Transcendence Theory of Drinfeld Modules

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BIRS Workshop on t-motives
September 28, 2009

World of positive characteristic

Let p be a fixedprime; q a fixed power of p.

$$
\begin{array}{ll}
A := \mathbb{F}_q[\theta] & \longleftrightarrow \mathbb{Z} \\
k := \mathbb{F}_q(\theta) & \longleftrightarrow \mathbb{Q} \\
k_\infty := \mathbb{F}_q((1/\theta)) & \longleftrightarrow \mathbb{R} \\
\bar{k} \text{ inside } \overline{k_\infty} & \longleftrightarrow \overline{\mathbb{Q}} \\
\mathbb{C}_\infty := \widehat{\overline{k_\infty}} & \longleftrightarrow \mathbb{C} \\
|f|_\infty := q^{\deg f} & \longleftrightarrow |\cdot|
\end{array}
$$

Drinfeld $\mathbb{F}_q[t]$-modules

Let $F : x \mapsto x^q$ be the Frobenius endomorphism of $\mathbb{G}_a/\mathbb{F}_q$.

Let $\bar{k}[F]$ be the twisted polynomial ring :

$$Fc = c^q F, \text{ for all } c \in \bar{k}.$$

A Drinfeld $\mathbb{F}_q[t]$-module ρ of rank r (over $\bar{k}$) is a $\mathbb{F}_q$-algebra homomorphism $\rho : \mathbb{F}_q[t] \to \bar{k}[F]$ given by $(\Delta \neq 0)$

$$\rho_t = \theta + g_1 F + \cdots + g_{r-1} F^{r-1} + \Delta F^r,$$

Drinfeld exponential $\exp_\rho(z) = \sum_{h=0}^{\infty} c_h z^{q^h}, c_h \in \bar{k}$, on $\mathbb{C}_\infty$ linearizes this t-action :

$$\begin{array}{ccc} \mathbb{C}_\infty & \xrightarrow{\exp_\rho} & \mathbb{G}_a(\mathbb{C}_\infty) = \mathbb{C}_\infty \\ \theta(\cdot) \downarrow & & \downarrow \rho_t \\ \mathbb{C}_\infty & \xrightarrow{\exp_\rho} & \mathbb{G}_a(\mathbb{C}_\infty) = \mathbb{C}_\infty \end{array}$$

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Periods of Drinfeld modules

Kernel of $\exp_\rho$ is a discrete free $\mathbb{F}_q[\theta]$-module $\Lambda_\rho \subset \mathbb{C}_\infty$ of rank r. Moreover

$$\exp_\rho(z) = z \prod_{\lambda \neq 0 \in \Lambda_\rho} (1 - \frac{z}{\lambda}).$$

The nonzero elements in Λ_ρ are the **periods** of the Drinfeld module ρ. They are all transcendental over $\bar{k}$(1986). In fact, any $u \in \mathbb{C}_\infty$ such that $\exp_\rho(u) \in \bar{k}$ are transcendental, these are called **Drinfeld logarithms** (of algebraic points) w.r.t ρ.

Morphisms of Drinfeld modules $h : \rho_1 \to \rho_2$ are the twisting polynomials $h \in \bar{k}[F]$ satisfying $(\rho_2)_t \circ h = h \circ (\rho_1)_t$.

Isomorphisms from ρ_1 to ρ_2 are given by constant polynomials $h \in \bar{k} \subset \bar{k}[F]$ such that $h\,\Lambda_{\rho_1} = \Lambda_{\rho_2}$.

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Algebraic relations among periods

The endomorphism ring of Drinfeld module ρ can be identified with

$$R_\rho = \{\alpha \in \bar{k} |\ \alpha \Lambda_\rho \subset \Lambda_\rho\}.$$

The field of fractions of R_ρ, denoted by K_ρ, is called the field of multiplications of ρ. One has that $[K_\rho : k]$ always divides the rank of the Drinfeld module ρ.
Drinfeld module ρ of rank r is said to be without Complex Multiplications CM, if $K_\rho = k$, and with "full" CM if $[K_\rho : k] = r$.

If ρ has CM, there are non-trivial algebraic relations among its periods coming from the endomorphisms.

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Detour to Drinfeld A-modules

Now let k be any function field with field of constants $\mathbb{F}_q$.
Fix a place and call it ∞.
Take A to be the ring of functions in k regular away from ∞.

