On values of Modular Forms at Algebraic Points

Jing Yu

National Taiwan University, Taipei, Taiwan

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In value distribution theory the exponential function $e^z$ is a key. This function also enjoys the following extraordinary properties:

(Hermite-Lindemann-Weierstrass 1880) For $\alpha \neq 0 \in \overline{\mathbb{Q}}$, $e^\alpha$ is **transcendental**. Moreover, if algebraic numbers $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$ then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are **algebraically independent**, i.e. for any polynomial $P \neq 0 \in \overline{\mathbb{Q}}(x_1, \ldots, x_n)$,

$$P(e^{\alpha_1}, \ldots, e^{\alpha_n}) \neq 0$$

**Tools** for proving this come from **Complex Analysis**.

Let $\mathcal{H}$ be the complex upper half plane.
We are also interested in values of “natural” holomorphic functions taking at “**algebraic points**” of $\mathcal{H}$.
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Let $\mathcal{H}$ be the complex upper half plane. We are also interested in values of “natural” holomorphic functions taking at **algebraic points** of $\mathcal{H}$.
The modular function $j : SL_2(\mathbb{Z}) \backslash \mathcal{H} \cong \mathbb{C}$ which parametrizes isomorphism classes of complex elliptic curves. This function $j$ can be employed for proving the Picard theorem.

This $j$ also has beautiful transcendence property:
(Siegel-Schneider 1930) If $\alpha \in \overline{\mathbb{Q}} \cap \mathcal{H}$ and $\alpha$ is not quadratic, then $j(\alpha)$ is transcendental.
If $\alpha$ is (imaginary) quadratic, then $j(\alpha)$ is actually an algebraic integer, as known to Kronecker.

Call $\alpha \in \mathcal{H}$ **algebraic point** if $j(\alpha) \in \overline{\mathbb{Q}}$. Thus unless an algebraic point $\alpha \in \mathcal{H}$ is imaginary quadratic number, it must be a transcendental number.
Elliptic curves correspond to algebraic points can all be defined over $\overline{\mathbb{Q}}$. 

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**Values at Algebraic Points**
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Elliptic curves correspond to algebraic points can all be defined over $\overline{\mathbb{Q}}$. 
(Meromorphic) Modular form $f : \mathcal{H} \rightarrow \mathbb{C} \cup \{\infty\}$, of weight $k$, here $k$ is a fixed integer, satisfying for all $z \in \mathcal{H}$

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z),$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Modular forms are required to be meromorphic at $\infty$, i.e. with Fourier expansion:

$$f(z) = \sum_{n=n_0}^{\infty} a_n e^{2\pi i n z}.$$

Call $f$ arithmetic modular form if all coefficients $a_n \in \mathbb{Q}$.

Note one can replace $SL_2(\mathbb{Z})$ by its congruence subgroups $\Gamma$, and requiring $f$ to be meromorphic at all “cusps”.
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Reformulating works of Siegel-Schneider, one has

**Theorem.** Let $f$ be arithmetic modular form of nonzero weight $k$. Let $\alpha \in \mathcal{H}$ is an algebraic point which is neither zero nor pole of $f$, then $f(\alpha)$ is transcendental.

**Open Problem.** Let $f$ as above, $\alpha_1, \ldots, \alpha_n \in \mathcal{H}$ be algebraic points which are neither zeros nor poles of $f$. Suppose that the $\alpha_i$ are pairwise non-isogenous, are the values $f(\alpha_1), \ldots, f(\alpha_n)$ algebraically independent?

Here $\alpha$ and $\beta \in \mathcal{H}$ are said to be non-isogenous, if the elliptic curves they correspond are not isogenous.

Note that the value $f(\alpha)$ is always an algebraic multiple of the $k$-th power of a period (of the elliptic curve corresponding to $\alpha$) dividing by $\pi$.

Can prove the linearly independence over $\mathbb{Q}$ of these values.
Reformulating works of Siegel-Schneider, one has

**Theorem.** Let $f$ be arithmetic modular form of \textbf{nonzero} weight $k$. Let $\alpha \in \mathcal{H}$ is an algebraic point which is neither zero nor pole of $f$, then $f(\alpha)$ is transcendental.

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Can prove the linearly independence over \( \overline{\mathbb{Q}} \) of these values.
World of Positive characteristic

\[ F_q := \text{the finite field of } q \text{ elements.} \]
\[ k := F_q(\theta) := \text{the rational function field in the variable } \theta \text{ over } F_q. \]
\[ \bar{k} := \text{fixed algebraic closure of } k. \]

\[ k_\infty := F_q(((\frac{1}{\theta}))), \text{ completion of } k \text{ with respect to the infinite place.} \]
\[ \overline{k_\infty} := \text{a fixed algebraic closure of } k_\infty \text{ containing } \bar{k}. \]
\[ \mathbb{C}_\infty := \text{completion of } \overline{k_\infty} \text{ with respect to the canonical extension of the infinite place.} \]

Non-archimedean analytic function theory on \( \mathbb{C}_\infty \), and on Drinfeld upper-half space \( \mathcal{H}_\infty \) which is \( \mathbb{C}_\infty - k_\infty \).

