On Algebraic Independence of Special Zeta Values in Characteristic p

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t-motives

Let t, σ be variables independent of θ.
Let $\bar{k}(t)[\sigma, \sigma^{-1}]$ be **noncommutative** ring of Laurent polynomials in σ with coefficients in $\bar{k}(t)$, subject to the relation

$$\sigma f := f^{(-1)} \sigma \text{ for all } f \in \bar{k}(t).$$

Here $f^{(-1)}$ is the rational function obtained from $f \in \bar{k}(t)$ by twisting all its coefficients $a \in \bar{k}$ to $a^{\frac{1}{q}}$.

A pre-t-motive M over $\mathbb{F}_q$ is a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module which is finite-dimensional over $\bar{k}(t)$.
Let $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(M)$ be a $\bar{k}(t)$-basis of M.
Multiplying by σ on M is represented by $\sigma(\mathbf{m}) = \Phi\mathbf{m}$ for some matrix $\Phi \in \mathrm{GL}_r(\bar{k}(t))$.

The category of pre-t-motives over $\mathbb{F}_q$ forms a rigid abelian $\mathbb{F}_q(t)$-linear tensor category.

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Frobenius difference equations

From a pre-t-motive, one associates a "system of Frobenius difference equation" which has solutions in series of t.

Consider the operator on $\mathbb{C}_\infty$ by $x \mapsto x^{\frac{1}{q}}$.
Then extend this operator to $\mathbb{C}_\infty((t))$ as follows,
for $f = \sum_i a_i t^i \in \mathbb{C}_\infty((t))$ define $f^{(-1)} := \sum_i a_i^{q^{-1}} t^i$.

More generally, for matrix B with entries in $\mathbb{C}_\infty((t))$ define twisting $B^{(-1)}$ by the rule $B^{(-1)}{}_{ij} = B_{ij}{}^{(-1)}$.

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We view Ψ as giving a "fundamental" solution of the system of **Frobenius** difference equations described by the algebraic matrix Φ coming from M.

Note that if $\Psi' \in \mathrm{Mat}_r(\mathbb{C}_\infty((t)))$ is also a solution of the Frobenius system from Φ, then $\Psi'^{-1}\Psi \in \mathrm{GL}_r(\mathbb{F}_q(t))$.

A power series $f = \sum_{i=0}^{\infty} a_i t^i \in \mathbb{C}_\infty[[t]]$ that converges everywhere and satisfies

$$[k_\infty(a_0, a_1, a_2, \dots) : k_\infty] < \infty$$

is called an entire power series. As a function of t it takes values in $\overline{k_\infty}$, when restricted to $\overline{k_\infty}$. The ring of the entire power series is denoted by $\mathbb{E}$.

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Rigid analytic trivialization

$|\cdot|_\infty :=$ a fixed absolute value for the completed field $\mathbb{C}_\infty$.
$\mathbb{T} := \{f \in \mathbb{C}_\infty[[t]] \mid f$ converges on $|t|_\infty \leq 1\}$.
$\mathbb{L} :=$ the fraction field of $\mathbb{T}$.

Pre t-motive M is called rigid analytically trivial if there exists $\Psi \in \mathrm{GL}_r(\mathbb{L})$ such that

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Such matrix Ψ is called a rigid analytic trivialization of the pre t-motive in question.

The category $\mathcal{R}$ of rigid analytically trivial pre-t-motives over $\mathbb{F}_q$ forms a neutral Tannakian category over $\mathbb{F}_q(t)$.

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Tannakian duality

Given object M in $\mathcal{R}$ and let $\mathcal{T}_M$ be the strictly full Tannakian subcategory of $\mathcal{R}$ generated by M. That is, $\mathcal{T}_M$ consists of all objects of $\mathcal{R}$ isomorphic to subquotients of finite direct sums of

$$M^{\otimes u} \otimes (M^{\vee})^{\otimes v} \text{ for various } u, v,$$

where $M^{\vee}$ is the dual of M. By Tannakian **duality**, $\mathcal{T}_M$ is representable by an affine algebraic group scheme Γ_M over $\mathbb{F}_q(t)$. Such Γ_M is called the **motivic Galois group** of M.

Given rigid analytically trivial pre-t-motive M, the motivic Galois group Γ_M is isomorphic over $\mathbb{F}_q(t)$ to the linear algebraic Galois group Γ_Ψ of the associated Frobenius difference equation.

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This algebraic Galois group Γ_Ψ from solution Ψ has the key property

$$\dim \Gamma_\Psi = \mathrm{tr.deg}_{\overline{k}(t)}\ \overline{k}(t)(\Psi).$$

If furthermore $\Psi \in \mathrm{Mat}_r(\mathbb{E})$ and satisfies

$$\mathrm{tr.deg}_{\overline{k}(t)} \overline{k}(t)(\Psi) = \mathrm{tr.deg}_{\bar{k}} \overline{k}(\Psi(\theta)),$$

then we say that M has the **GP** property. It follows that

$$\dim \Gamma_M = \mathrm{tr.deg}_{\bar{k}}\ \overline{k}(\Psi(\theta)).$$

Pre-t-motives having the **GP** property first come from Anderson-Brownawell-Papanikolas 2004, through reformulating the submodule theorem of Yu 1997 which plays the role of Wüstholz subgroup theorem (1989).

Papanikolas theory 2008

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The Carlitz motive C. Let $C = \bar{k}(t)$ with σ-action:

$$\sigma f = (t - \theta) f^{(-1)}, \;\; f \in C.$$

Here $\Phi = (t - \theta)$. Analytic solution Ψ of the equation $\Psi^{(-1)} = (t - \theta)\Psi$ is given by

$$\Psi_C(t) = (-\theta)^{-q/(q-1)} \prod_{i=1}^{\infty} (1 - t/\theta^{q^i}).$$

Note Galois group here is $\Gamma_C = \mathbb{G}_m$ which has dimension 1.
Therefore $\Psi_C(\theta) = \frac{-1}{\tilde{\pi}}$ is transcendental over $\bar{k}$,

Motivic Galois theory

We are interested in finitely generated extension of $\bar{k} = \overline{\mathbb{F}_q(\theta)}$ generated by a set S of special values, denoted by K_S. In particular we want to determine all algebraic relations among elements of S. From known algebraic relations, we can guess the transcendence degree of K_S over $\bar{k}$, and the goal is to prove that is indeed the specific degree in question.

To proceed, we construct a t-motive M_S for this purpose, so that it has the **GP** property and its "periods" $\Psi_S(\theta)$ from rigid analytic trivialization generate also the field K_S, then computing the dimension of this motivic Galois group Γ_{M_S}.

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Suppose we have pre-t-motive M_1 (M_2) with **GP** property for set of values S_1 (S_2 respectively), and we are able to determine the Galois group Γ_{M_1} (Γ_{M_2} respectively). To handle the set $S_1 \cup S_2$, we form the direct sum of pre-t-motive $M = M_1 \oplus M_2$. Then the dimension of the Galois group Γ_M equals to the transcendence degree over $\bar{k}$ of the compositum of the field K_{S_1} and K_{S_2} which is $K_{S_1 \cup S_2}$. We have surjective morphisms from Γ_M onto both Γ_{M_1} and Γ_{M_2}. On many occasions this makes it possible to deduce the dimension of Γ_M from the algebraic group structures of Γ_{M_1} and Γ_{M_2}. As an example

The End. Thank You.