Towards a Unified Coordinate Method for Compressible Multifluid Flows

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Outline

- Motivation
- Inviscid compressible flow model
  - Generalized coordinates Euler equations
  - Grid movement conditions
  - Equations of state
  - Transport eqs. for multifluid problems
- Finite volume numerical method
  - Godunov-type f-wave formulation of LeVeque *et al.*
- Numerical examples
- Future direction

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- Some basic facts
  - Lagrangian method resolve material or slip lines sharply if no grid tangling
  - Generalized curvilinear grid is often superior to Cartesian when employed in numerical methods for complex fixed or moving geometries

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  - Falling liquid drop problem
  - Shock-bubble interaction
  - Flying Aluminum-plate problem
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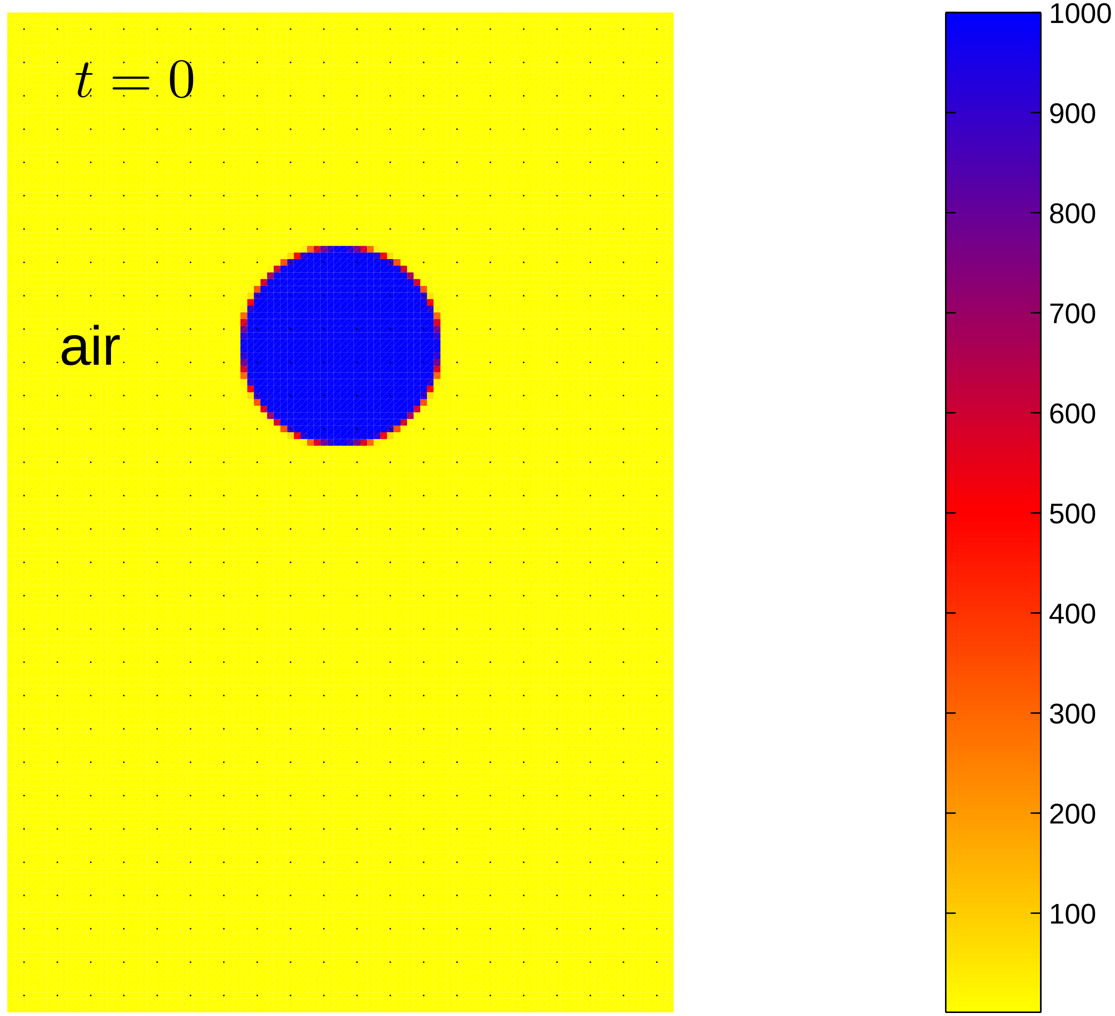
  - Falling rigid object in water tank
- Search for more robust method (work present here is preliminary)

Falling Liquid Drop Problem

- Interface capturing with gravity



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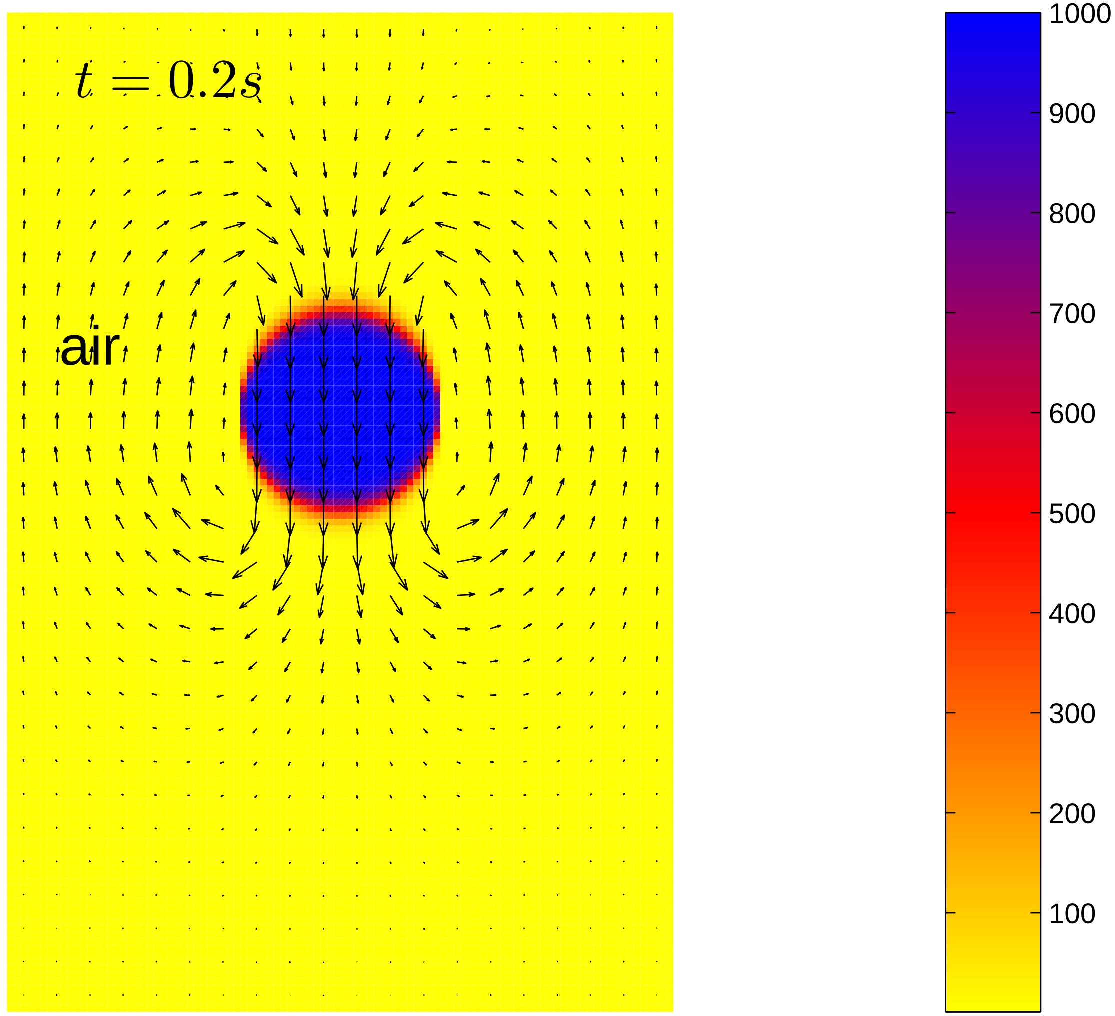
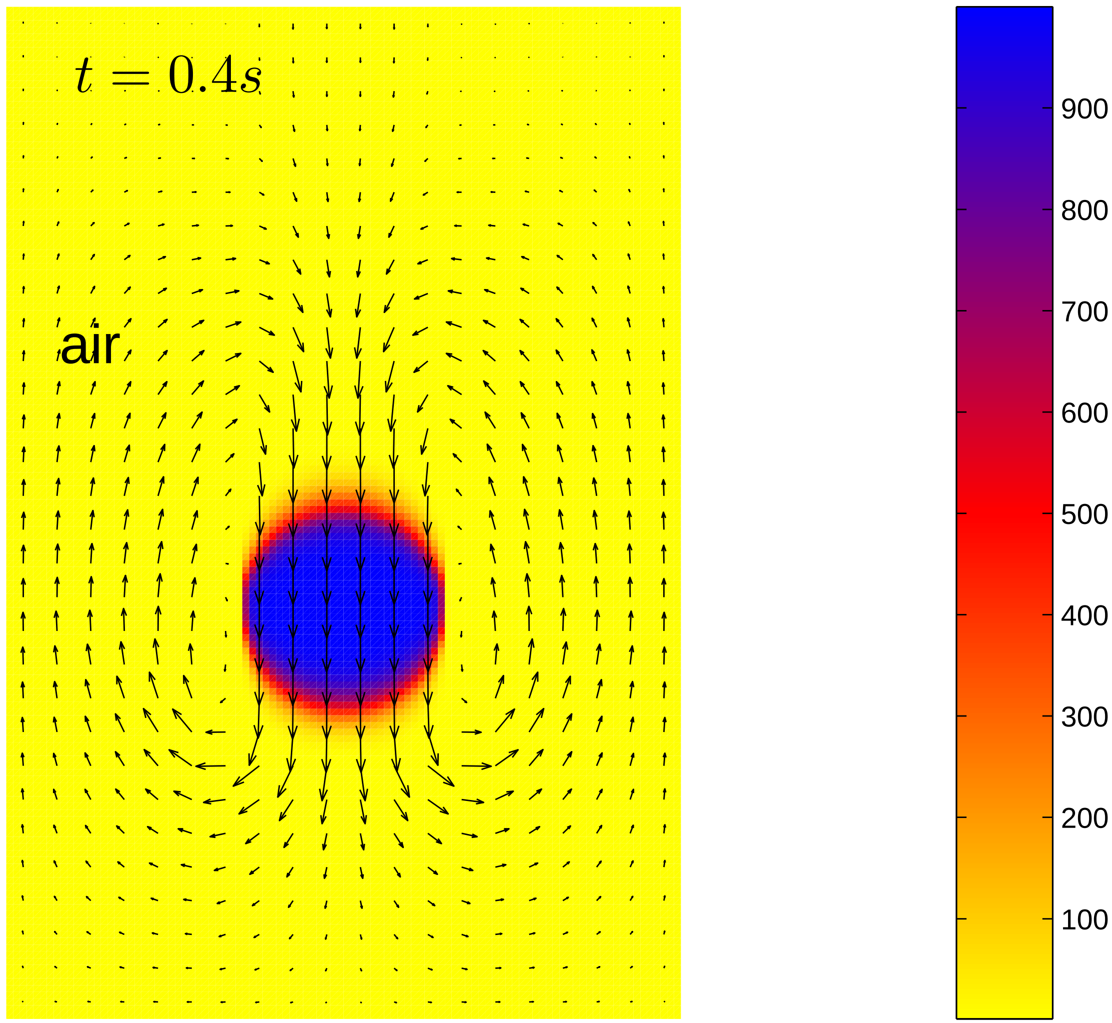


Falling Liquid Drop Problem

- Interface diffused badily



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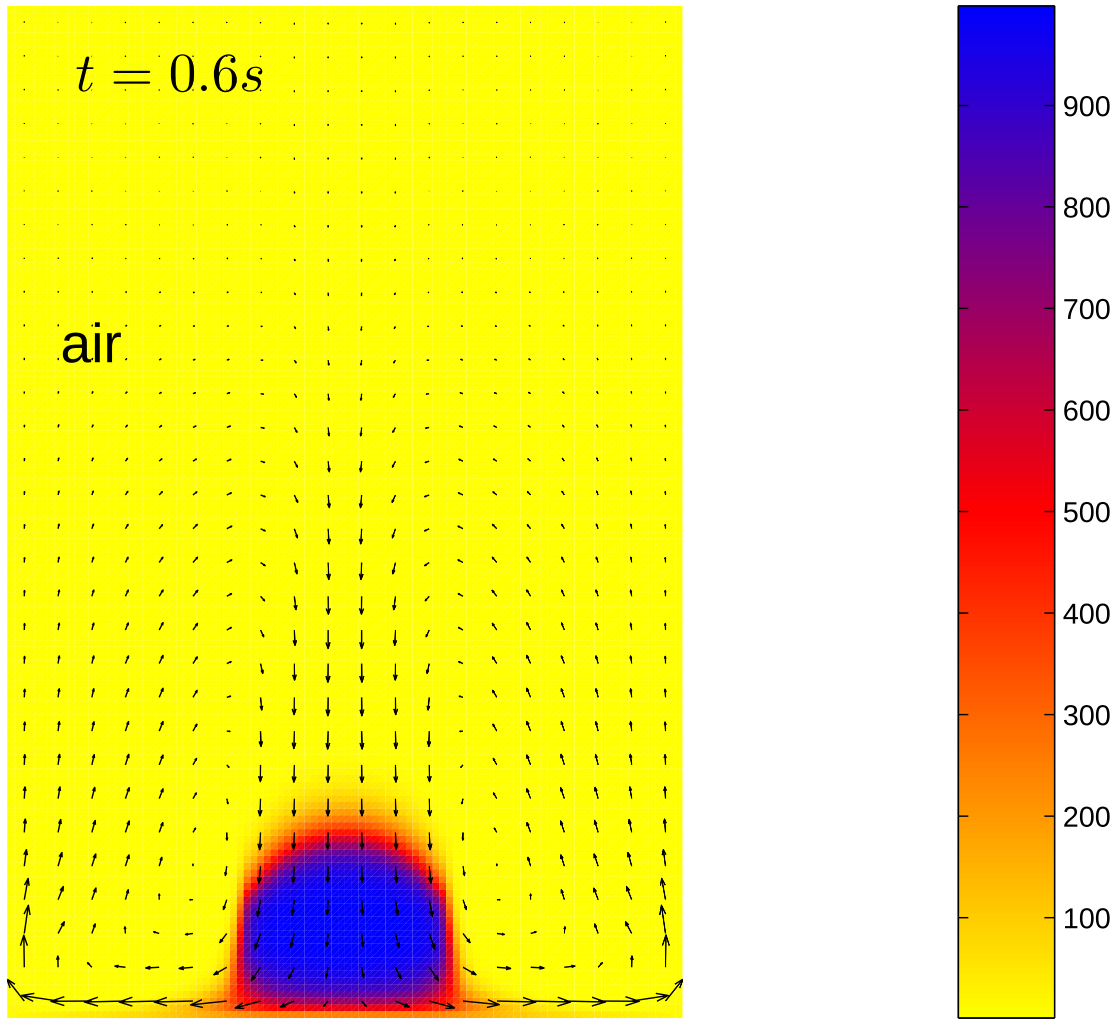

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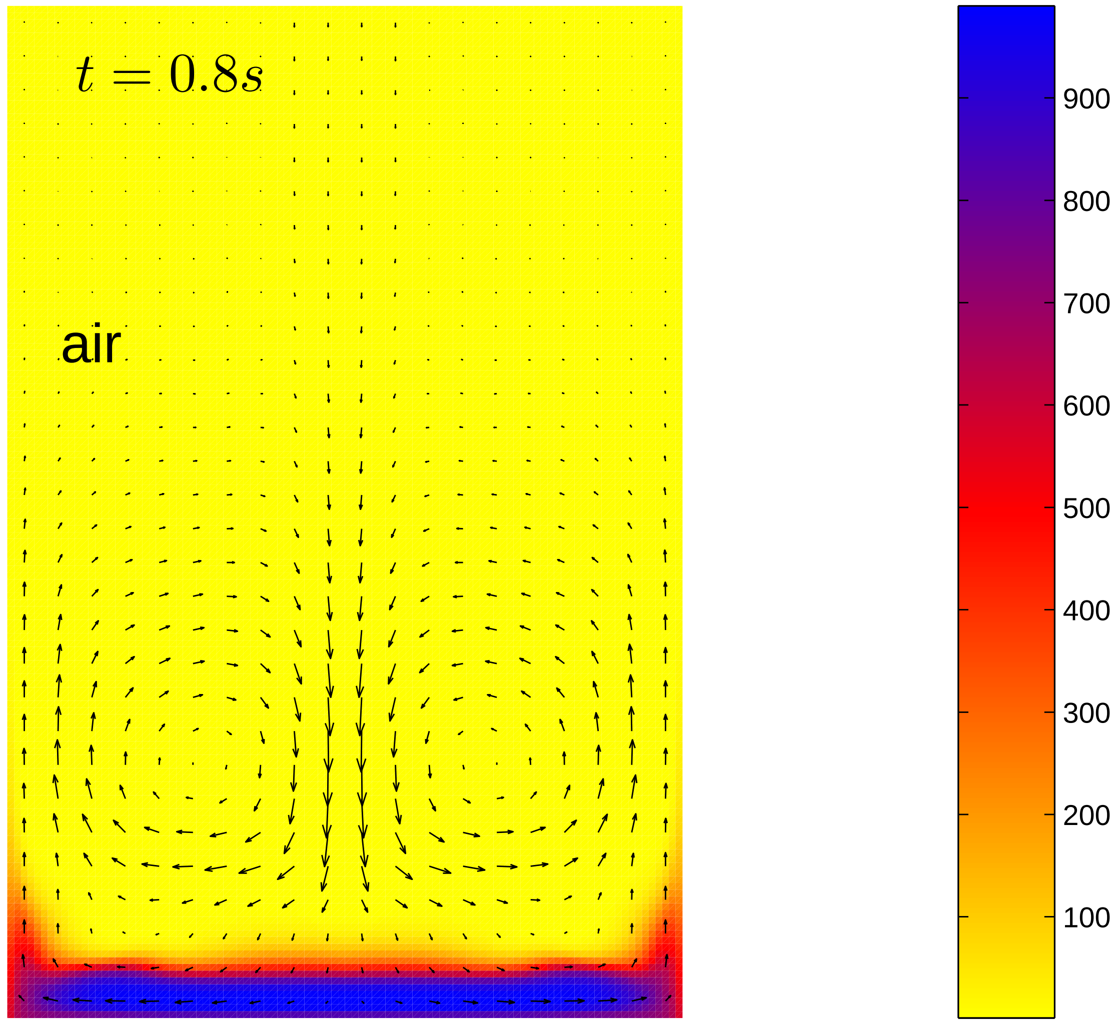