Natural non-archimedean analytic functions come from Drinfeld modules theory.
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Non-archimedean analytic function theory on \( C_\infty \), and on Drinfeld upper-half space \( \mathcal{H}_\infty \) which is \( C_\infty - k_\infty \).

Natural non-archimedean analytic functions come from Drinfeld modules theory.
Let \( \tau : x \mapsto x^q \) be the Frobenius endomorphism of \( \mathbb{G}_a / \mathbb{F}_q \).

Let \( \mathbb{C}_\infty[\tau] \) be the twisted polynomial ring:

\[
\tau c = c^q \tau, \text{ for all } c \in \mathbb{C}_\infty.
\]

A Drinfeld \( \mathbb{F}_q[t] \)-module \( \rho \) of rank \( r \) (over \( \mathbb{C}_\infty \)) is a \( \mathbb{F}_q \)-linear ring homomorphism (Drinfeld 1974) \( \rho : \mathbb{F}_q[t] \to \mathbb{C}_\infty[\tau] \) given by \((\Delta \neq 0)\)

\[
\rho_t = \theta + g_1 \tau + \cdots + g_{r-1} \tau^{r-1} + \Delta \tau^r,
\]

Drinfeld exponential \( \exp^\rho(z) = \sum_{h=0}^{\infty} c_h z^{q^h}, c_h \in \bar{k}, \) on \( \mathbb{C}_\infty \)
linearizes this \( t \)-action:

\[
\begin{align*}
\mathbb{C}_\infty & \xrightarrow{\exp^\rho} \mathbb{G}_a(\mathbb{C}_\infty) = \mathbb{C}_\infty \\
\theta(\cdot) & \downarrow \quad \downarrow \rho_t \\
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Drinfeld modules

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Transcendence theory

Analogue of Hermite-Lindemann-Weierstrass, and Siegel-Schneider:

**Theorem 1.** (Yu 1986) Let $\rho$ be a Drinfeld $\mathbb{F}_q[t]$-module defined over $\bar{k}$, with associated exponential map $\exp_{\rho}(z)$ on $\mathbb{C}_\infty$. If $\alpha \neq 0 \in \bar{k}$, then $\exp_{\rho}(\alpha)$ is transcendental over $k$.

**Theorem 2.** (A. Thiery 1995) Suppose the Drinfeld module $\rho$ is of rank 1. If $\alpha_1, \ldots, \alpha_n \in \bar{k}$ are linearly independent over $k$, then $\exp_{\rho}(\alpha_1), \ldots, \exp_{\rho}(\alpha_n)$ are algebraically independent over $k$.

Drinfeld upper-half space $\mathcal{H}_\infty$ parametrizes isomorphism classes of rank 2 Drinfeld modules, let $j = g_1^{q+1}/\Delta$:

$$j : \text{GL}_2(\mathbb{F}_q[\theta]) \backslash \mathcal{H}_\infty \cong \mathbb{C}_\infty.$$

Call $\alpha \in \mathcal{H}_\infty$ **algebraic point** if $j(\alpha) \in \bar{k}$. Drinfeld modules corresponding to algebraic points can be defined over $\bar{k}$.
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The \( j \) values and periods

We have the following

**Theorem 3.** (Yu 1986) If \( \alpha \in \bar{k} \cap \mathcal{H}_\infty \) and \( \alpha \) is not quadratic over \( k \), then \( j(\alpha) \) is transcendental over \( k \). Moreover, for those \( \alpha \) quadratic over \( k \), \( j(\alpha) \) are integral over \( \mathbb{F}_q[\theta] \).

If Drinfeld module \( \rho \) is of rank \( r \), kernel of \( \exp_\rho \) is a discrete free \( \mathbb{F}_q[\theta] \)-module \( \Lambda_\rho \subset \mathbb{C}_\infty \) of rank \( r \). Moreover

\[
\exp_\rho(z) = z \prod_{\lambda \neq 0 \in \Lambda_\rho} (1 - \frac{z}{\lambda}).
\]

The nonzero elements in \( \Lambda_\rho \) are the **periods** of the Drinfeld module \( \rho \). They are all transcendental over \( \bar{k} \) by Theorem 1.

Morphisms of Drinfeld modules \( f : \rho_1 \rightarrow \rho_2 \) are the twisting polynomials \( f \in \bar{k}[\tau] \) satisfying \( (\rho_2)_t \circ f = f \circ (\rho_1)_t \).
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Isomorphisms from $\rho_1$ to $\rho_2$ are given by constant polynomials $f \in \bar{k} \subset \bar{k}[\tau]$ such that $f \Lambda_{\rho_1} = \Lambda_{\rho_2}$.

The endomorphism ring of Drinfeld module $\rho$ can be identified with

$$R_\rho = \{ \alpha \in \bar{k} \mid \alpha \Lambda_\rho \subset \Lambda_\rho \}.$$

The field of fractions of $R_\rho$, denoted by $K_\rho$, is called the field of multiplications of $\rho$. One has that $[K_\rho : k]$ always divides the rank of the Drinfeld module $\rho$.

