1 Benchmark problem

The physical Model

$$\begin{cases} \partial_\tau(h) \quad + \partial_\xi(\kappa h) & = -\alpha_\rho \mathcal{M} \\ \partial_\tau(hu) + \partial_\xi(hu \otimes \kappa + p) & = -\alpha_\rho u\mathcal{M} + h\mathcal{S}(u,\theta) \\ \partial_\tau x \quad = -(\theta - \Theta_n)\mathcal{E}(h,u,\theta)\boldsymbol{\eta}(\theta) \end{cases} \tag{1}$$

where h is the , u is the fluid velocity,$x(\xi, t)$ is the position interface position, ξ is the lagrangian coordinate, Θ_n is the neutral angle, $\theta = (\theta, \phi)$ is the local inclination angles, $\boldsymbol{\eta}(\theta)$ is the local unit normal to the interface. θ is the angle between the z axis and $\boldsymbol{\eta}(\theta)$, ϕ is the angle between the x axis and $\boldsymbol{\eta}(\theta)$.

$$\kappa(\theta) = \frac{u}{\cos\theta} \quad \text{and} \quad \boldsymbol{\eta}(\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \tag{2}$$

The pressure is defined according to the $Mohr - Coulomb$ behavior

$$p(h,\theta,\kappa) = \frac{h^2\beta(\kappa,\theta)}{2}, \quad with \quad \beta(\kappa,\theta) = \left(\cos^2\theta\right)\mathcal{K}(\kappa)$$

where the factor $\mathcal{K}(\kappa)$ is define, for $\varepsilon = \frac{H_{ref}}{L_{ref}}$, as

$$\mathcal{K}(\kappa) = \begin{cases} \mathcal{K}_-(\varepsilon,\theta_\phi,\theta_\delta) & for \quad \partial_\xi\kappa \geq 0 \\ \mathcal{K}_+(\varepsilon,\theta_\phi,\theta_\delta) & for \quad \partial_\xi\kappa < 0 \end{cases}$$

$$\mathcal{K}_\pm(\varepsilon,\theta_\phi,\theta_\delta) = \varepsilon\left(\frac{2\left(1 \pm \sqrt{1 - \frac{\cos^2\theta_\phi}{\cos^2\theta_\delta}}\right)}{\cos^2\theta_\phi} - 1\right)$$

where θ_ϕ and θ_δ are respectively the internal and the basal friction angles.

1.1 Model for the net driving acceleration

$$\mathcal{S}(u,\theta) = \cos\theta\left(\sin\theta - \varepsilon^\beta sgn(u)\mu\cos\theta\right)$$

where $\mu = 0.9 * tan(\theta_f)$ with θ_f the friction angle.

1.2 Erosion/deposition

$$\mathcal{E}(h,u,\theta) = \mathcal{F}_e(h,u,\theta)\,\mathcal{F}_h(h)$$
$$\mathcal{F}_e(h,u,\theta) = \alpha_e\left(\frac{(1-\tanh\left(e_\alpha(||\kappa||-\kappa_{th})\right))}{2}\mathcal{H}(\theta-\Theta_n)+\mathcal{H}(\Theta_n-\theta)\right)$$

Acordind to the relation $\mathcal{H}(\Theta_n-\theta) = 1-\mathcal{H}(\theta-\Theta_n)$, we can also use the simplified form

$$\mathcal{F}_e(h,u,\theta) = \alpha_e\left(1-\frac{(1+\tanh\left(e_\alpha(||\kappa||-\kappa_{th})\right))}{2}\mathcal{H}(\theta-\Theta_n)\right)$$

$$\mathcal{F}_h(h) = h+\alpha_h\sqrt{h}, \qquad \kappa_{th} = \alpha_v(\theta-\Theta)^2$$

1.3 Benchmark Data

$$\theta_\phi = 34^o, \quad \Theta_n = 34^o, \quad \theta_\delta = 23^o \quad \theta_f = 33^o$$
$$\alpha_h = 0.05, \quad \alpha_e = 2.0, \quad \alpha_v = 1.0, \quad \alpha_\rho = 0.9, \quad e_\alpha = 20, \quad \varepsilon = 1.0.$$

1.4 Initial condition

Computational domain $[-3,3]$. Initial uniform mesh according to x and mesh sizes of : 100, 200, 400.

The initial basal surface

$$\mathcal{B}(\tau=0,x) = \mathcal{B}_0\exp\left(-\frac{32x^2}{9}\right)$$

The initial layer profile

$$h(\tau=0,x)\cos\theta_0(x) = L_0\exp\left(-\frac{32x^2}{9}\right)-\mathcal{B}_0\exp\left(-\frac{32x^2}{9}\right)$$

$L_0 = 2,$

	$Test1$	$Test2$	$Test3$
$\mathcal{B}_0 =$	0	0.2	0.5