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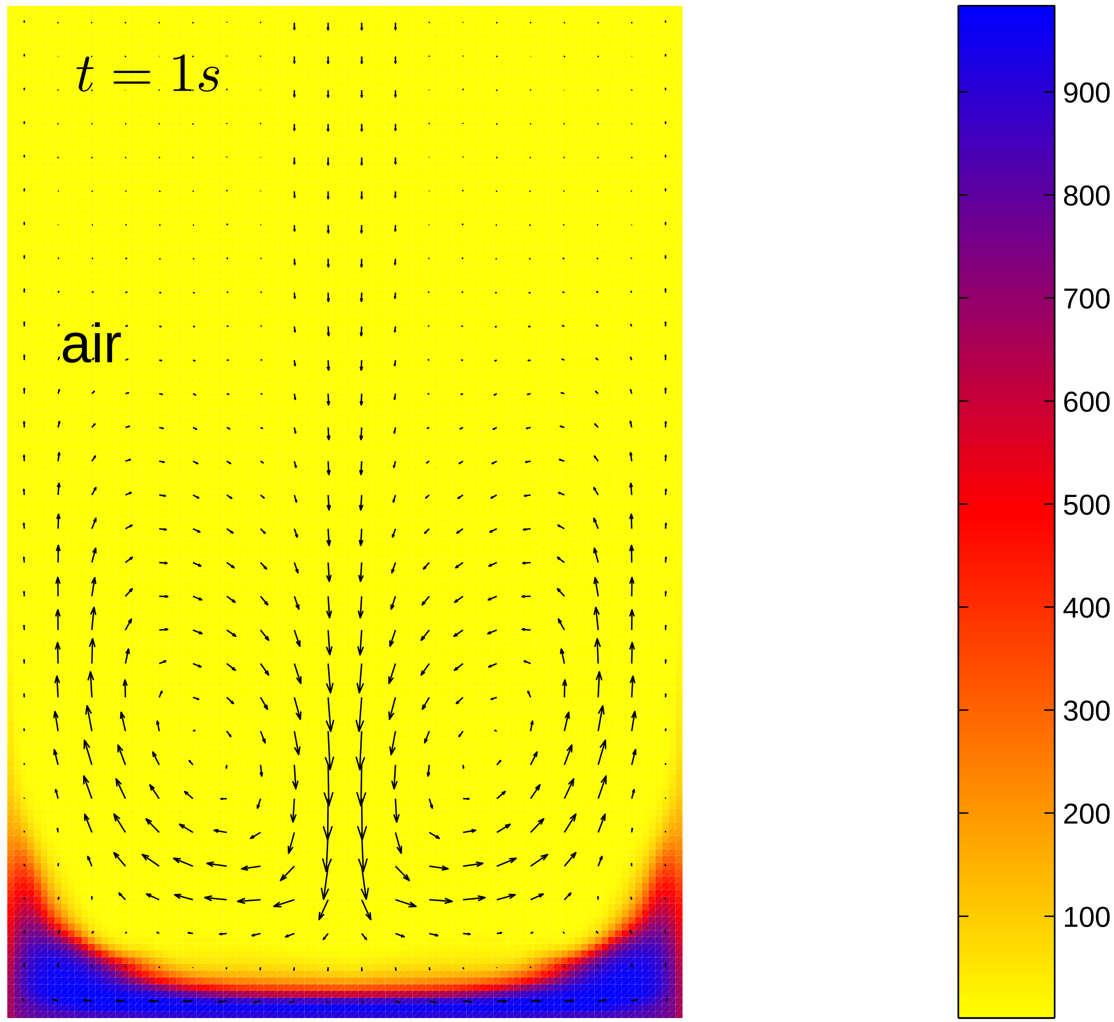
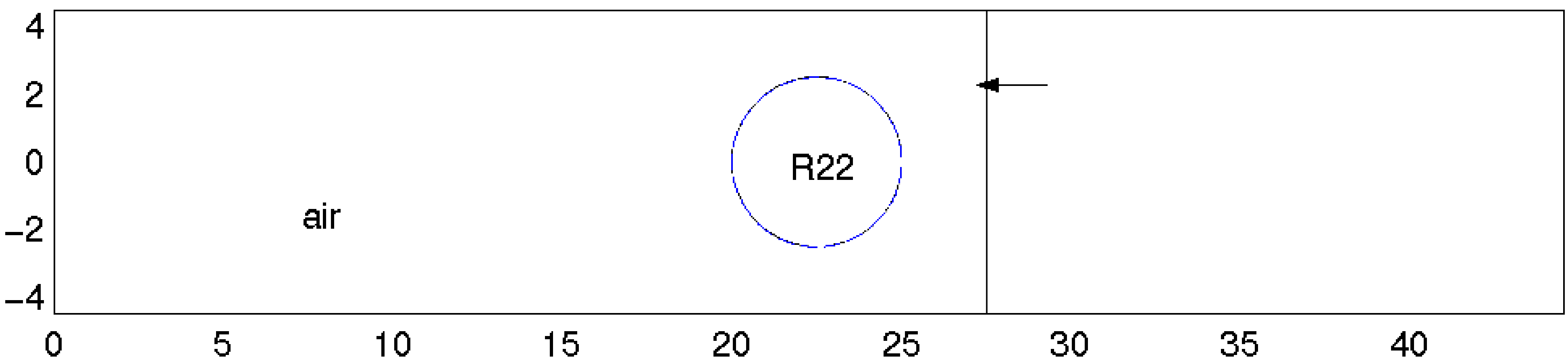


Shock-Bubble Interaction

- Volume tracking for material interface



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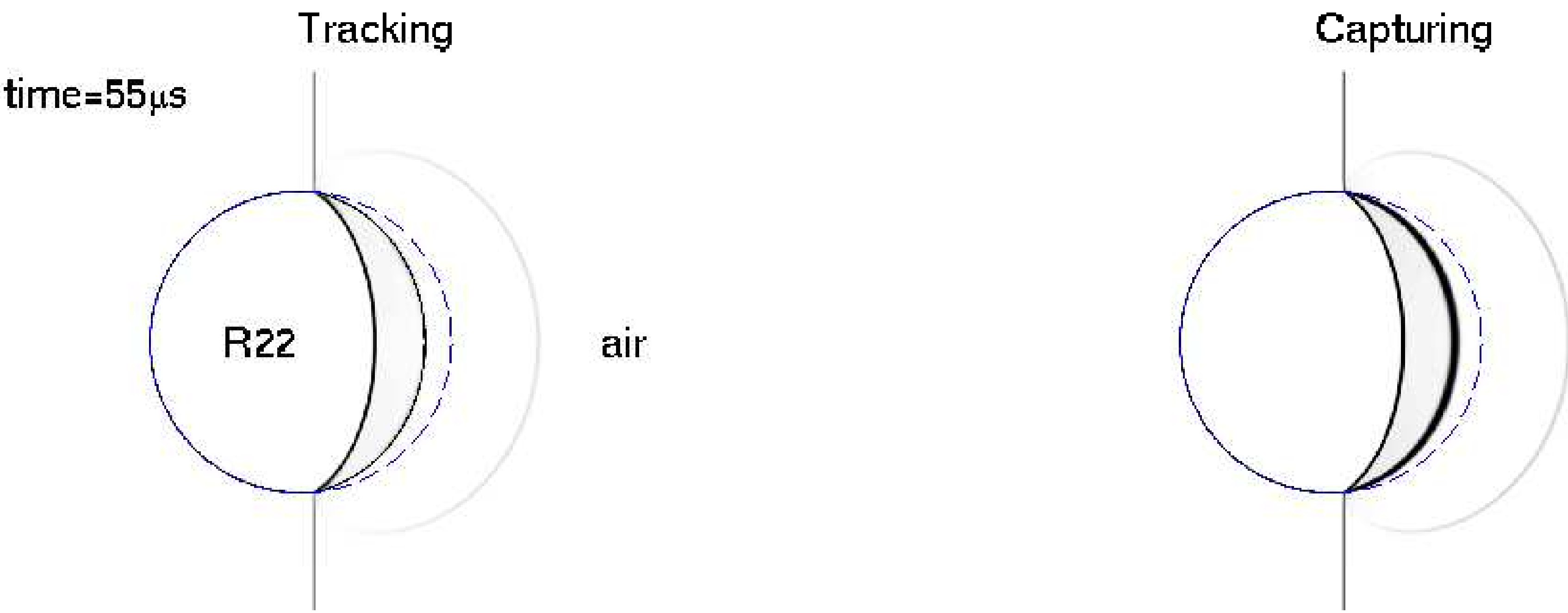

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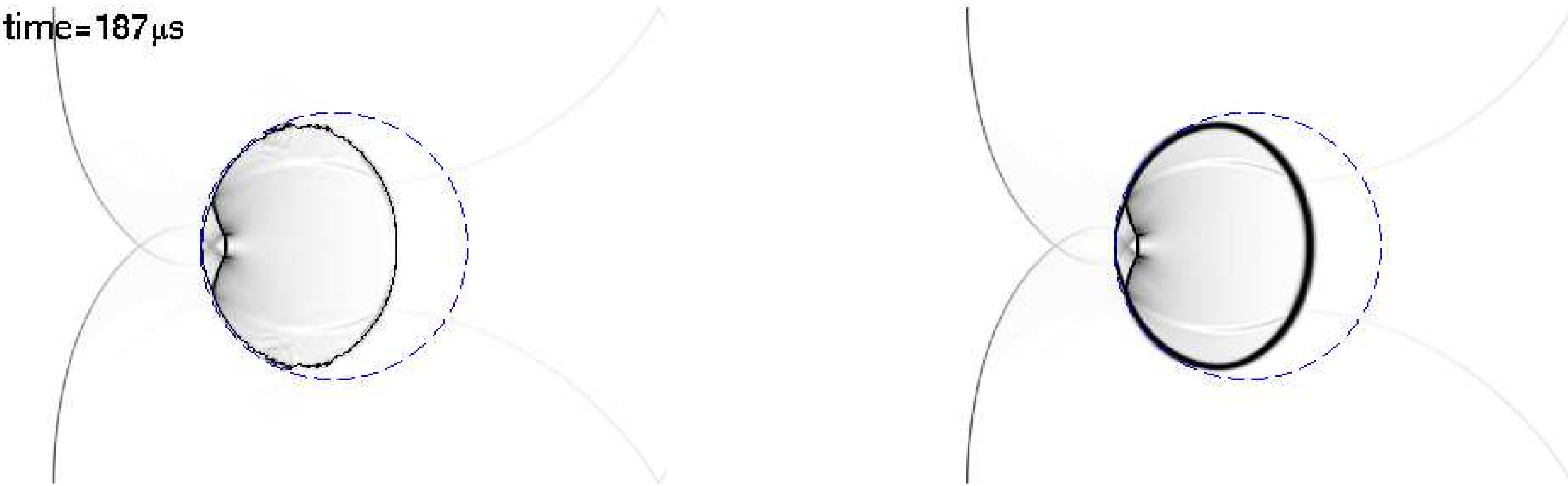

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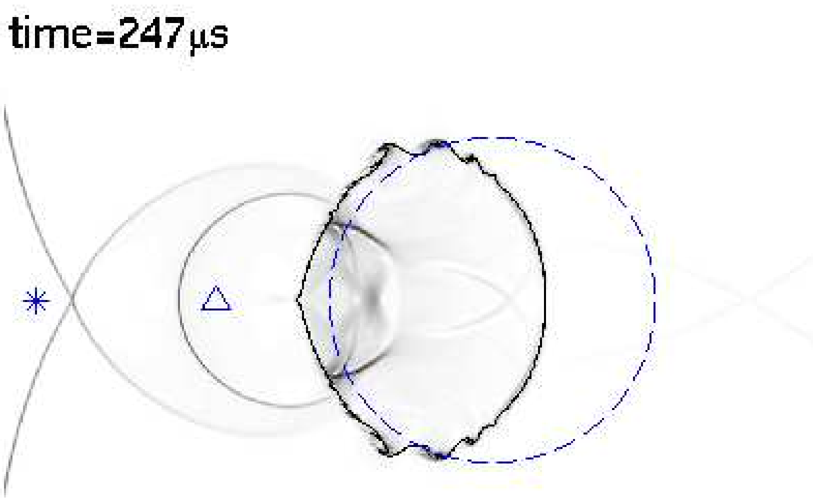

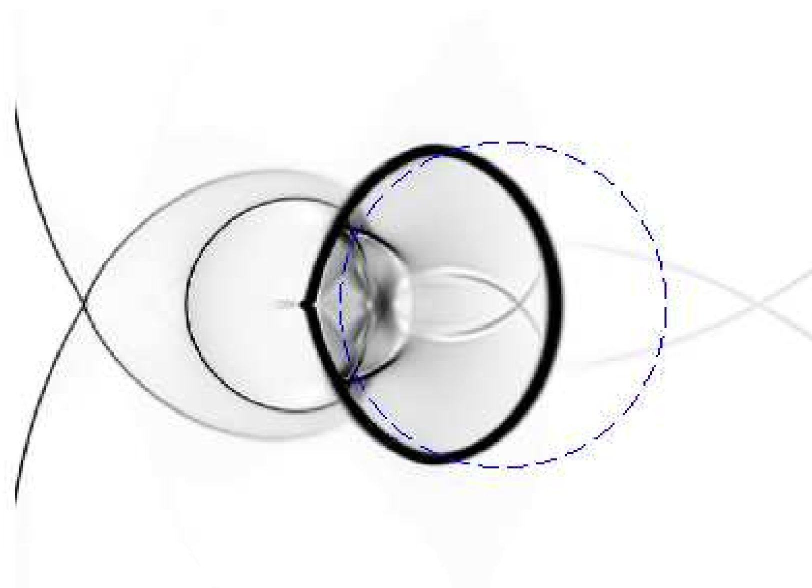

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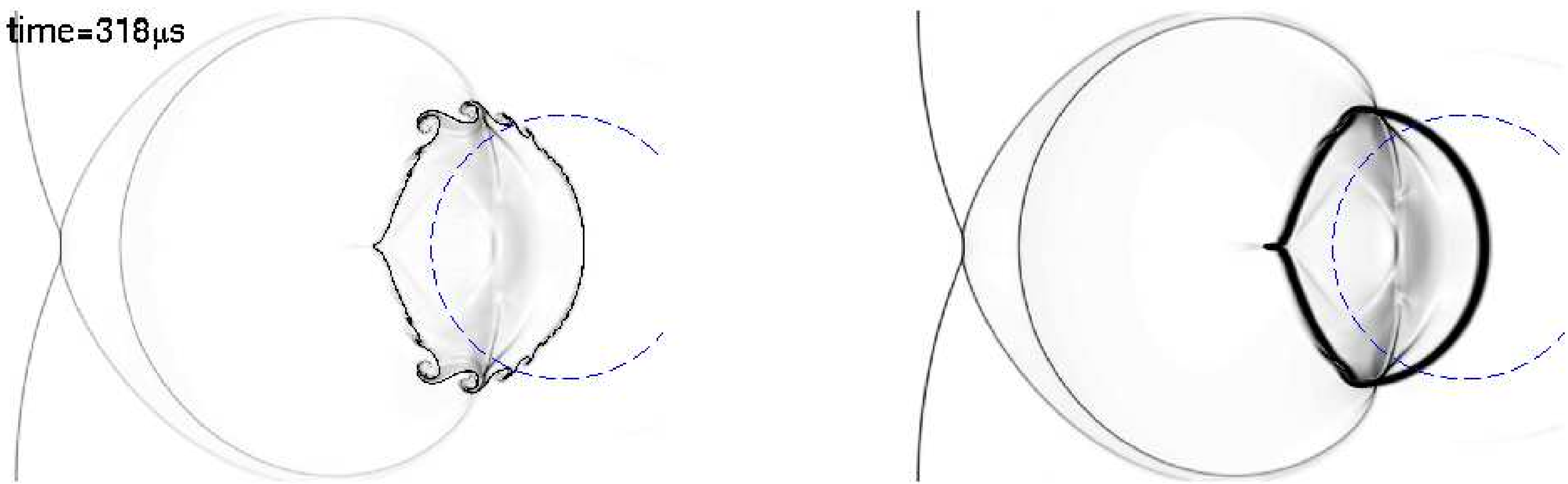