A Drinfeld A-module ρ is simply an A-action on $\mathbb{G}_a$ defined over $\bar{k}$ which linearizes to the scalar A-action on $\mathrm{Lie}\, \mathbb{G}_a$.

Take any non-constant "t" in A. Then ρ can be viewed as Drinfeld $\mathbb{F}_q[t]$-module with "complex multiplications" by A.

For the purpose of transcendence theory, the study of Drinfeld A-modules can thus be reduced to the study of Drinfeld $\mathbb{F}_q[t]$-modules.

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Periods of the 2nd kind

To introduce quasi-periods, we consider certain (bi-)derivations.

A $\mathbb{F}_q$-linear map from $\delta : \mathbb{F}_q[t] \longrightarrow \bar{k}[F]F$ is called a derivation of the Drinfeld module ρ if, for all $a, b \in \mathbb{F}_q[t]$, the following holds

$$\delta_{ab} = a(\theta)\delta_a + \delta_a \rho_b.$$

Given derivation δ of ρ, there is $\mathbb{F}_q$-linear entire function

$F_\delta(z) = \sum_{h=1}^{\infty} b_h z^{q^h}, b_h \in \bar{k}$, on $\mathbb{C}_\infty$,

satisfying the following difference equation :

$$F_\delta(\theta z) - \theta F_\delta(z) = \delta_t(\exp_\rho(z)).$$

This $F_\delta(z)$ is quasi-periodic in the sense

$$F_\delta(z + \lambda) = F_\delta(z) + F_\delta(\lambda), \text{ for } \lambda \in \Lambda_\rho.$$

$$\int_\lambda \delta := F_\delta(\lambda) \text{ is } \mathbb{F}_q[\theta] - \text{linear in } \lambda \in \Lambda_\rho.$$

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Periods and quasi-periods

The values $F_\delta(\lambda)$, $\lambda \in \Lambda_\rho$, are called the **quasi-periods** of ρ w.r.t. the derivation δ. All nonzero quasi-periods are also transcendental over $\bar{k}$ (1990).

The set of all derivations of ρ modulo "strictly inner" derivations is a $\bar{k}$-vector space of dimension $r = \operatorname{rank} \rho$. This gives the de Rham cohomology of the Drinfeld module ρ.

δ is called strictly inner derivation if there exists $m \in \bar{k}[F]F$ so that

$$\delta = \delta^{(m)} : a \longmapsto m\rho_a - a(\theta)m, \quad \text{for all } a \in \mathbb{F}_q[t].$$

Strictly inner derivations only give zero quasi-periods.

Consider the derivation $\delta^{(1)} : a \mapsto a(\theta) - \rho_a$, then $F_{\delta^{(1)}}(z) = z - \exp_\rho(z)$. Hence periods of ρ are just quasi-periods w.r.t. the 1st kind derivation $\delta^{(1)}$.

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The period matrix

The de Rham isomorphism says that $\delta \mapsto (\lambda \mapsto \int_\lambda \delta)$ gives a natural isomorphism from the de Rham cohomology of ρ onto a $\bar{k}$-structure of the space $\mathrm{Hom}_{\mathbf{A}}(\Lambda_\rho, \mathbb{C}_\infty)$.

Let $\{[\delta_0 = [\delta^{(1)}], [\delta_1], \ldots, [\delta_{r-1}]\}$ be a basis of the de Rham cohomology of ρ. Let $\{\lambda_1, \ldots, \lambda_r\}$ be a fixed A-basis of Λ_ρ. Then **period matrix** of ρ corresponding to this choices of basis is

$$P_\rho = \left(\int_{\lambda_i} \delta_j \right)$$

$$= \begin{pmatrix} \lambda_1 & F_1(\lambda_1) & \cdots & F_{r-1}(\lambda_1) \\ \lambda_2 & F_1(\lambda_2) & \cdots & F_{r-1}(\lambda_2) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_r & F_1(\lambda_r) & \cdots & F_{r-1}(\lambda_r) \end{pmatrix},$$

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t-motives

Analogue of Legendre's relation (Anderson, Gekeler) says $\det P_\rho = \alpha\tilde{\pi}$, with $\alpha \neq 0 \in \bar{k}$.
Here $\tilde{\pi}$ is period of the rank one Carlitz module.

Let t, σ be variables independent of θ.
Let $\bar{k}(t)[\sigma, \sigma^{-1}]$ be **noncommutative** ring of Laurent polynomials in σ with coefficients in $\bar{k}(t)$, subject to the relation

$$\sigma f := f^{(-1)}\sigma \text{ for all } f \in \bar{k}(t).$$

Here $f^{(-1)}$ is the rational function obtained from $f \in \bar{k}(t)$ by twisting all its coefficients $a \in \bar{k}$ to $a^{\frac{1}{q}}$.