Drinfeld module $\rho$ of rank 2 is said to be without Complex Multiplications if $K_\rho = k$, and with CM if $[K_\rho : k] = 2$. If $\rho$ has CM, there are non-trivial algebraic relations among its periods.
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Drinfeld modular forms

Modular form \( f : \mathcal{H}_\infty \rightarrow \mathbb{C}_\infty \cup \{\infty\} \), of weight \( k \) and type \( m \), here \( k \) is a fixed integer, \( m \in \mathbb{Z}/(q - 1)\mathbb{Z} \), satisfying for all \( z \in \mathcal{H}_\infty \)

\[
f\left(\frac{az + b}{cz + d}\right) = (\det \gamma)^m (cz + d)^k f(z),
\]

\( \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_q[\theta]). \)

Modular forms are required to be “rigid” meromorphic functions, and at \( \infty \) with “Fourier” expansion:

\[
f(z) = \sum_{n=n_0}^{\infty} a_n q_\infty(z)^n, \quad q_\infty(z) = \sum_{a \in \mathbb{F}_q[\theta]} \frac{1}{z - a}.
\]

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Note one can replace \( GL_2(\mathbb{F}_q[\theta]) \) by its congruence subgroups \( \Gamma \), and requiring \( f \) to be meromorphic at all “cusps”.

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Here one also proves

**Theorem.** (Yu) Let $f$ be arithmetic modular form of **nonzero** weight $k$. Let $\alpha \in \mathcal{H}_\infty$ is an algebraic point which is neither zero nor pole of $f$, then $f(\alpha)$ is transcendental over $k$.

**Open Problem.** Let $f$ as above, $\alpha_1, \ldots, \alpha_n \in \mathcal{H}_\infty$ be algebraic points which are neither zeros nor poles of $f$. Suppose that the $\alpha_i$ are pairwise non-isogenous, are the values $f(\alpha_1), \ldots, f(\alpha_n)$ algebraically independent over $k$?

Here $\alpha$ and $\beta \in \mathcal{H}_\infty$ are said to be non-isogenous, if the Drinfeld modules they correspond are not isogenous.

Again the value $f(\alpha)$ is equal to an element of $\bar{k}$ times $k$-th power of a period (of the rank 2 Drinfeld modules defined over $\bar{k}$ corresponding to $\alpha$) dividing by the Carlitz period (of rank 1 Drinfeld $\mathbb{F}_q[t]$-module).
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Again the value \( f(\alpha) \) is equal to an element of \( \bar{k} \) times \( k \)-th power of a period (of the rank 2 Drinfeld modules defined over \( \bar{k} \) corresponding to \( \alpha \)) dividing by the Carlitz period (of rank 1 Drinfeld \( \mathbb{F}_q[t] \)-module).
The CM points are those \( \alpha \in \mathcal{H}_\infty \) which are quadratic over \( k \), hence correspond to Drinfeld modules with CM.

**Theorem.** (C.-Y. Chang 2010) Let \( f \) be arithmetic modular form of **nonzero** weight. Let \( \alpha_1, \ldots, \alpha_n \in \mathcal{H}_\infty \) be CM points which are neither zeros nor poles of \( f \). Suppose that the \( \alpha_i \) are pairwise non-isogenous, then the values \( f(\alpha_1), \ldots, f(\alpha_n) \) are algebraically independent over \( k \).

Here CM points \( \alpha \) and \( \beta \in \mathcal{H}_\infty \) are non-isogenous, precisely when they belong to different quadratic extension of \( k \).

Method for proving algebraic independence in positive characteristic, developed in the last 10 years, by Anderson, Brownawell, Chang, Papanikolas, and Yu. Crucial step by Papanikolas 2008.
Motivic transcendence theory

Realizing a program of Grothendieck in positive characteristic.

We are interested in finitely generated extension of $\overline{k}$ generated by a set $S$ of special values, denoted by $K_S$. In particular we want to determine all algebraic relations among elements of $S$.

From known algebraic relations, one can guess the transcendence degree of $K_S$ over $\overline{k}$, and the goal is to prove that is indeed the specific degree in question.

To proceed, we construct a $t$-motive $M_S$ for this purpose, so that it has the GP property and its “periods” $\Psi_S(\theta)$ from “rigid analytic trivialization ”generate also the field $K_S$, then computing the dimension of the motivic Galois (algebraic) group $\Gamma_{M_S}$.

GP property of the motive $M_S$ requires:

$$\dim \Gamma_{M_S} = \text{tr.deg}_{\overline{k}(\Psi_S(\theta))} \overline{k}(\Psi_S(\theta)).$$
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To proceed, we construct a $t$-motive $M_S$ for this purpose, so that it has the GP property and its “periods” $\Psi_S(\theta)$ from “rigid analytic trivialization” generate also the field $K_S$, then computing the dimension of the motivic Galois (algebraic) group $\Gamma_{M_S}$.

GP property of the motive $M_S$ requires:

$$\dim \Gamma_{M_S} = \text{tr.deg}_{\bar{k}} \bar{k}(\Psi_S(\theta)).$$

Jing Yu, NTU, Taiwan

Values at Algebraic Points
The End. Thank You.