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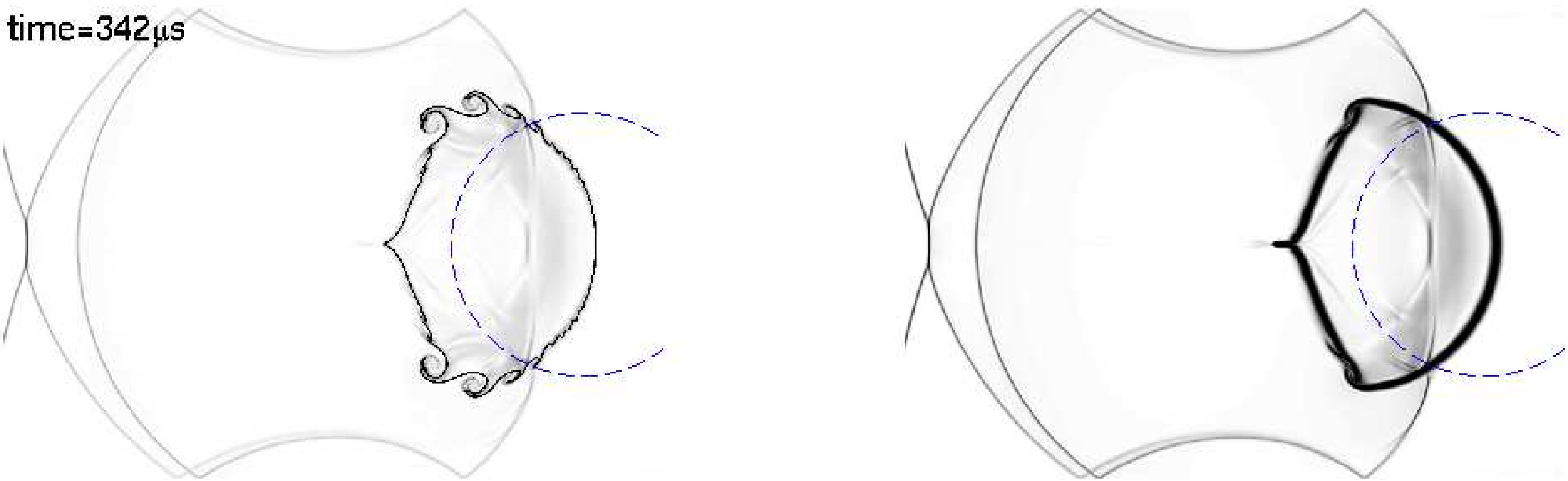

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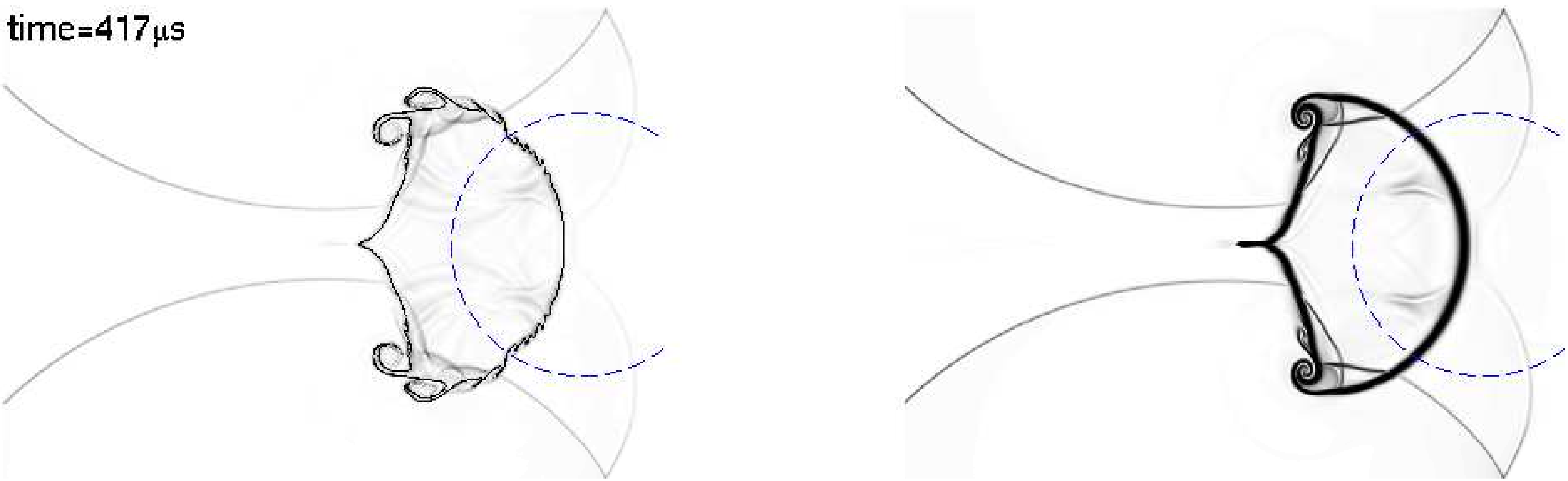
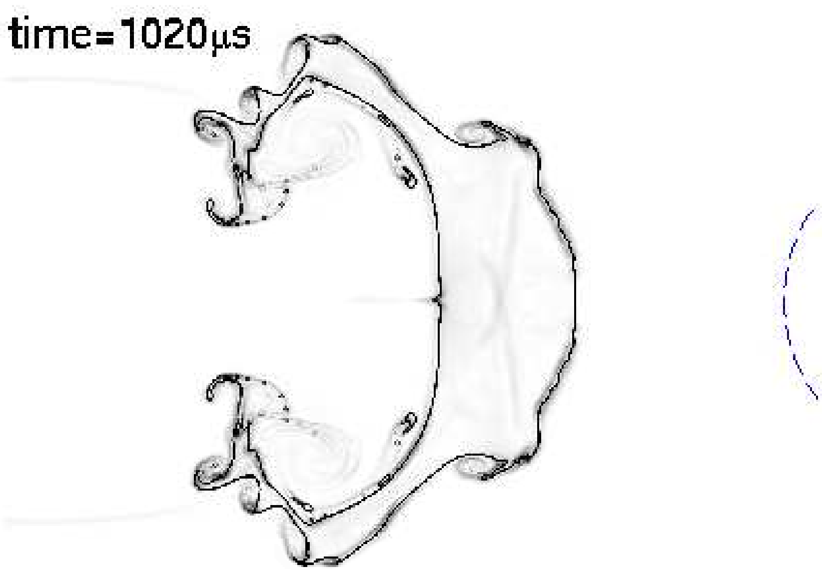
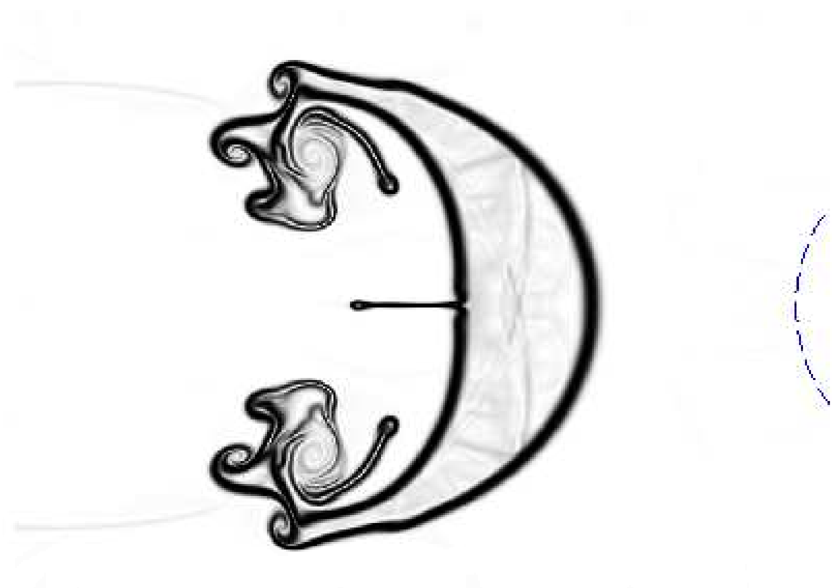


Shock-Bubble Interaction

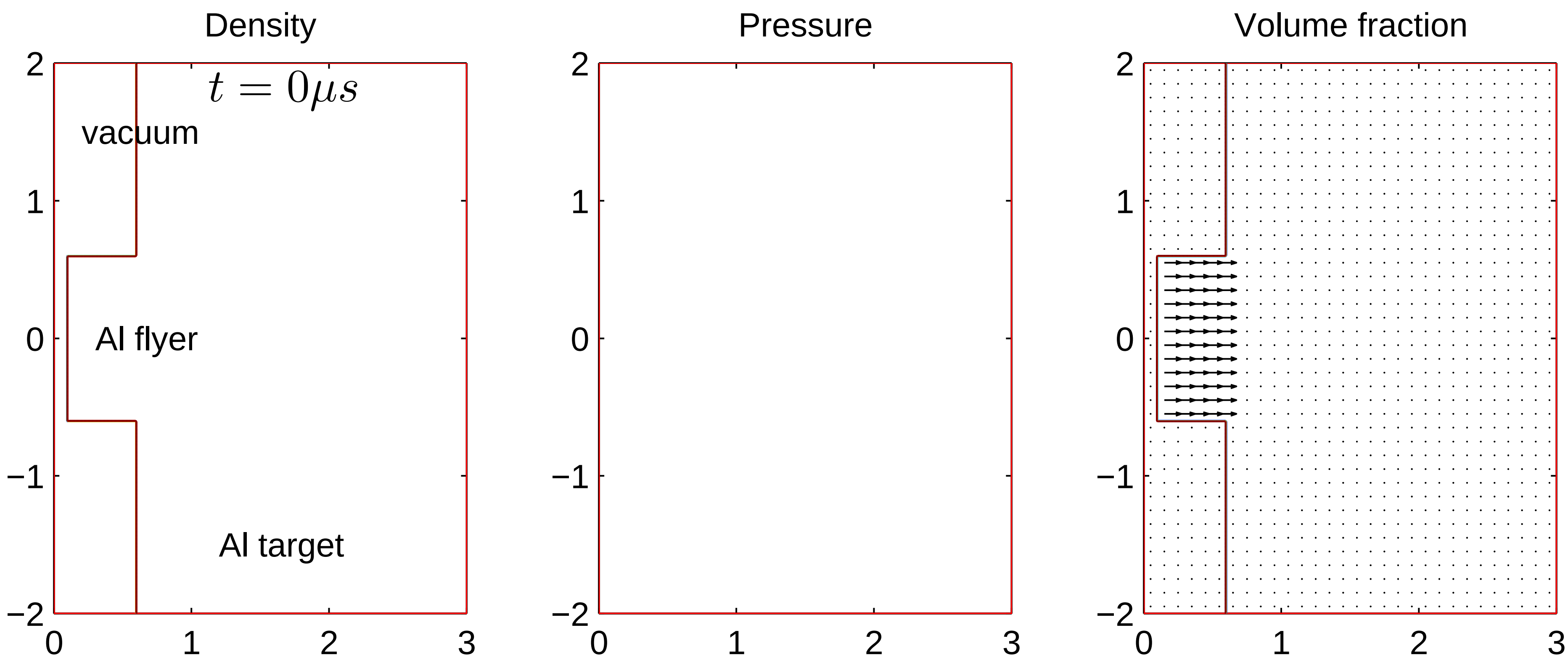
- Small moving irregular cells: stability & accuracy





Flying Aluminum-plate problem

- Vacuum-Al interface tracking



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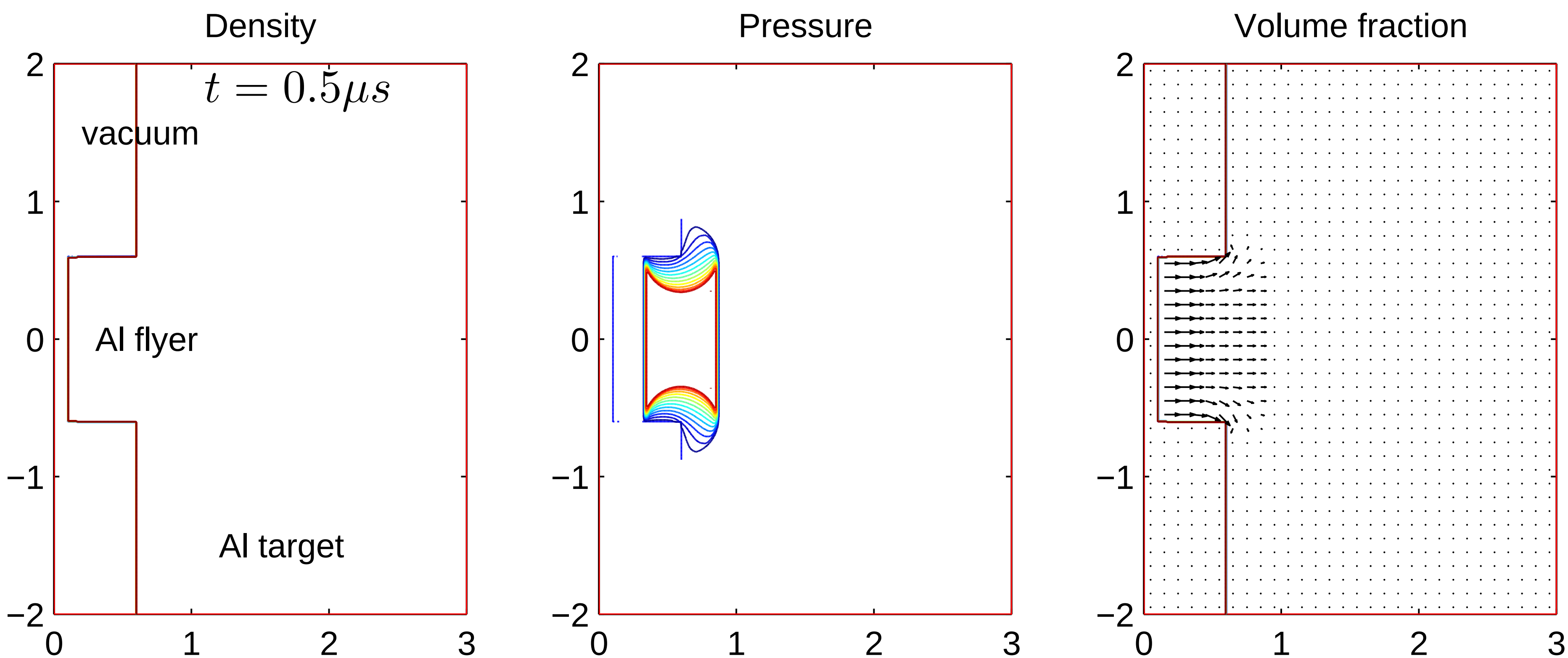

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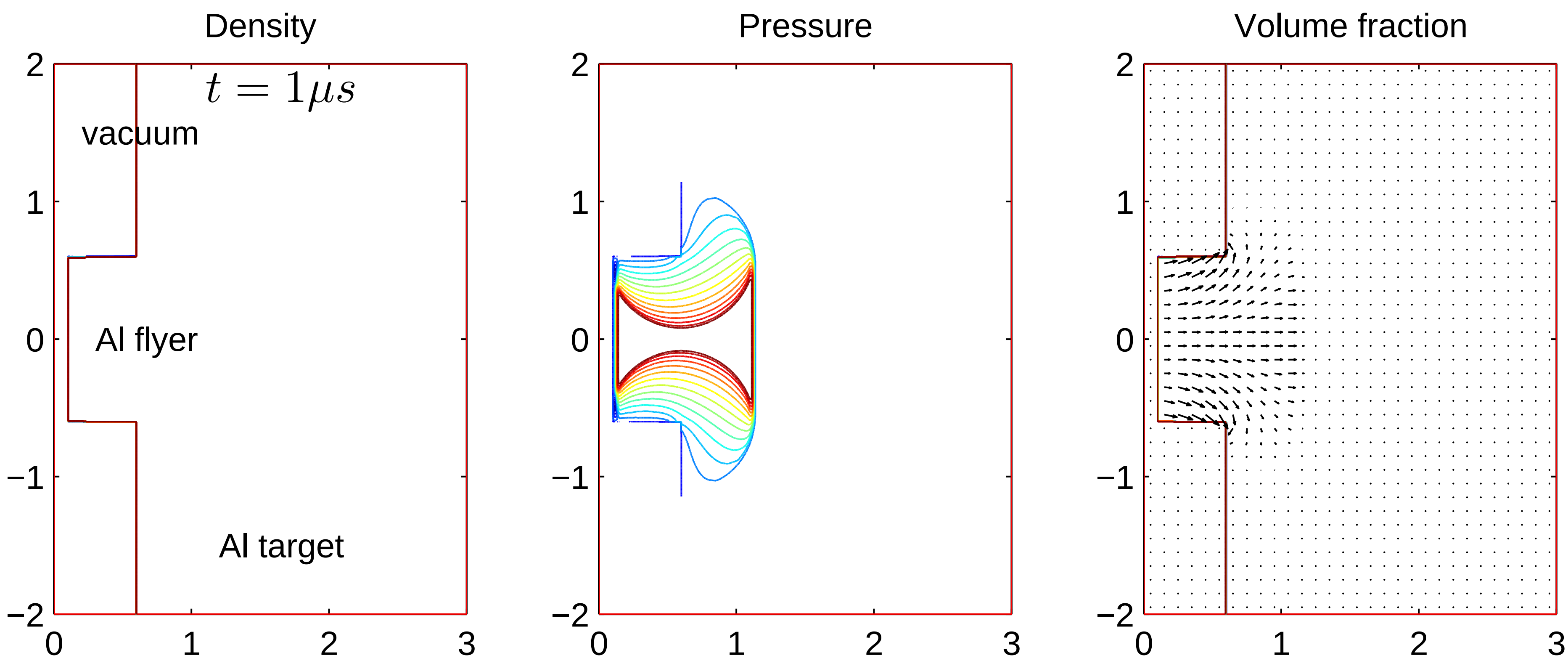

