (over $\bar{k}$) is a $\mathbb{F}_q$-linear ring homomorphism (Drinfeld 1974) $\rho : \mathbb{F}_q[t] \to \bar{k}[\tau]$ given by $(\Delta \neq 0)$

$$\rho_t = \theta + g_1\tau + \cdots + g_{r-1}\tau^{r-1} + \Delta\tau^r,$$

Drinfeld exponential $\exp_\rho(z) = \sum_{h=0}^{\infty} c_h z^{q^h}, c_h \in \bar{k}$, on $\mathbb{C}_\infty$ linearizes this t-action :

$$\begin{tikzcd}
\mathbb{C}_\infty \arrow[r, "\exp_\rho"] \arrow[d, "\theta(\cdot)"'] & \mathbb{G}_a(\mathbb{C}_\infty) = \mathbb{C}_\infty \arrow[d, "\rho_t"] \\
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\end{array}$$

Periods of Drinfeld modules

Kernel of $\exp_\rho$ is a discrete free $\mathbb{F}_q[\theta]$-module $\Lambda_\rho \subset \mathbb{C}_\infty$ of rank r. Moreover

$$\exp_\rho(z) = z \prod_{\lambda \neq 0 \in \Lambda_\rho} (1 - \frac{z}{\lambda}).$$

The nonzero elements in Λ_ρ are the **periods** of the Drinfeld module ρ. They are all transcendental over $\bar{k}$ (Yu 1986).

Morphisms of Drinfeld modules $f : \rho_1 \to \rho_2$ are the twisting polynomials $f \in \bar{k}[\tau]$ satisfying $(\rho_2)_t \circ f = f \circ (\rho_1)_t$.

Isomorphisms from ρ_1 to ρ_2 are given by constant polynomials $f \in \bar{k} \subset \bar{k}[\tau]$ such that $f\,\Lambda_{\rho_1} = \Lambda_{\rho_2}$.

The endomorphism ring of Drinfeld module ρ can be identified with

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Algebraic relations among periods

The field of fractions of R_ρ, denoted by K_ρ, is called the field of multiplications of ρ. One has that $[K_\rho : k]$ always divides the rank of the Drinfeld module ρ.
Drinfeld module ρ of rank r is said to be without Complex Multiplications CM, if $K_\rho = k$, and with "full" CM if $[K_\rho : k] = r$. If ρ has CM, there are non-trivial algebraic relations among its periods.

In late 1980's, quasi-periods for Drinfeld modules are introduced by Anderson, Deligne, Gekeler, and Yu.

All nonzero quasi-periods are also transcendental over $\bar{k}$ (Yu 1990), and there are algebraic relations between periods, quasi-periods and the Carlitz period $\tilde{\pi}$ (Anderson, Gekeler 1989), as analogue of the Legendre relation.

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The period matrix

Let ρ be a Drinfeld module of rank r, and let $\{\delta_1, \dots, \delta_{r-1}\}$ be a basis of the de Rham cohomology of ρ. Let $F_i(z)$ be the quasi-periodic function associated to δ_i, $i = 1, \dots, r-1$, and $\{\lambda_1, \dots, \lambda_r\}$ be a fixed basis of Λ_ρ. Then **period matrix** of ρ corresponding to this choice of basis is

$$P_\rho = \begin{pmatrix} \lambda_1 & F_1(\lambda_1) & \cdots & F_{r-1}(\lambda_1) \\ \lambda_2 & F_1(\lambda_2) & \cdots & F_{r-1}(\lambda_2) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_r & F_1(\lambda_r) & \cdots & F_{r-1}(\lambda_r) \end{pmatrix}$$

Analogue of Legendre's relation amounts to $\det P_\rho = \alpha\tilde{\pi}$, with $\alpha \neq 0 \in \bar{k}$.

Let Drinfeld $\mathbb{F}_q[t]$-module ρ of rank r (over $\bar{k}$) be given by

$$\rho_t = \theta + g_1\tau + \cdots + g_{r-1}\tau^{r-1} + \tau^r,$$

We associate to ρ a dimension r pre-t-motive M_ρ via the matrix

$$\Phi_\rho = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ (t-\theta) & -g_1^{(-1)} & \cdots & \cdots & -g_{r-1}^{(-1)} \end{pmatrix}$$

To solve the Frobenius difference equation, let $\{\lambda_1, \ldots, \lambda_r\}$ be a fixed basis of Λ_ρ, and $\delta_i : t \mapsto \tau^i, i = 1, \ldots, r-1$, be chosen basis of the de Rham cohomology of ρ. Then the solution Ψ_ρ can be explicitly written down which specializes (setting $t = \theta$) to the period matrix $P\rho$.

Galois games

Suppose we have pre-t-motive M_1 (M_2) with **GP** property for set of values S_1 (S_2 respectively), and we are able to determine the Galois group Γ_{M_1} (Γ_{M_2} respectively). To handle the set $S_1 \cup S_2$, we form the direct sum of pre-t-motive $M = M_1 \oplus M_2$. Then the dimension of the Galois group Γ_M equals to the transcendence degree over $\bar{k}$ of the compositum of the field K_{S_1} and K_{S_2} which is $K_{S_1 \cup S_2}$. We have surjective morphisms from Γ_M onto both Γ_{M_1} and Γ_{M_2}. On many occasions this makes it possible to deduce the dimension of Γ_M from the algebraic group structures of Γ_{M_1} and Γ_{M_2}. As an example

Theorem (2008, Chang-Yu) Let ρ be a Drinfeld modules with full CM, then its periods and quasi-periods are algebraically independent over $\bar{k}$ from the values $\zeta_C(m), (q-1) \nmid m$.

Similar phenomena should hold also in the classical world!

The End. Thank You.