A pre-t-motive M over $\mathbb{F}_q$ is a left $\bar{k}(t)[\sigma, \sigma^{-1}]$-module which is finite-dimensional over $\bar{k}(t)$.

Let $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(M)$ be a $\bar{k}(t)$-basis of M. Multiplying by σ on M is represented by $\sigma(\mathbf{m}) = \Phi\mathbf{m}$ for some matrix $\Phi \in \mathrm{GL}_r(\bar{k}(t))$.

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Motives associated to Drinfeld modules

The category of **pre-t-motives** over $\mathbb{F}_q$ forms an abelian $\mathbb{F}_q(t)$-linear tensor category.

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$$\rho_t = \theta + g_1 F + \cdots + g_{r-1} F^{r-1} + F^r,$$

We associate to ρ a dimension r pre-t-motive M_ρ via the matrix

$$\Phi_\rho = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ (t-\theta) & -g_1^{1/q} & \cdots & \cdots & -g_{r-1}^{1/q^{r-1}} \end{pmatrix}$$

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Anderson generating function

Now fix an A-basis $\{\lambda_1, \ldots, \lambda_r\}$ of the period lattice Λ_ρ. For each $1 \le i \le r$, consider the sequence of t-division points:

$$\exp_\rho(\lambda_i/\theta), \exp_\rho(\lambda_i/\theta^2), \exp_\rho(\lambda_i/\theta^3), \ldots.$$

The **Anderson generating functions** is: for $1 \le i \le r$,

$$f_i(t) := \sum_{j=0}^{\infty} \exp_\rho(\lambda_i/\theta^{j+1})t^j = \lambda_i/(\theta - t) + \sum_{j=1}^{\infty} c_j {\lambda_i}^{q^j}/(\theta^{q^j} - t).$$

We observe that

$$\mathrm{Res}_{t=\theta} f_i = -\lambda_i = -\int_{\lambda_i} \delta^{(1)}.$$

Let δ_j be the derivation given by $t \mapsto F^j$ for $1 \le j \le r-1$. For $\ell \in \mathbb{N}$, let $f_i^{(\ell)}$ be the series obtained from f_i by changing all coefficients to its q^ℓ-th roots, then also

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A Frobenius difference equation

$$\widehat{\Psi} := \begin{bmatrix} f_1 & f_2 & \cdots & f_r \\ f_1^{(1)} & f_2^{(1)} & \cdots & f_r^{(1)} \\ \vdots & \vdots & & \vdots \\ f_1^{(r-1)} & f_2^{(r-1)} & \cdots & f_r^{(r-1)} \end{bmatrix}.$$

$$L := \begin{bmatrix} g_1 & g_2^{(-1)} & g_3^{(-2)} & \cdots & g_{r-1}^{(-r+2)} & 1 \\ g_2 & g_3^{(-1)} & g_4^{(-2)} & \cdots & 1 & \\ \vdots & \vdots & & & & \\ g_{r-1} & 1 & & & & \\ 1 & & & & & \end{bmatrix}$$

and set $\Psi := (L^{-1}\{[\widehat{\Psi}^{(1)}]^{-1}\})^{\mathsf{t}}$. Then $\Psi(\theta)$ gives essentially the period matrix P_ρ of the Drinfeld module ρ. Moreover

$$\Psi^{(-1)} = \Phi\Psi.$$

Linear independence (over $\bar{k}$) theory

Method of Schneider-Lang in positive characteristic.

$\mathbb{F}_q$-linear functions as functions satisfying algebraic differential equations:

$$f(z) = \sum_{h=0}^{\infty} c_h z^{q^h}, c_h \in \bar{k}.$$

Method of Baker-Wüstholz for t-modules.

Analogue of Wüstholz subgroup theorem :

Let $G = (\mathbb{G}_a^d, \phi)$ be a t-module defined over $\bar{k}$. Let $\mathbf{u}$ be a point in $\operatorname{Lie} G(\mathbb{C}_\infty)$ such that $\exp_G(\mathbf{u}) \in G(\bar{k})$. Then the smallest vector subspace in $\operatorname{Lie} G$ defined over $\bar{k}$ which is invariant under $d(\phi_t)$ and which contains $\mathbf{u}$ must be the tangent space at the origin of a t-submodule of G.