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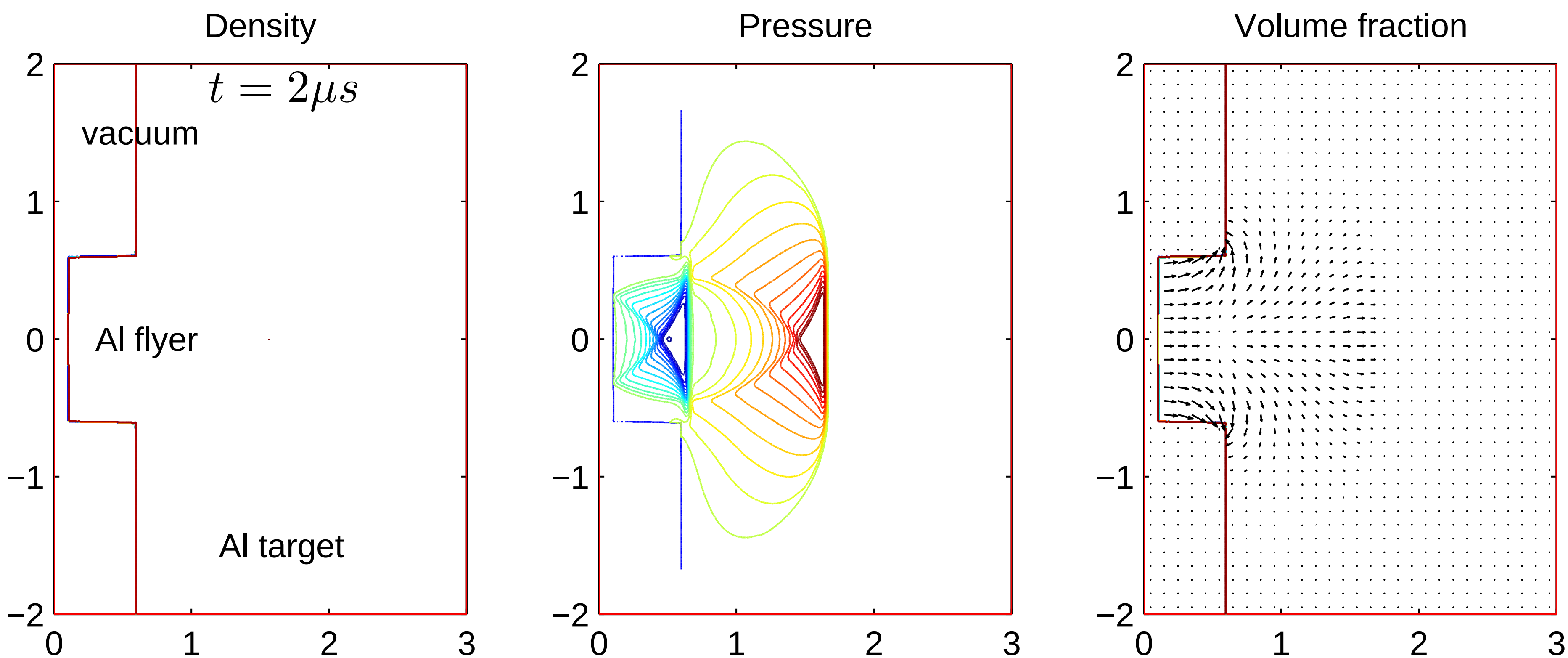
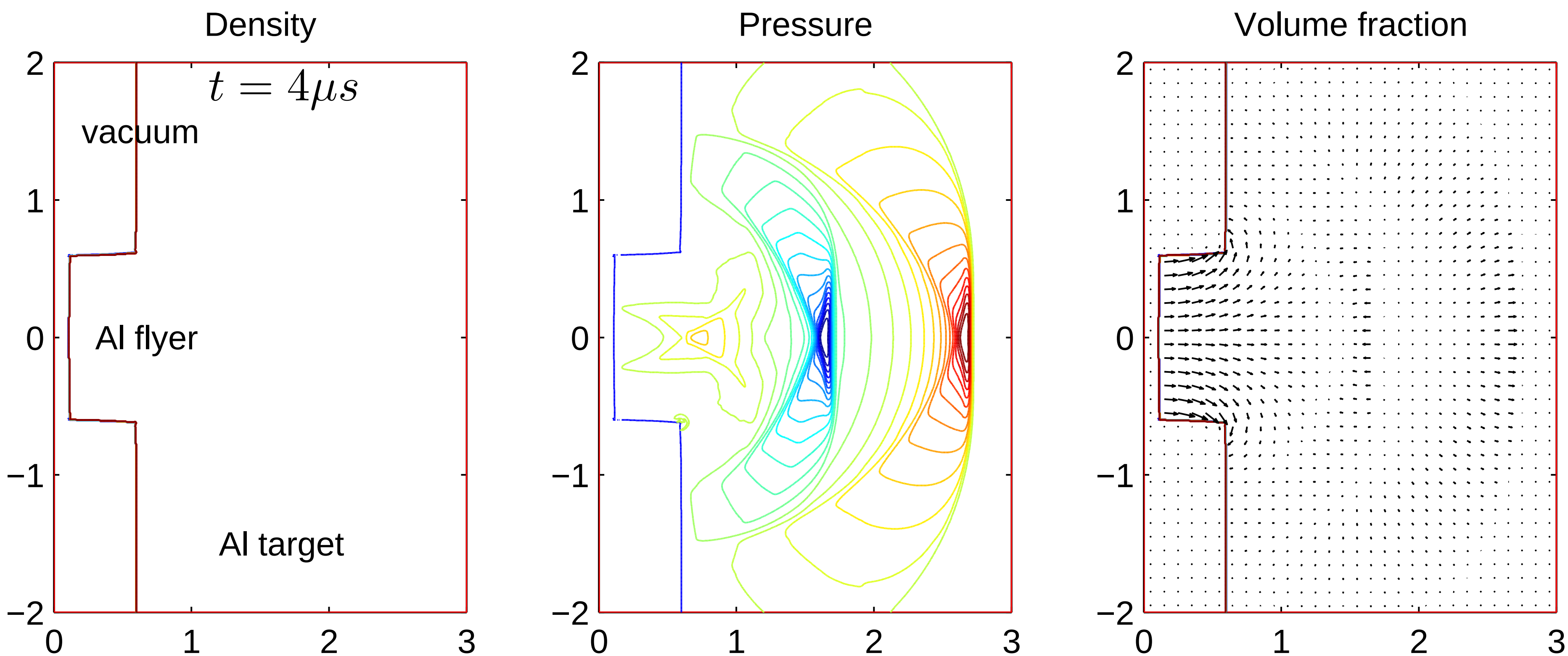


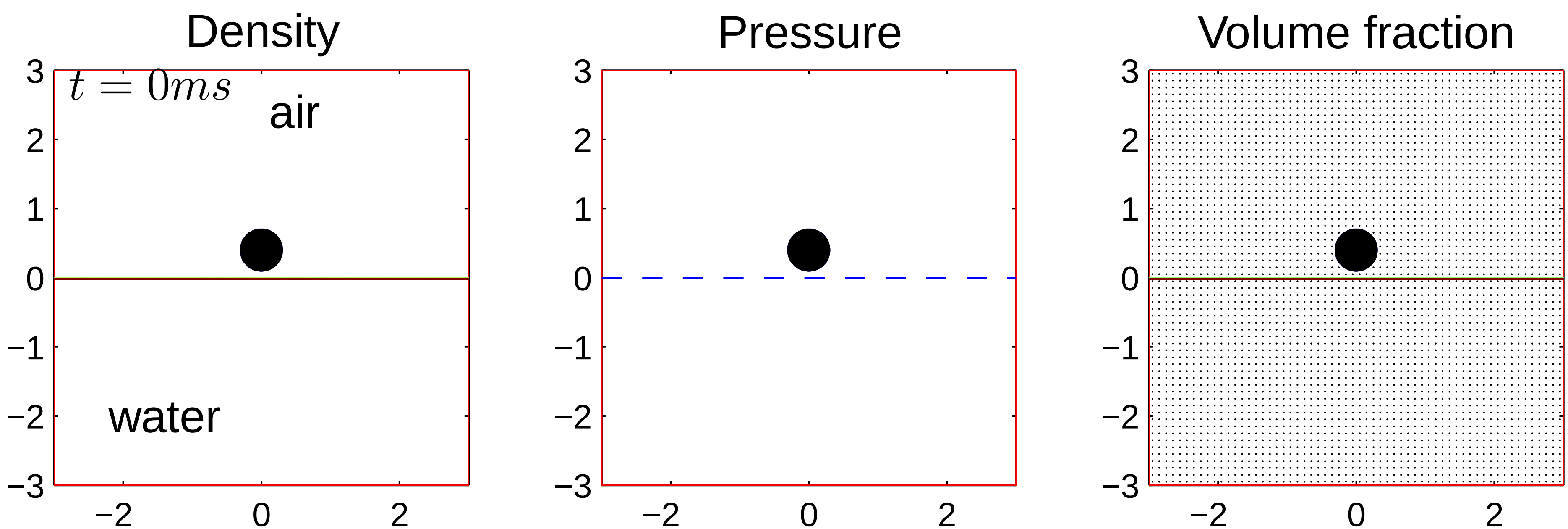
Flying Aluminum-plate problem

- Small moving irregular cells: stability & accuracy



Falling Rigid Object in Water Tank

- Moving boundary tracking & interface capturing



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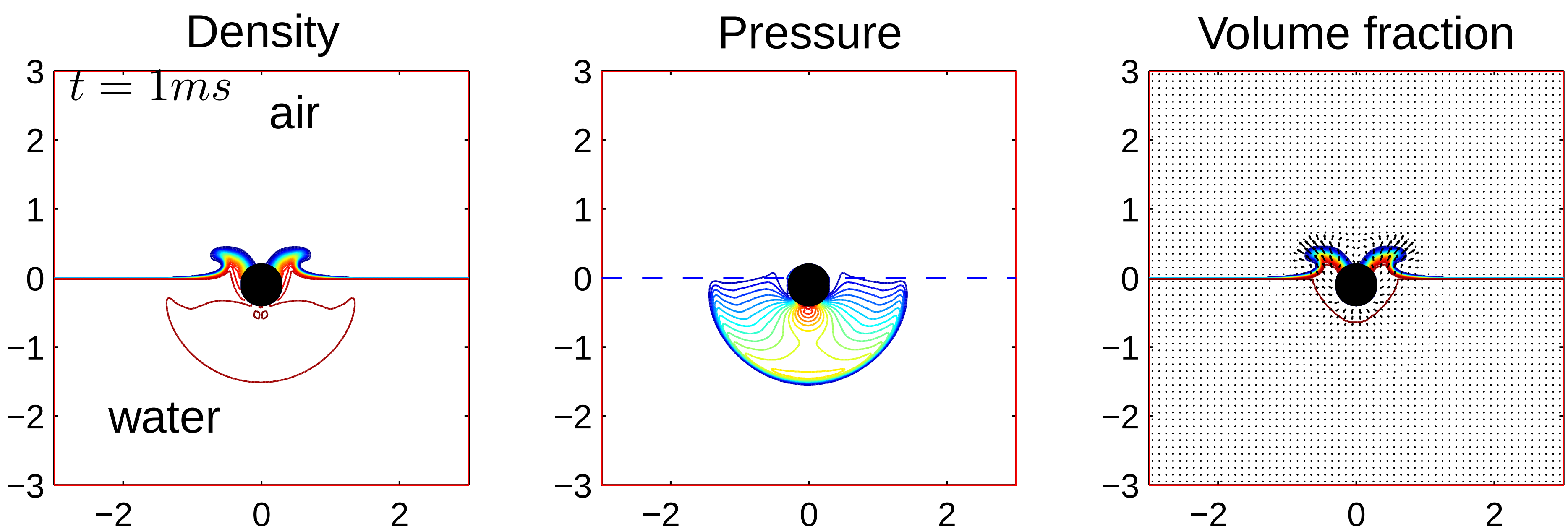

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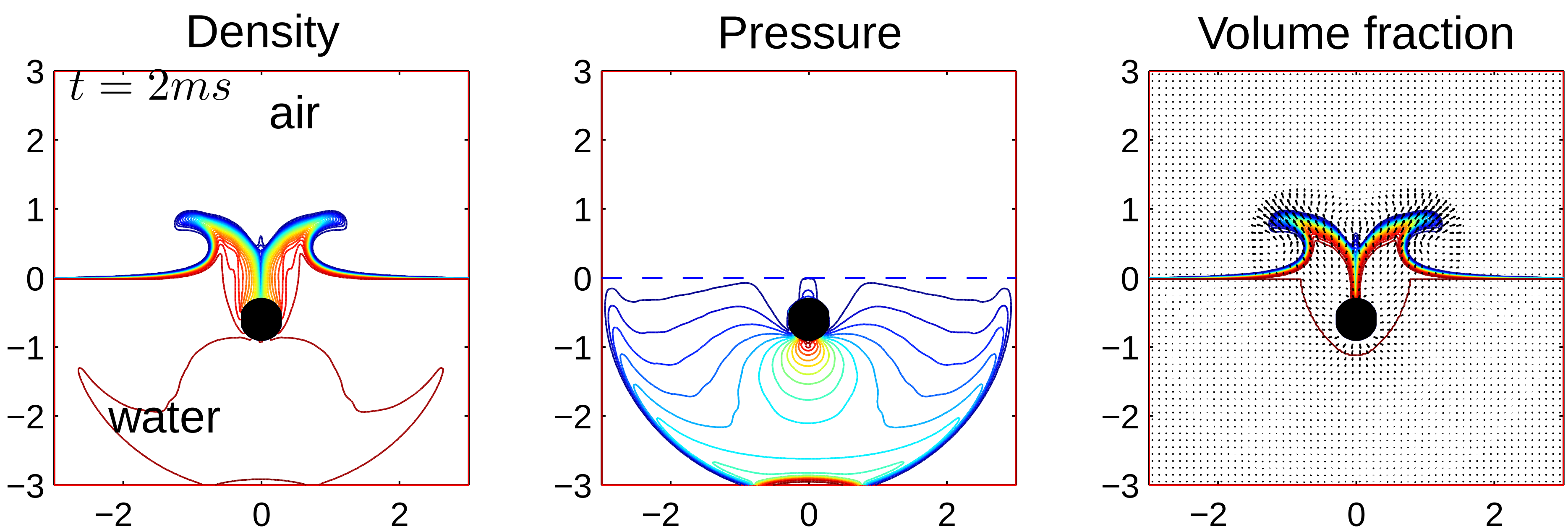

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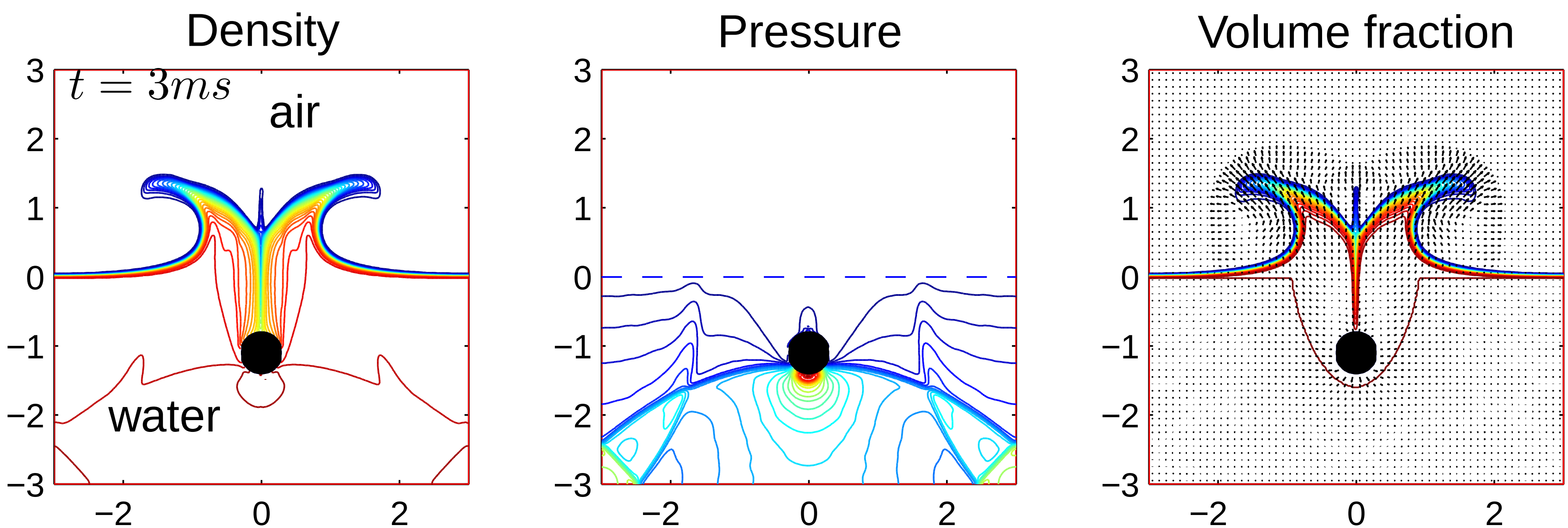

Euler Eqs. in Generalized Coord.

With gravity effect included, for example, 2D compressible Euler eqs. in Cartesian coordinates take

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + \frac{\partial g(q)}{\partial y} = \psi(q)$$

where

$$q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \quad f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ Eu + pu \end{bmatrix}, \quad g(q) = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ Ev + pv \end{bmatrix}, \quad \psi = \begin{bmatrix} 0 \\ 0 \\ \rho g \\ \rho g v \end{bmatrix}$$

ρ : density, (u, v) : vector of particle velocity

p : pressure, $E = \rho[e + (u^2 + v^2)/2]$: total energy

$e(\rho, p)$: internal energy, ψ : gravitational source term

Euler in General. Coord. (Cont.)

- Introduce transformation $(t,\ x,\ y) \leftrightarrow (\tau,\ \xi,\ \eta)$ via

$$\begin{pmatrix} dt \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x_\tau & x_\xi & x_\eta \\ y_\tau & y_\xi & y_\eta \end{pmatrix} \begin{pmatrix} d\tau \\ d\xi \\ d\eta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} d\tau \\ d\xi \\ d\eta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \xi_t & \xi_x & \xi_y \\ \eta_t & \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \end{pmatrix}$$

- Basic grid-metric relations:

$$\begin{pmatrix} 1 & 0 & 0 \\ \xi_t & \xi_x & \xi_y \\ \eta_t & \eta_x & \eta_y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x_\tau & x_\xi & x_\eta \\ y_\tau & y_\xi & y_\eta \end{pmatrix}^{-1} = \frac{1}{J} \begin{bmatrix} x_\xi y_\eta - x_\eta y_\xi & 0 & 0 \\ -x_\tau y_\eta + y_\tau x_\eta & y_\eta & -y_\xi \\ x_\tau y_\xi - y_\tau x_\xi & -x_\eta & x_\xi \end{bmatrix}$$

  - $J = x_\xi y_\eta - x_\eta y_\xi$: grid Jacobian

Euler in General. Coord. (Cont.)

With these notations, Euler eqs. in generalized coord. are

$$\frac{\partial \tilde{q}}{\partial \tau} + \frac{\partial \tilde{f}}{\partial \xi} + \frac{\partial \tilde{g}}{\partial \eta} = \tilde{\psi}$$

where

$$\tilde{q} = J \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \tilde{f} = J \begin{bmatrix} \rho U \\ \rho u U + \xi_x p \\ \rho v U + \xi_y p \\ EU + pU - \xi_t p \end{bmatrix}, \tilde{g} = J \begin{bmatrix} \rho V \\ \rho u V + \eta_x p \\ \rho v V + \eta_y p \\ EV + pV - \eta_t p \end{bmatrix}, \tilde{\psi} = J \begin{bmatrix} 0 \\ 0 \\ \rho g \\ \rho g v \end{bmatrix}$$

with contravariant velocities U & V defined by

$$U = \xi_t + \xi_x u + \xi_y v \quad \& \quad V = \eta_t + \eta_x u + \eta_y v$$

Euler in General. Coord. (Cont.)

Model system in quasi-linear form

$$\frac{\partial \tilde{q}}{\partial \tau} + A\frac{\partial \tilde{q}}{\partial \xi} + B\frac{\partial \tilde{q}}{\partial \eta} = \tilde{\psi}$$

$$A = \frac{\partial \tilde{f}}{\partial \tilde{q}} = \begin{bmatrix} \xi_t & \xi_x & \xi_y & 0 \\ \xi_x p_\rho - u\mathcal{U} & \xi_x u(1-p_E) + U & \xi_y u - \xi_x v p_E & \xi_x p_E \\ \xi_y p_\rho - v\mathcal{U} & \xi_x v - \xi_y u p_E & \xi_y v(1-p_E) + U & \xi_y p_E \\ (p_\rho - H)\mathcal{U} & \xi_x H - u\mathcal{U} p_E & \xi_y H - v\mathcal{U} p_E & U + p_E\mathcal{U} \end{bmatrix}$$

$$B = \frac{\partial \tilde{g}}{\partial \tilde{q}} = \begin{bmatrix} \eta_t & \eta_x & \eta_y & 0 \\ \eta_x p_\rho - u\mathcal{V} & \eta_x u(1-p_E) + V & \eta_y u - \eta_x v p_E & \eta_x p_E \\ \eta_y p_\rho - v\mathcal{V} & \eta_x v - \eta_y u p_E & \eta_y v(1-p_E) + V & \eta_y p_E \\ (p_\rho - H)\mathcal{V} & \eta_x H - u\mathcal{V} p_E & \eta_y H - v\mathcal{V} p_E & V + p_E\mathcal{V} \end{bmatrix}$$

with $H = (E+p)/\rho$, $\quad \mathcal{U} = U - \xi_t = \xi_x u + \xi_y v$, $\quad \mathcal{V} = V - \eta_t = \eta_x u + \eta_y v$

Euler in General. Coord. (Cont.)

Eigen-structure of matrix A is

$$\Lambda_A = \text{diag}\left(U - c\sqrt{\xi_x^2+\xi_y^2},\ U,\ U,\ U + c\sqrt{\xi_x^2+\xi_y^2}\right)$$

$$R_A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ u-\alpha_1 c & u & \alpha_2 & u+\alpha_1 c \\ v-\alpha_2 c & v & -\alpha_1 & v+\alpha_2 c \\ H-\mathcal{U}_1 c & H-c^2/p_E & -\mathcal{U}_2 & H+\mathcal{U}_1 c \end{bmatrix}$$

$$L_A = \begin{bmatrix} (p_\rho + c\mathcal{U}_1)/2c^2 & -(\alpha_1 c + up_E)/2c^2 & -(\alpha_2 c + vp_E)/2c^2 & p_E/2c^2 \\ 1-p_\rho/c^2 & up_E/c^2 & vp_E/c^2 & -p_E/c^2 \\ \mathcal{U}_2 & \alpha_2 & -\alpha_1 & 0 \\ (p_\rho - c\mathcal{U}_1)/2c^2 & (\alpha_1 c - up_E)/2c^2 & (\alpha_2 c - vp_E)/2c^2 & p_E/2c^2 \end{bmatrix}$$

with $(\alpha_1,\alpha_2) = (\xi_x,\xi_y)/\sqrt{\xi_x^2+\xi_y^2}$, $\quad \mathcal{U}_1 = \alpha_1 u + \alpha_2 v$, $\quad \mathcal{U}_2 = -\alpha_2 u + \alpha_1 v$

Euler in General. Coord. (Cont.)

Eigen-structure of matrix B is

$$\Lambda_B = \text{diag}\left(V - c\sqrt{\eta_x^2+\eta_y^2},\ V,\ V,\ V + c\sqrt{\eta_x^2+\eta_y^2}\right)$$

$$R_B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ u - \beta_1 c & u & \beta_2 & u + \beta_1 c \\ v - \beta_2 c & v & -\beta_1 & v + \beta_2 c \\ H - \mathcal{V}_1 c & H - c^2/p_E & -\mathcal{V}_2 & H + \mathcal{V}_1 c \end{bmatrix}$$

$$L_B = \begin{bmatrix} (p_\rho + c\mathcal{V}_1)/2c^2 & -(\beta_1 c + u p_E)/2c^2 & -(\beta_2 c + v p_E)/2c^2 & p_E/2c^2 \\ 1 - p_\rho/c^2 & u p_E/c^2 & v p_E/c^2 & -p_E/c^2 \\ \mathcal{V}_2 & \beta_2 & -\beta_1 & 0 \\ (p_\rho - c\mathcal{V}_1)/2c^2 & (\beta_1 c - u p_E)/2c^2 & (\beta_2 c - v p_E)/2c^2 & p_E/2c^2 \end{bmatrix}$$

with $(\beta_1, \beta_2) = (\eta_x, \eta_y)/\sqrt{\eta_x^2+\eta_y^2}$, $\quad \mathcal{V}_1 = \beta_1 u + \beta_2 v$, $\quad \mathcal{V}_2 = -\beta_2 u + \beta_1 v$

Grid Movement Conditions

Continuity on mixed derivatives of grid coordinates gives geometrical conservation laws

$$\frac{\partial}{\partial \tau}\begin{pmatrix} x_\xi \\ y_\xi \\ x_\eta \\ y_\eta \end{pmatrix} + \frac{\partial}{\partial \xi}\begin{pmatrix} -x_\tau \\ -y_\tau \\ 0 \\ 0 \end{pmatrix} + \frac{\partial}{\partial \eta}\begin{pmatrix} 0 \\ 0 \\ -x_\tau \\ -y_\tau \end{pmatrix} = 0$$

with (x_τ, y_τ) to be specified as, for example,

- Eulerian case: $(x_\tau, y_\tau) = \vec{0}$
- Lagrangian case: $(x_\tau, y_\tau) = (u, v)$
- Lagrangian-like case: $(x_\tau, y_\tau) = h_0(u, v)$ or $(h_0 u, k_0 v)$
  - $h_0 \in [0, 1]$ & $k_0 \in [0, 1]$ (fixed piecewise const.)

Grid Movement (Cont.)

- General 1-parameter case: $(x_\tau, y_\tau) = h(u, v)$, $\quad h \in [0, 1]$

  At given time instance, h can be chosen based on

  - Grid-angle preserving condition (Hui *et al.* JCP 1999)

$$\frac{\partial}{\partial \tau} \cos^{-1}\left(\frac{\nabla \xi}{|\nabla \xi|} \cdot \frac{\nabla \eta}{|\nabla \eta|}\right) = \frac{\partial}{\partial \tau} \cos^{-1}\left(\frac{-y_\eta x_\eta - y_\xi x_\xi}{\sqrt{y_\xi^2 + y_\eta^2}\sqrt{x_\xi^2 + x_\eta^2}}\right)$$

$$= \cdots$$

$$= \mathcal{A} h_\xi + \mathcal{B} h_\eta + \mathcal{C} h = 0 \quad \text{(1st order PDE)}$$

with

$$\mathcal{A} = \sqrt{x_\eta^2 + y_\eta^2}\,(v x_\xi - u y_\xi)\,, \quad \mathcal{B} = \sqrt{x_\xi^2 + y_\xi^2}\,(u y_\eta - v x_\eta)$$

$$\mathcal{C} = \sqrt{x_\xi^2 + y_\xi^2}\,(u_\eta y_\eta - v_\eta x_\eta) - \sqrt{x_\eta^2 + y_\eta^2}\,(u_\xi y_\xi - v_\xi x_\xi)$$

Grid Movement (Cont.)

- General 1-parameter case: $(x_\tau, y_\tau) = h(u, v)$, $\quad h \in [0, 1]$

  Or alternatively, based on
  - Mesh-area preserving condition

$$\begin{aligned} \frac{\partial J}{\partial \tau} &= \frac{\partial}{\partial \tau}(x_\xi y_\eta - x_\eta y_\xi) \\ &= x_{\xi\tau}\, y_\eta + x_\xi\, y_{\eta\tau} - x_{\eta\tau}\, y_\xi - x_\eta\, y_{\xi\tau} \\ &= \cdots \\ &= \mathcal{A} h_\xi + \mathcal{B} h_\eta + \mathcal{C} h = 0 \quad \text{(1st order PDE)} \end{aligned}$$

  with

$$\mathcal{A} = u y_\eta - v x_\eta, \quad \mathcal{B} = v x_\xi - u y_\xi, \quad \mathcal{C} = u_\xi y_\eta + v_\eta x_\xi - u_\eta y_\xi - v_\xi x_\eta$$

Grid Movement (Cont.)

To ensure $h \in [0,1]$, transformed variable $\tilde{h} = \kappa(h)$ is used, *e.g.*, Hui *et al.* employed $\kappa = \ln(\varepsilon h |\vec{u}|)$, ε normalized constant, yielding

$$\tilde{\mathcal{A}}\tilde{h}_\xi + \tilde{\mathcal{B}}\tilde{h}_\eta + \tilde{\mathcal{C}} = 0$$

- Grid-angle preserving case

$$\tilde{\mathcal{A}} = \sqrt{x_\eta^2 + y_\eta^2}\,(x_\xi \sin\theta - y_\xi \cos\theta)\,, \quad \tilde{\mathcal{B}} = \sqrt{x_\xi^2 + y_\xi^2}\,(y_\eta \cos\theta - x_\eta \sin\theta)$$
$$\tilde{\mathcal{C}} = \sqrt{x_\xi^2 + y_\xi^2}\,[y_\eta(\cos\theta)_\eta - x_\eta(\sin\theta)_\eta] - \sqrt{x_\eta^2 + y_\eta^2}\,[y_\xi(\cos\theta)_\xi - x_\xi(\sin\theta)_\xi]$$

- Mesh-area preserving case

$$\tilde{\mathcal{A}} = y_\eta \cos\theta - x_\eta \sin\theta, \quad \tilde{\mathcal{B}} = x_\xi \sin\theta - y_\xi \cos\theta$$
$$\tilde{\mathcal{C}} = y_\eta(\cos\theta)_\xi - x_\eta(\sin\theta)_\xi + x_\xi(\sin\theta)_\eta - y_\xi(\cos\theta)_\eta$$

where $\vec{u} = (u,v) = |\vec{u}|(\cos\theta, \sin\theta)$

Grid Movement: Remarks

- Numerics: h- or $\tilde{h}$-equation constraint geometrical laws

$$\frac{\partial}{\partial \tau}\begin{pmatrix} x_\xi \\ y_\xi \\ x_\eta \\ y_\eta \end{pmatrix} - \frac{\partial}{\partial \xi}\begin{pmatrix} hu \\ hv \\ 0 \\ 0 \end{pmatrix} - \frac{\partial}{\partial \eta}\begin{pmatrix} 0 \\ 0 \\ hu \\ hv \end{pmatrix} = 0$$

- Usability: Mesh-area evolution equation

$$\frac{\partial J}{\partial \tau} - \frac{\partial}{\partial \xi}\left[h\left(u y_\eta - v x_\eta\right)\right] - \frac{\partial}{\partial \eta}\left[h\left(v x_\xi - u y_\xi\right)\right] = 0$$

- Initial & boundary conditions for h- or $\tilde{h}$-equation ?

Grid Movement: 2 Free Degrees

- 2-parameter case of Hui *et al.* (2005): $(x_\tau, y_\tau) = (U_g, V_g)$
  - Imposed conditions
    1. Grid-angle preserving
    2. Specialized grid-material line matching (see next)

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- Other 2-parameter case: $(x_\tau, y_\tau) = (hu, kv)$
  - Novel imposed conditions for $h \in [0, 1]$ & $k \in [0, 1]$?

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Roadmap of current work:

$$\boxed{(x_\tau, y_\tau) = h_0(u, v)} \longrightarrow \boxed{(x_\tau, y_\tau) = h(u, v)} \longrightarrow \cdots$$

Novel Conditions for h & k

- Mesh-area preserving case

$$\begin{aligned}\frac{\partial J}{\partial \tau} &= \frac{\partial}{\partial \tau}\left(x_\xi y_\eta - x_\eta y_\xi\right)\\ &= x_{\xi\tau}\, y_\eta + x_\xi\, y_{\eta\tau} - x_{\eta\tau}\, y_\xi - x_\eta\, y_{\xi\tau}\\ &= \cdots\\ &= (\mathcal{A}_1 h_\xi + \mathcal{B}_1 h_\eta + \mathcal{C}_1 h) + (\mathcal{A}_2 k_\xi + \mathcal{B}_2 k_\eta + \mathcal{C}_2 k) = 0,\end{aligned}$$

yielding, for example,

$$\begin{aligned}\mathcal{A}_1 h_\xi + \mathcal{B}_1 h_\eta + \mathcal{C}_1 h = 0\\ \mathcal{A}_2 k_\xi + \mathcal{B}_2 k_\eta + \mathcal{C}_2 k = 0\end{aligned}$$

with

$$\begin{aligned}&\mathcal{A}_1 = u y_\eta, &&\mathcal{B}_1 = u y_\xi, &&\mathcal{C}_1 = u_\xi y_\eta - u_\eta y_\xi\\ &\mathcal{A}_2 = -v x_\eta, &&\mathcal{B}_2 = v x_\xi, &&\mathcal{C}_2 = v_\eta x_\xi - v_\xi x_\eta\end{aligned}$$

Single-Fluid Model

With $(x_\tau, y_\tau) = h_0(u, v)$, our model system for single-phase flow reads

$$\frac{\partial}{\partial \tau}\begin{pmatrix} J\rho \\ J\rho u \\ J\rho v \\ JE \\ x_\xi \\ y_\xi \\ x_\eta \\ y_\eta \end{pmatrix} + \frac{\partial}{\partial \xi}\begin{pmatrix} J\rho U \\ J\rho u U + y_\eta p \\ J\rho v U - x_\eta p \\ JEU + (y_\eta u - x_\eta v)p \\ -h_0 u \\ -h_0 v \\ 0 \\ 0 \end{pmatrix} + \frac{\partial}{\partial \eta}\begin{pmatrix} J\rho V \\ J\rho u V - y_\xi p \\ J\rho v V + x_\xi p \\ JEV + (x_\xi v - y_\xi u)p \\ 0 \\ 0 \\ -h_0 u \\ -h_0 v \end{pmatrix} = \tilde{\psi}$$

where $U = (1 - h_0)(y_\eta u - x_\eta v)$ & $V = (1 - h_0)(x_\xi v - y_\xi u)$

Single-Fluid Model: Remarks

- Hyperbolicity (under thermodyn. stability cond.)
  - In Cartesian coordinates, model is hyperbolic