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A t-module of dimension d is a pair $(\mathbb{G}_a^d, \phi)$, consisting of $\mathbb{F}_q$-algebra homomorphism

$$\phi : \mathbb{F}_q[t] \longrightarrow \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_a^d) \cong \mathrm{Mat}_d(\bar{k}[F]).$$

given by

$$\phi_t = \theta I + N + g_1 F + \cdots + g_r F^r,$$

where $N \in \mathrm{Mat}_d(\bar{k})$ is nilpotent.

One also has the exponential map $\exp_G$ for t-module G:

$$\begin{array}{ccc} \mathbb{C}_\infty^d & \xrightarrow{\exp_G} & \mathbb{G}_a^d(\mathbb{C}_\infty) = \mathbb{C}_\infty^d \\ {\scriptstyle d(\phi_t)}\downarrow & & \downarrow{\scriptstyle \phi_t} \\ \mathbb{C}_\infty^d & \xrightarrow{\exp_G} & \mathbb{G}_a^d(\mathbb{C}_\infty) = \mathbb{C}_\infty^d \end{array}$$

Anderson's t-modules

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$$\begin{array}{ccc} \mathbb{C}_\infty^d & \xrightarrow{\exp_G} & \mathbb{G}_a^d(\mathbb{C}_\infty) = \mathbb{C}_\infty^d \\ {\scriptstyle d(\phi_t)}\downarrow & & \downarrow{\scriptstyle \phi_t} \\ \mathbb{C}_\infty^d & \xrightarrow{\exp_G} & \mathbb{G}_a^d(\mathbb{C}_\infty) = \mathbb{C}_\infty^d \end{array}$$

Linear independence Theorem

The t-submodule theorem says that all linear relations satisfied by a logarithmic vector of an algebraic point on t-module should come from algebraic relations inside the t-module under consideration. Structure of t-modules is "rigid". Usually it is possible to analyze the t-submodules in question.

Using the t-submodule theorem, one obtains:

Let ρ be Drinfeld module of rank r with field of multiplications K_ρ. Let $[\delta_1] = [\delta^{(1)}, \ldots, [\delta_r]$ be a basis of the de Rham cohomology of ρ, with corresponding quasi-periodic functions $F_{\delta_1}, \ldots, F_{\delta_r}$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$, be logarithms with $\exp_\rho(\mathbf{u}_i) \in \bar{k}$ for each i. Suppose that these $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent over K_ρ. Then the $rn + 1$ elements, $1, \mathbf{u}_i, F_{\delta_j}(\mathbf{u}_i), i = 1, \ldots, n, j = 2, \ldots, r$, are linearly independent over $\bar{k}$.

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Construction of t-modules

First, a t-module G_ρ of dimension $r = \operatorname{rank}\rho$:

$$(\phi_\rho)_t := \begin{bmatrix} \rho_t & 0 & 0\cdots & 0\cdots & 0 \\ (\delta_2)_t & \theta F^0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \\ (\delta_r)_t & 0 & & \cdots & \theta F^0 \end{bmatrix}.$$

This has exponential map :

$$\exp_{G_\rho} : \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_r \end{pmatrix} \longmapsto \begin{pmatrix} \exp_\rho(z_1) \\ z_2 + F_{\delta_2}(z_1) \\ \vdots \\ z_r + F_{\delta_r}(z_1) \end{pmatrix}$$

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From linear independence to algebraic independence

$$\mathbf{u} = (1, \mathbf{u}_1, -F_{\delta_2}(\mathbf{u}_1), \cdots, -F_{\delta_r}(\mathbf{u}_1), \cdots, \mathbf{u}_n, -F_{\delta_2}(\mathbf{u}_n), \cdots, -F_{\delta_r}(\mathbf{u}_n)).$$

The algebraic point $\exp_G(\mathbf{u})$ corresponding to this vector is

$$(1, \exp_\rho(\mathbf{u}_1), 0, \cdots, \exp_\rho(\mathbf{u}_2), 0, \cdots, \cdots, \exp_\rho(\mathbf{u}_n), 0, \cdots).$$

The hypothesis that $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent over K_ρ implies precisely that this algebraic point on G does not fall in any proper t-submodule of G.

Extensive efforts of using the t-submodule theorem to prove linear independence results by many people in the late 1990's, e.g. A-B-P concerning the independence of geometric Gamma values, lead to a "motivic" way for attacking **algebraic independence** in positive characteristic.

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The End. Thank You.