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- Canonical form
  - In Cartesian coordinates

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + \frac{\partial g(q)}{\partial y} = \psi(q)$$

  - In generalized coordinates

$$\frac{\partial q}{\partial \tau} + \frac{\partial f(q,\Xi)}{\partial \xi} + \frac{\partial g(q,\Xi)}{\partial \eta} = \psi(q),$$

Ξ: grid metrics

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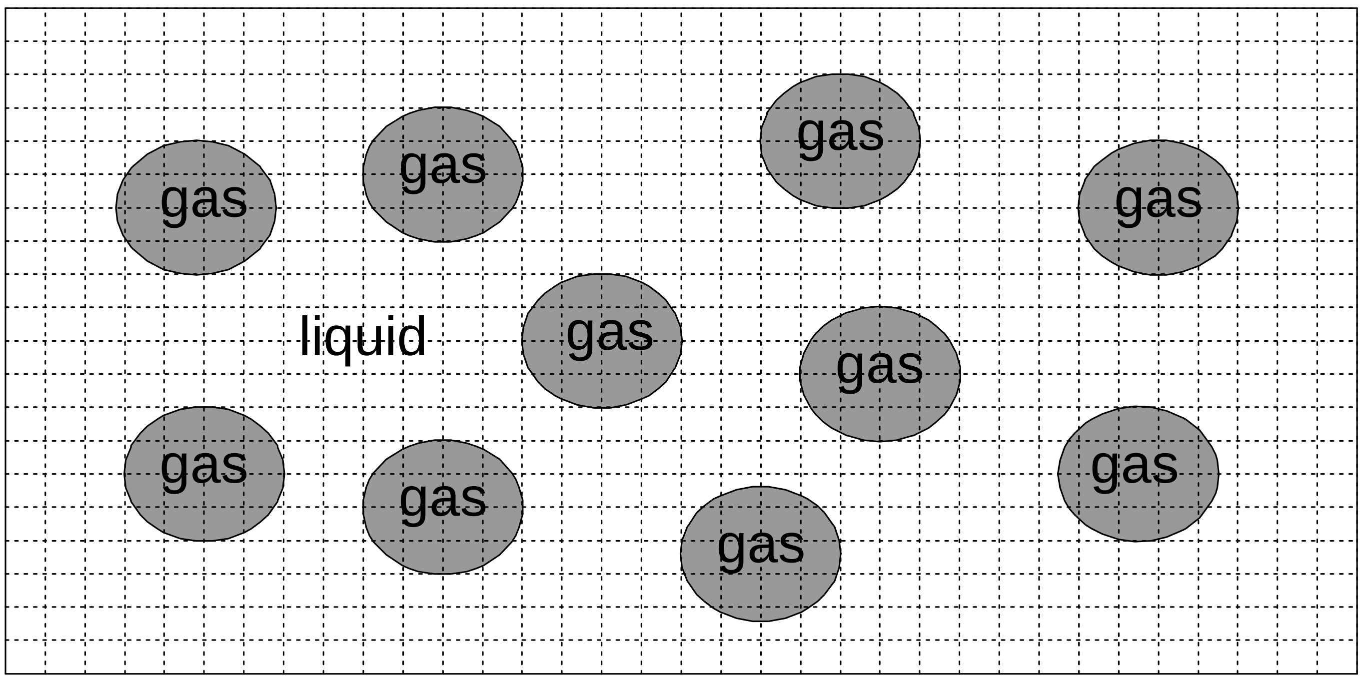
$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + \frac{\partial g(q)}{\partial y} = \psi(q)$$

  - In generalized coordinates : spatially varying fluxes

$$\frac{\partial q}{\partial \tau} + \frac{\partial f(q,\Xi)}{\partial \xi} + \frac{\partial g(q,\Xi)}{\partial \eta} = \psi(q), \qquad \Xi\text{: grid metrics}$$

Extension to Multifluid

- Assume homogeneous (1-pressure & 1-velocity) flow;
  i.e., across interfaces $p_\iota = p$ & $\vec{u}_\iota = \vec{u}$, $\forall$ fluid phase ι



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- Assume homogeneous (1-pressure & 1-velocity) flow; *i.e.*, across interfaces $p_\iota = p$ & $\vec{u}_\iota = \vec{u}$, $\forall$ fluid phase ι
- Mathematical model: Fluid-mixture type
  - Use basic conservation (or balance) laws for single & multicomponent fluid mixtures
  - Introduce additional transport equations for problem-dependent material quantities near numerically diffused interfaces, yielding direct computation of pressure from EOS

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- Sample examples
  - Barotropic 2-phase flow
  - Hybrid barotropic & non-barotropic 2-phase flow

Barotropic 2-Phase Flow

- Equations of state
  - Fluid component 1 & 2: Tait EOS

$$p(\rho) = (p_{0\iota} + \mathcal{B}_\iota)\left(\frac{\rho}{\rho_{0\iota}}\right)^{\gamma_\iota} - \mathcal{B}_\iota, \quad \iota = 1, 2$$

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  - Mixture pressure law (Shyue, JCP 2004)

$$p(\rho, \rho e) = \begin{cases} (p_{0\iota} + \mathcal{B}_\iota)\left(\dfrac{\rho}{\rho_{0\iota}}\right)^{\gamma_\iota} - \mathcal{B}_\iota & \text{if} \quad \alpha = 0 \text{ or } 1 \\ (\gamma - 1)\left(\rho e + \dfrac{\rho\mathcal{B}}{\rho_0}\right) - \gamma\mathcal{B} & \text{if} \quad \alpha \in (0, 1) \end{cases}$$

Here α denotes volume fraction of one chosen fluid component

Barotropic 2-Phase Flow

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  - Fluid component 1 & 2: Tait EOS

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variant form of

$$p(\rho, S) = \mathcal{A}(S)\,(p_0 + \mathcal{B})\left(\frac{\rho}{\rho_0}\right)^{\gamma} - \mathcal{B}$$

$\mathcal{A}(S) = e^{[(S-S_0)/C_V]}$, S, C_V: specific entropy & heat at constant volume

Barotropic 2-Phase Flow (Cont.)

- Transport equations for material quantities γ, $\mathcal{B}$, & ρ_0
  - γ-based equations

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\gamma - 1}\right) + U\frac{\partial}{\partial \xi}\left(\frac{1}{\gamma - 1}\right) + V\frac{\partial}{\partial \eta}\left(\frac{1}{\gamma - 1}\right) = 0$$

$$\frac{\partial}{\partial \tau}\left(\frac{\gamma\mathcal{B}}{\gamma - 1}\right) + U\frac{\partial}{\partial \xi}\left(\frac{\gamma\mathcal{B}}{\gamma - 1}\right) + V\frac{\partial}{\partial \eta}\left(\frac{\gamma\mathcal{B}}{\gamma - 1}\right) = 0$$

$$\frac{\partial}{\partial \tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right) + \frac{\partial}{\partial \xi}\left(J\frac{\mathcal{B}}{\rho_0}\rho U\right) + \frac{\partial}{\partial \eta}\left(J\frac{\mathcal{B}}{\rho_0}\rho V\right) = 0$$

Barotropic 2-Phase Flow (Cont.)

- Transport equations for material quantities γ, $\mathcal{B}$, & ρ_0
  - γ-based equations

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{1}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{1}{\gamma-1}\right)=0$$
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$$\frac{\partial}{\partial \tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right)+\frac{\partial}{\partial \xi}\left(J\frac{\mathcal{B}}{\rho_0}\rho U\right)+\frac{\partial}{\partial \eta}\left(J\frac{\mathcal{B}}{\rho_0}\rho V\right)=0$$

  - Above equations are derived from energy equation & make use of homogeneous equilibrium flow assumption together with mass conservation law

Barotropic 2-Phase Flow (Cont.)

- Transport equations for material quantities γ, $\mathcal{B}$, & ρ_0
  - γ-based equations

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{1}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{1}{\gamma-1}\right)=0$$

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$$\cancel{\frac{\partial}{\partial \tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right)+\frac{\partial}{\partial \xi}\left(J\frac{\mathcal{B}}{\rho_0}\rho U\right)+\frac{\partial}{\partial \eta}\left(J\frac{\mathcal{B}}{\rho_0}\rho V\right)=0}$$

- If we ignore $J\mathcal{B}\rho/\rho_0$ term, they are essentially equations proposed by Saurel & Abgrall (SISC 1999), but are written in generalized coord.

Barotropic 2-Phase Flow (Cont.)

- Transport equations for material quantities γ, $\mathcal{B}$, & ρ_0
  - γ-based equations

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{1}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{1}{\gamma-1}\right)=0$$

$$\frac{\partial}{\partial \tau}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)=0$$

$$\frac{\partial}{\partial \tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right)+\frac{\partial}{\partial \xi}\left(J\frac{\mathcal{B}}{\rho_0}\rho U\right)+\frac{\partial}{\partial \eta}\left(J\frac{\mathcal{B}}{\rho_0}\rho V\right)=0$$

  - α-based equations

$$\frac{\partial \alpha}{\partial \tau}+U\frac{\partial \alpha}{\partial \xi}+V\frac{\partial \alpha}{\partial \eta}=0, \qquad \text{with} \quad z=\sum_{\iota=1}^{2}\alpha_\iota z_\iota, \quad z=\frac{1}{\gamma-1}\&\frac{\gamma\mathcal{B}}{\gamma-1}$$

$$\frac{\partial}{\partial \tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right)+\frac{\partial}{\partial \xi}\left(J\frac{\mathcal{B}}{\rho_0}\rho U\right)+\frac{\partial}{\partial \eta}\left(J\frac{\mathcal{B}}{\rho_0}\rho V\right)=0$$

Barotropic 2-Phase Flow (Cont.)

- Transport equations for material quantities γ, $\mathcal{B}$, & ρ_0
  - γ-based equations

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{1}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{1}{\gamma-1}\right)=0$$

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$$\frac{\partial}{\partial \tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right)+\frac{\partial}{\partial \xi}\left(J\frac{\mathcal{B}}{\rho_0}\rho U\right)+\frac{\partial}{\partial \eta}\left(J\frac{\mathcal{B}}{\rho_0}\rho V\right)=0$$

  - α-based equations (Allaire *et al.* , JCP 2002)

$$\frac{\partial \alpha}{\partial \tau}+U\frac{\partial \alpha}{\partial \xi}+V\frac{\partial \alpha}{\partial \eta}=0 \qquad \text{with} \quad z=\sum_{\iota=1}^{2}\alpha_\iota z_\iota, \quad z=\frac{1}{\gamma-1}\,\&\,\frac{\gamma\mathcal{B}}{\gamma-1}$$

$$\frac{\partial}{\partial \tau}\left(J\rho_1\alpha\right)+\frac{\partial}{\partial \xi}\left(J\rho_1\alpha U\right)+\frac{\partial}{\partial \eta}\left(J\rho_1\alpha V\right)=0 \qquad \text{with} \quad z=\frac{\mathcal{B}}{\rho_0}\rho$$

Barotropic 2-Phase Flow (Cont.)

- Transport equations for material quantities γ, $\mathcal{B}$, & ρ_0
  - γ-based equations

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{1}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{1}{\gamma-1}\right)=0$$

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  - α-based equations (Kapila *et al.* , Phys. Fluid 2001)

$$\frac{\partial \alpha}{\partial \tau}+U\frac{\partial \alpha}{\partial \xi}+V\frac{\partial \alpha}{\partial \eta}=\alpha_1\alpha_2\left(\frac{\rho_1 c_1^2-\rho_2 c_2^2}{\sum_{k=1}^{2}\alpha_k\rho_k c_k^2}\right)\nabla\cdot\vec{u}$$

$\cdots$ will not be discussed here

Barotropic & Non-Barotropic Flow

- Equations of state
  - Fluid component 1: Tait EOS

$$p(\rho) = (p_0 + \mathcal{B}) \left(\frac{\rho}{\rho_0} \right)^{\gamma} - \mathcal{B}$$

  - Fluid component 2: Noble-Abel EOS

$$p(\rho, \rho e) = \left(\frac{\gamma - 1}{1 - b\rho} \right) \rho e \qquad (0 \le b \le 1/\rho)$$

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  - Mixture pressure law (Shyue, Shock Waves 2006)

$$p(\rho, \rho e) = \begin{cases} (p_0 + \mathcal{B}) \left(\frac{\rho}{\rho_0}\right)^{\gamma} - \mathcal{B} & \text{if} \quad \alpha = 1 \qquad \text{(fluid 1)} \\ \left(\frac{\gamma - 1}{1 - b\rho}\right) (\rho e - \mathcal{B}) - \mathcal{B} & \text{if} \quad \alpha \ne 1 \end{cases}$$

Baro. & Non-Baro. Flow (Cont.)

- Equations of state
  - Fluid component 1: Tait EOS

$$p(V) = \mathcal{A}(S_0)\,(p_0 + \mathcal{B})\left(\frac{V_0}{V}\right)^{\gamma} - \mathcal{B}, \qquad V = 1/\rho$$

  - Fluid component 2: Noble-Abel EOS

$$p(V,S) = \mathcal{A}(S)p_0\left(\frac{V_0 - b}{V - b}\right)^{\gamma}$$

  - Mixture pressure law

$$p(V,S) = \mathcal{A}(S)\,(p_0 + \mathcal{B})\left(\frac{V_0 - b}{V - b}\right)^{\gamma} - \mathcal{B}$$

Baro. & Non-Baro. Flow (Cont.)

- Equations of state
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$$p(V,S) = \mathcal{A}(S)\,(p_0 + \mathcal{B})\left(\frac{V_0 - b}{V - b}\right)^{\gamma} - \mathcal{B}$$

variant form of $$p(\rho,\rho e) = \left(\frac{\gamma - 1}{1 - b\rho}\right)(\rho e - \mathcal{B}) - \mathcal{B}$$

Baro. & Non-Baro. Flow (Cont.)

- Transport equations for material quantities γ, b, & $\mathcal{B}$
  - α-based equations

$$\frac{\partial \alpha}{\partial \tau} + U \frac{\partial \alpha}{\partial \xi} + V \frac{\partial \alpha}{\partial \eta} = 0$$

$$\frac{\partial}{\partial \tau} \left(J \rho_1 \alpha \right) + \frac{\partial}{\partial \xi} \left(J \rho_1 \alpha U \right) + \frac{\partial}{\partial \eta} \left(J \rho_1 \alpha V \right) = 0$$

with $z = \sum_{\iota=1}^{2} \alpha_\iota z_\iota$, $\quad z = \frac{1}{\gamma-1}$, $\quad \frac{b\rho}{\gamma-1}$, $\quad$ & $\frac{\gamma-b\rho}{\gamma-1}\mathcal{B}$

Baro. & Non-Baro. Flow (Cont.)

- Transport equations for material quantities γ, b, & $\mathcal{B}$
  - α-based equations

$$\frac{\partial \alpha}{\partial \tau} + U\frac{\partial \alpha}{\partial \xi} + V\frac{\partial \alpha}{\partial \eta} = 0$$

$$\frac{\partial}{\partial \tau}(J\rho_1\alpha) + \frac{\partial}{\partial \xi}(J\rho_1\alpha U) + \frac{\partial}{\partial \eta}(J\rho_1\alpha V) = 0$$

with $\boxed{z = \sum_{\iota=1}^{2} \alpha_\iota z_\iota, \quad z = \frac{1}{\gamma-1}, \quad \frac{b\rho}{\gamma-1}, \quad \& \ \frac{\gamma-b\rho}{\gamma-1}\mathcal{B}}$

Note:

$$\frac{1-b\rho}{\gamma-1}\ p + \frac{\gamma-b\rho}{\gamma-1}\mathcal{B} = \rho e = \sum_{\iota=1}^{2}\alpha_\iota\rho_\iota e_\iota$$

$$= \sum_{\iota=1}^{2}\alpha_\iota\left(\frac{1-b_\iota\rho_\iota}{\gamma_\iota-1}\ p_\iota + \frac{\gamma_\iota-b_\iota\rho_\iota}{\gamma_\iota-1}\mathcal{B}_\iota\right)$$

Baro. & Non-Baro. Flow (Cont.)

- Transport equations for material quantities γ, b, & $\mathcal{B}$
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Note: $$\frac{1-b\rho}{\gamma-1} p + \frac{\gamma-b\rho}{\gamma-1}\mathcal{B} = \rho e = \sum_{\iota=1}^{2} \alpha_\iota \rho_\iota e_\iota$$

$$= \sum_{\iota=1}^{2} \alpha_\iota \left(\frac{1-b_\iota\rho_\iota}{\gamma_\iota-1} p_\iota + \frac{\gamma_\iota-b_\iota\rho_\iota}{\gamma_\iota-1}\mathcal{B}_\iota \right)$$

Multifluid Model

With $(x_\tau, y_\tau) = h_0(u, v)$ & sample EOS described above, our α-based model for multifluid flow is

$$\frac{\partial}{\partial \tau}\begin{pmatrix} J\rho \\ J\rho u \\ J\rho v \\ JE \\ x_\xi \\ y_\xi \\ x_\eta \\ y_\eta \\ J\rho_1\alpha \end{pmatrix} + \frac{\partial}{\partial \xi}\begin{pmatrix} J\rho U \\ J\rho u U + y_\eta p \\ J\rho v U - x_\eta p \\ JEU + (y_\eta u - x_\eta v)p \\ -h_0 u \\ -h_0 v \\ 0 \\ 0 \\ J\rho_1 \alpha U \end{pmatrix} + \frac{\partial}{\partial \eta}\begin{pmatrix} J\rho V \\ J\rho u V - y_\xi p \\ J\rho v V + x_\xi p \\ JEV + (x_\xi v - y_\xi u)p \\ 0 \\ 0 \\ -h_0 u \\ -h_0 v \\ J\rho_1 \alpha V \end{pmatrix} = \tilde{\psi}$$

$$\frac{\partial \alpha}{\partial \tau} + U\frac{\partial \alpha}{\partial \xi} + V\frac{\partial \alpha}{\partial \eta} = 0,$$

plus α-averaged material quantities

Multifluid Model (Cont.)

For convenience, our multifluid model is written into

$$\frac{\partial q}{\partial \tau} + f\left(\frac{\partial}{\partial \xi}, q, \Xi\right) + g\left(\frac{\partial}{\partial \eta}, q, \Xi\right) = \tilde{\psi}$$

with

$$q = [J\rho, J\rho u, J\rho v, JE, x_\xi, y_\xi, x_\eta, y_\eta, J\rho_1 \alpha, \alpha]^T$$

$$f = \left[\frac{\partial}{\partial \xi}(J\rho U), \frac{\partial}{\partial \xi}(J\rho u U + y_\eta p), \frac{\partial}{\partial \xi}(J\rho v U - x_\eta p), \frac{\partial}{\partial \xi}(JEU + (y_\eta u - x_\eta v)p),\right.$$
$$\left.\frac{\partial}{\partial \xi}(-h_0 u), \frac{\partial}{\partial \xi}(-h_0 v), 0, 0, \frac{\partial}{\partial \xi}(J\rho_1 \alpha U), U\frac{\partial \alpha}{\partial \xi}\right]^T$$

$$g = \left[\frac{\partial}{\partial \eta}(J\rho V), \frac{\partial}{\partial \eta}(J\rho u V - y_\xi p), \frac{\partial}{\partial \eta}(J\rho v V + x_\xi p), \frac{\partial}{\partial \eta}(JEV + (x_\xi v - y_\xi u)p),\right.$$
$$\left.0, 0, \frac{\partial}{\partial \eta}(-h_0 u), \frac{\partial}{\partial \eta}(-h_0 v), \frac{\partial}{\partial \eta}(J\rho_1 \alpha V), V\frac{\partial \alpha}{\partial \eta}\right]^T$$

Multifluid model: Remarks

- As before, under thermodyn. stability condition, our multifluid model in generalized coordinates is hyperbolic when $h_0 \neq 1$, & is weakly hyperbolic when $h_0 = 1$
- Our model system is written in quasi-conservative form with spatially varying fluxes in generalized coordinates
- Our grid system is a time-varying grid
- Extension of the model to general non-barotropic multifluid flow can be made in an analogous manner

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Numerical approximation ?

Numerical Approximation

- Equations to be solved are

$$\frac{\partial q}{\partial \tau} + f\left(\frac{\partial}{\partial \xi}, q, \Xi\right) + g\left(\frac{\partial}{\partial \eta}, q, \Xi\right) = \tilde{\psi}$$

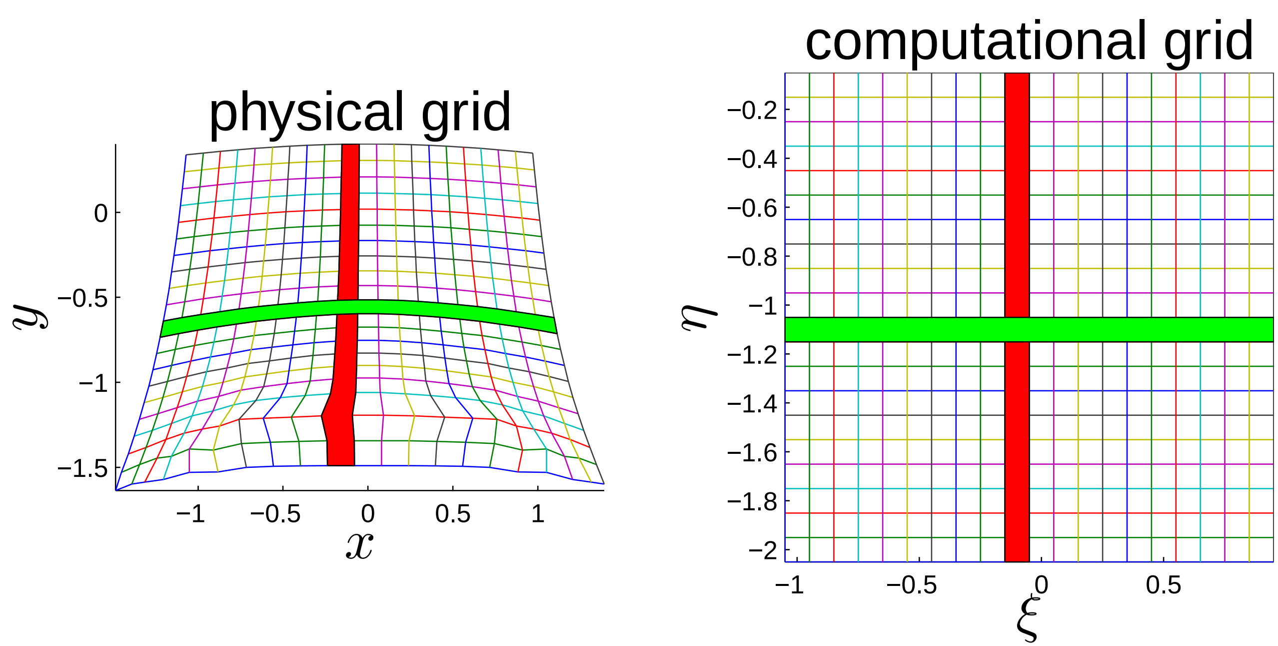
- A simple dimensional-splitting approach based on f-wave formulation of LeVeque *et al.* is used
  - Solve one-dimensional generalized Riemann problem (defined below) at each cell interfaces
  - Use resulting jumps of fluxes (decomposed into each wave family) of Riemann solution to update cell averages
  - Introduce limited jumps of fluxes to achieve high resolution

Numerical Approximation (Cont.)

Employ finite volume formulation of numerical solution

$$Q_{ij}^n \approx \frac{1}{\Delta\xi\Delta\eta}\int_{C_{ij}} q(\xi,\eta,\tau_n)\,dA$$

that gives approximate value of cell average of solution q over cell $C_{ij} = [\xi_i, \xi_{i+1}] \times [\eta_j, \eta_{j+1}]$ at time τ_n



Generalized Riemann Problem

Generalized Riemann problem of our multifluid model at cell interface $\xi_{i-1/2}$ consists of the equation

$$\frac{\partial q}{\partial \tau} + F_{i-\frac{1}{2},j}\left(\partial_\xi, q, \Xi\right) = 0$$

together with flux function

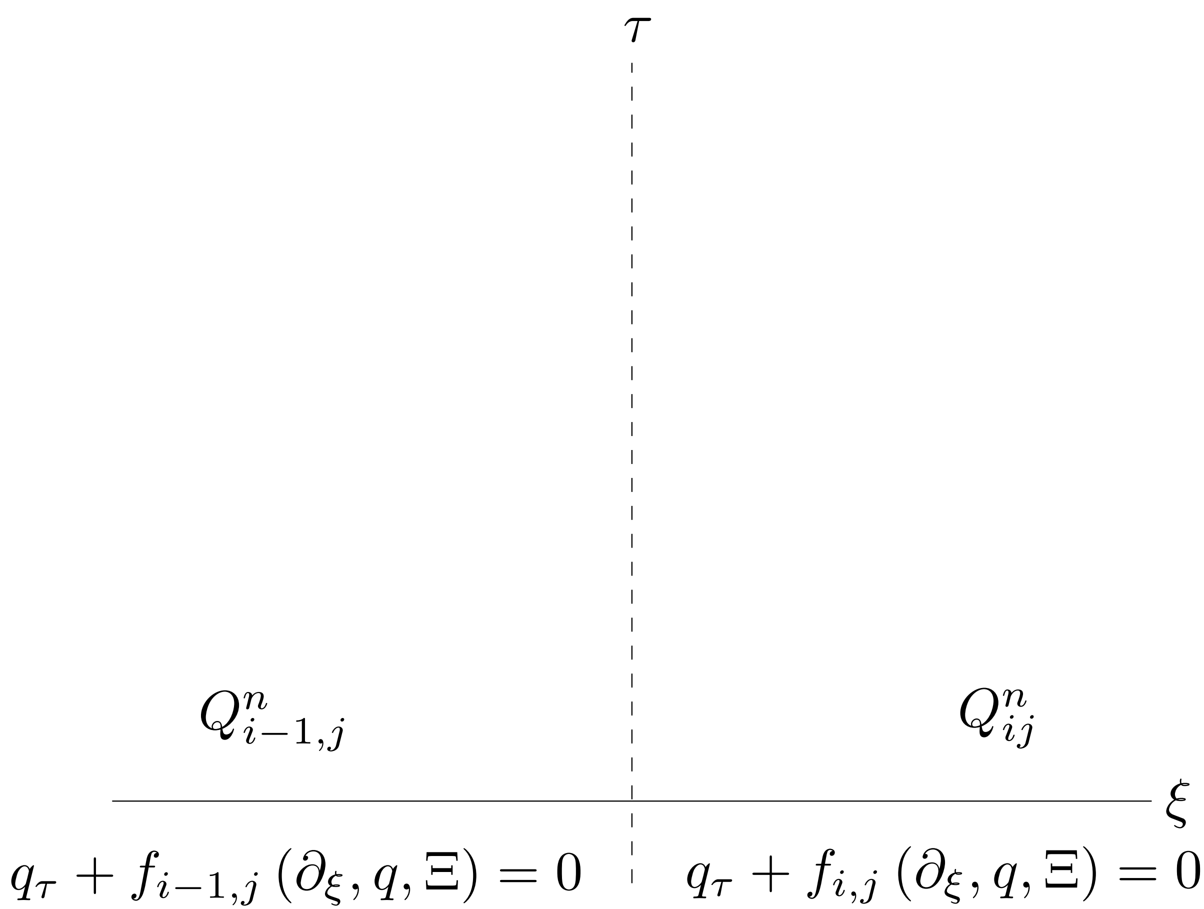
$$F_{i-\frac{1}{2},j} = \begin{cases} f_{i-1,j}\left(\partial_\xi, q, \Xi\right) & \text{for} \quad \xi < \xi_{i-1/2} \\ f_{ij}\left(\partial_\xi, q, \Xi\right) & \text{for} \quad \xi > \xi_{i-1/2} \end{cases}$$

and piecewise constant initial data

$$q(\xi, 0) = \begin{cases} Q^n_{i-1,j} & \text{for} \quad \xi < \xi_{i-1/2} \\ Q^n_{ij} & \text{for} \quad \xi > \xi_{i-1/2} \end{cases}$$

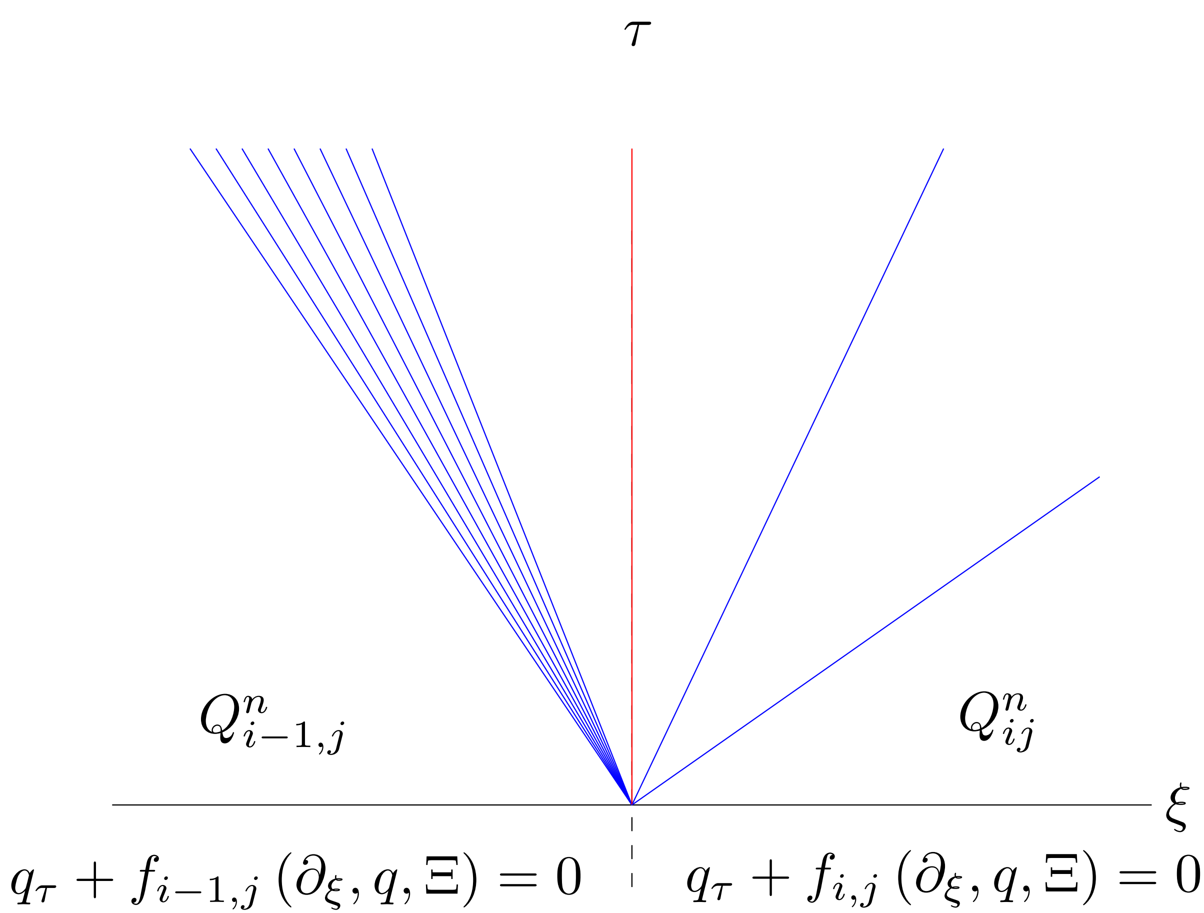
General. Riemann Problem (Cont.)

Generalized Riemann problem at time $\tau = 0$



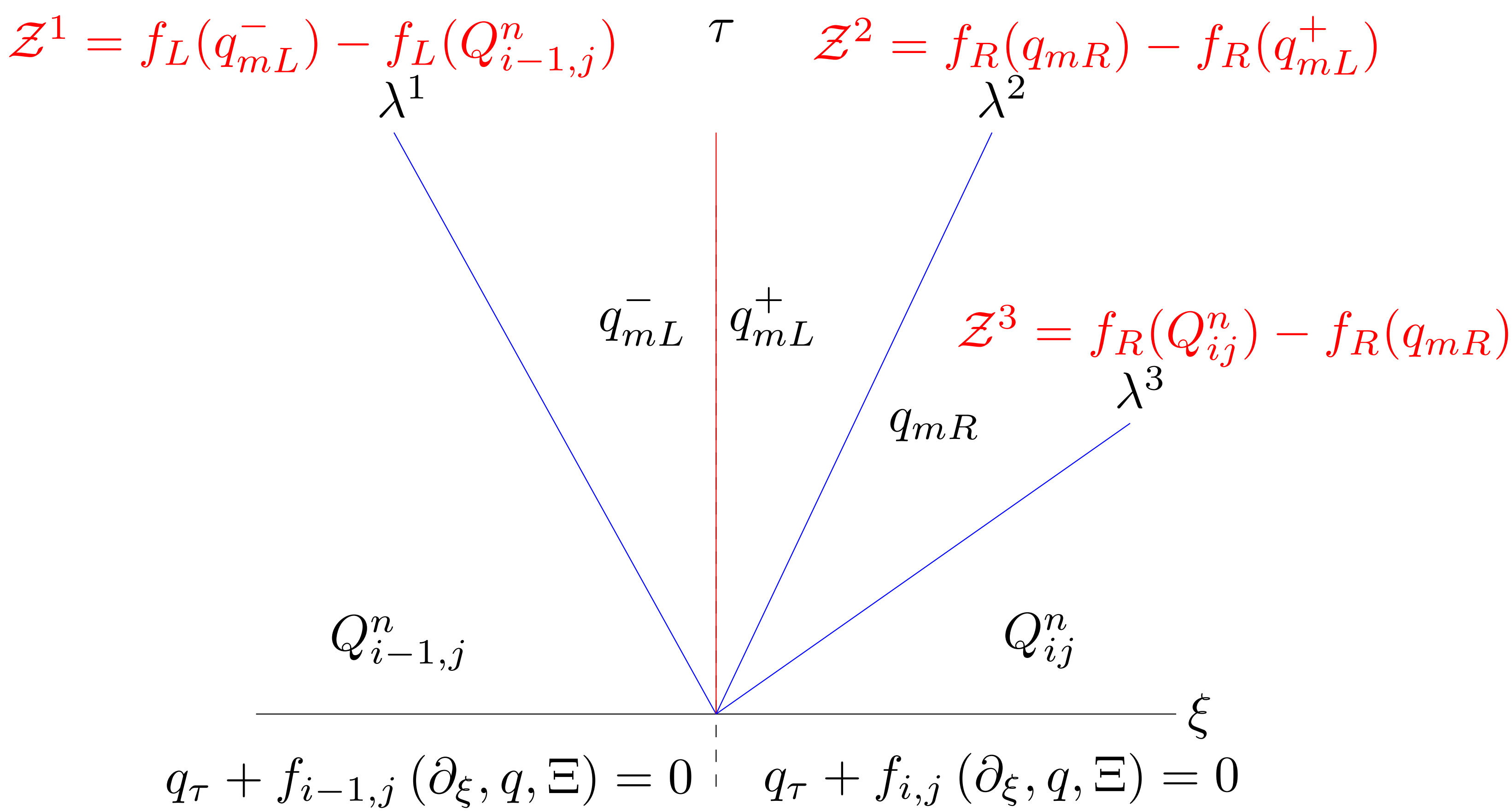
General. Riemann Problem (Cont.)

Exact generalized Riemann solution: basic structure



General. Riemann Problem (Cont.)

Shock-only approximate Riemann solution: basic structure



Numerical Approximation (Cont.)

Basic steps of a dimensional-splitting scheme

- ξ-sweeps: solve

$$\frac{\partial q}{\partial \tau} + f\left(\frac{\partial}{\partial \xi}, q, \Xi\right) = 0$$

updating Q_{ij}^{n} to $Q_{i,j}^{*}$

- η-sweeps: solve

$$\frac{\partial q}{\partial \tau} + g\left(\frac{\partial}{\partial \eta}, q, \Xi\right) = 0$$

updating Q_{ij}^{*} to $Q_{i,j}^{n+1}$

Numerical Approximation (Cont.)

That is to say,

- ξ-sweeps: we use

$$Q_{ij}^{*} = Q_{ij}^{n} - \frac{\Delta\tau}{\Delta\xi}\left(\mathcal{F}^{-}_{i+\frac{1}{2},j} - \mathcal{F}^{+}_{i-\frac{1}{2},j}\right) - \frac{\Delta\tau}{\Delta\xi}\left(\tilde{\mathcal{Z}}_{i+\frac{1}{2},j} - \tilde{\mathcal{Z}}_{i-\frac{1}{2},j}\right)$$

with $\displaystyle \tilde{\mathcal{Z}}_{i-\frac{1}{2},j} = \frac{1}{2}\sum_{p=1}^{m_w} \text{sign}\left(\lambda^{p}_{i-\frac{1}{2},j}\right)\left(1 - \frac{\Delta\tau}{\Delta\xi}\left|\lambda^{p}_{i-\frac{1}{2},j}\right|\right)\tilde{\mathcal{Z}}^{p}_{i-\frac{1}{2},j}$

- η-sweeps: we use

$$Q_{ij}^{n+1} = Q_{ij}^{*} - \frac{\Delta\tau}{\Delta\eta}\left(\mathcal{G}^{-}_{i,j+\frac{1}{2}} - \mathcal{G}^{+}_{i,j-\frac{1}{2}}\right) - \frac{\Delta\tau}{\Delta\eta}\left(\tilde{\mathcal{Z}}_{i,j+\frac{1}{2}} - \tilde{\mathcal{Z}}_{i,j-\frac{1}{2}}\right)$$

with $\displaystyle \tilde{\mathcal{Z}}_{i,j-\frac{1}{2}} = \frac{1}{2}\sum_{p=1}^{m_w} \text{sign}\left(\lambda^{p}_{i,j-\frac{1}{2}}\right)\left(1 - \frac{\Delta\tau}{\Delta\eta}\left|\lambda^{p}_{i,j-\frac{1}{2}}\right|\right)\tilde{\mathcal{Z}}^{p}_{i,j-\frac{1}{2}}$

Numerical Approx.: Remarks

- Flux-based wave decomposition

$$f_{i,j} - f_{i-1,j} = \sum_{p=1}^{m_w} \mathcal{Z}^p_{i-1/2} = \sum_{p=1}^{m_w} \lambda^p_{i-1/2} \mathcal{W}^p_{i-1/2}$$

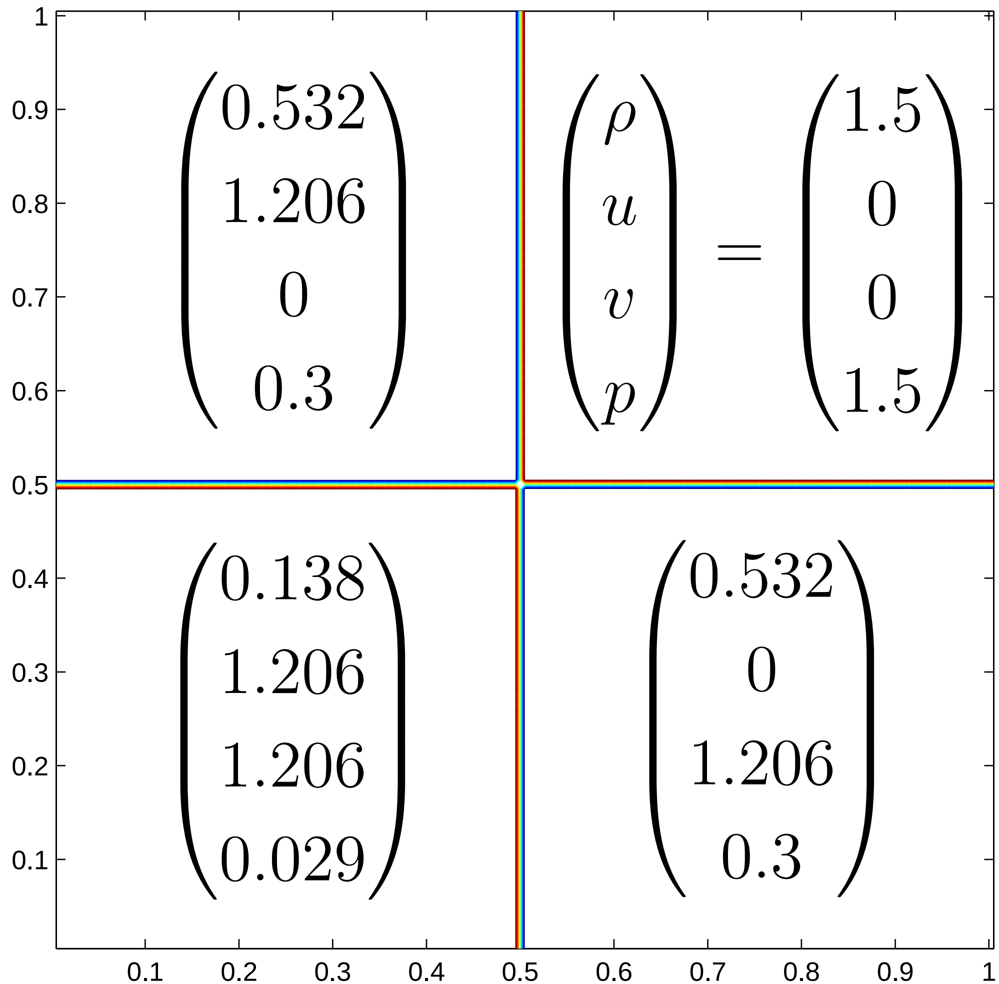
- Some care should be taken on the limited jump of fluxes $\tilde{\mathcal{W}}^p$, for $p = 2$ (contact wave), in particular to ensure correct pressure equilibrium across material interfaces
- MUSCL-type (slope limited) high resolution extension is not simple as one might think of for multifluid problems
- Splitting of discontinuous fluxes at cell interfaces: significance ?
- First order or high resolution method for geometric conservation laws: significance to grid uniformity ?

Numerical Examples: 2D

- 2D Riemann problem
- Underwater explosion
- Shock-bubble interaction
  - Helium bubble case
  - Refrigerant bubble case

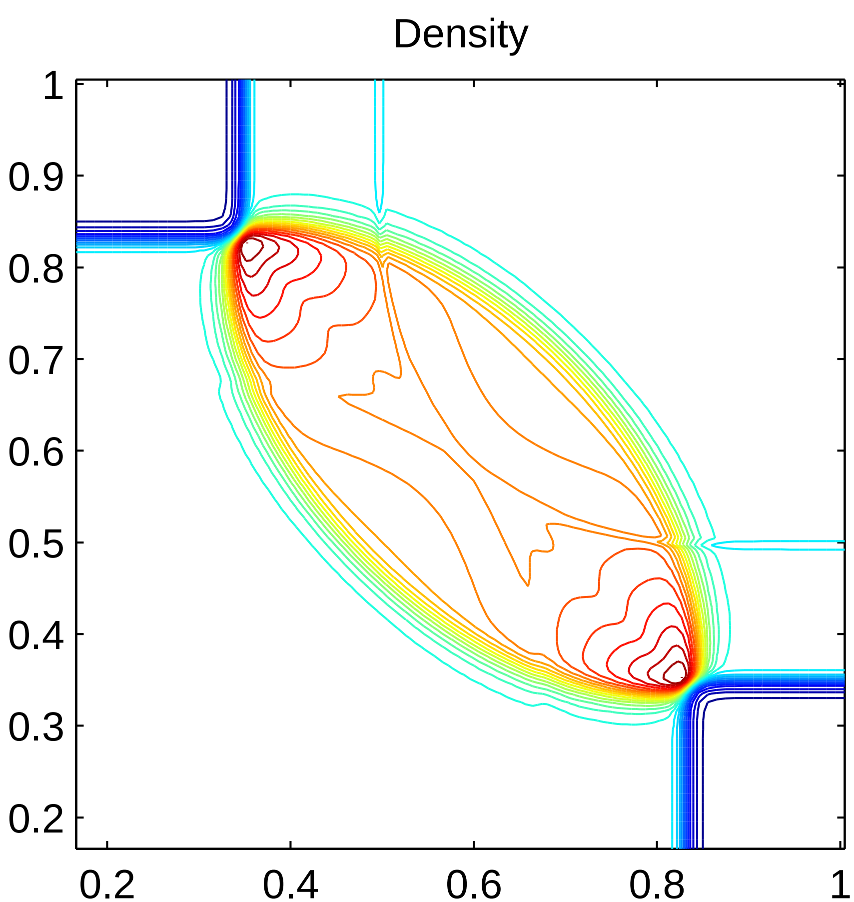
2D Riemann Problem

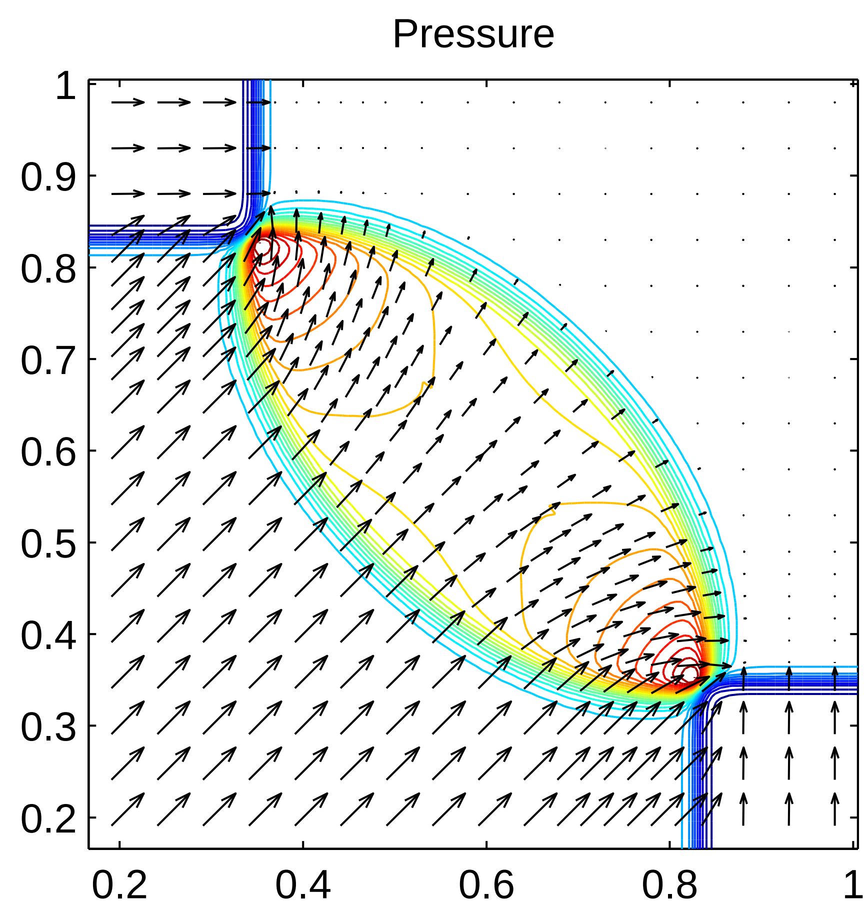
Initial condition for 4-shock wave pattern



2D Riemann problem (Cont.)

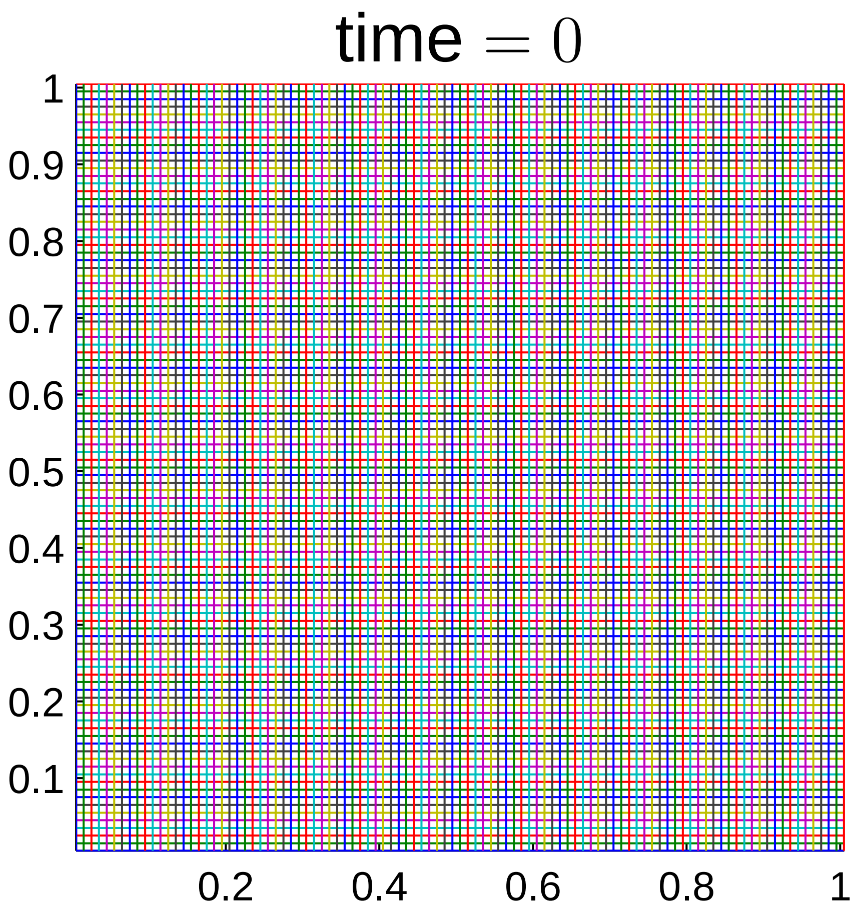
- Numerical contours for density and pressure

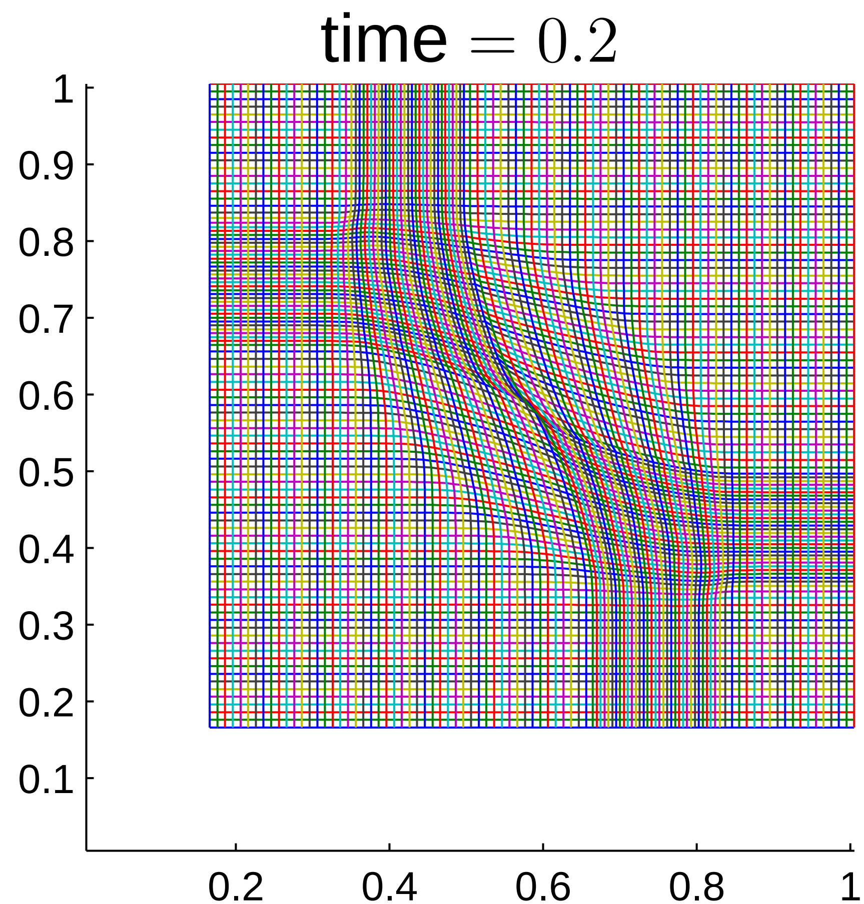




2D Riemann problem (Cont.)

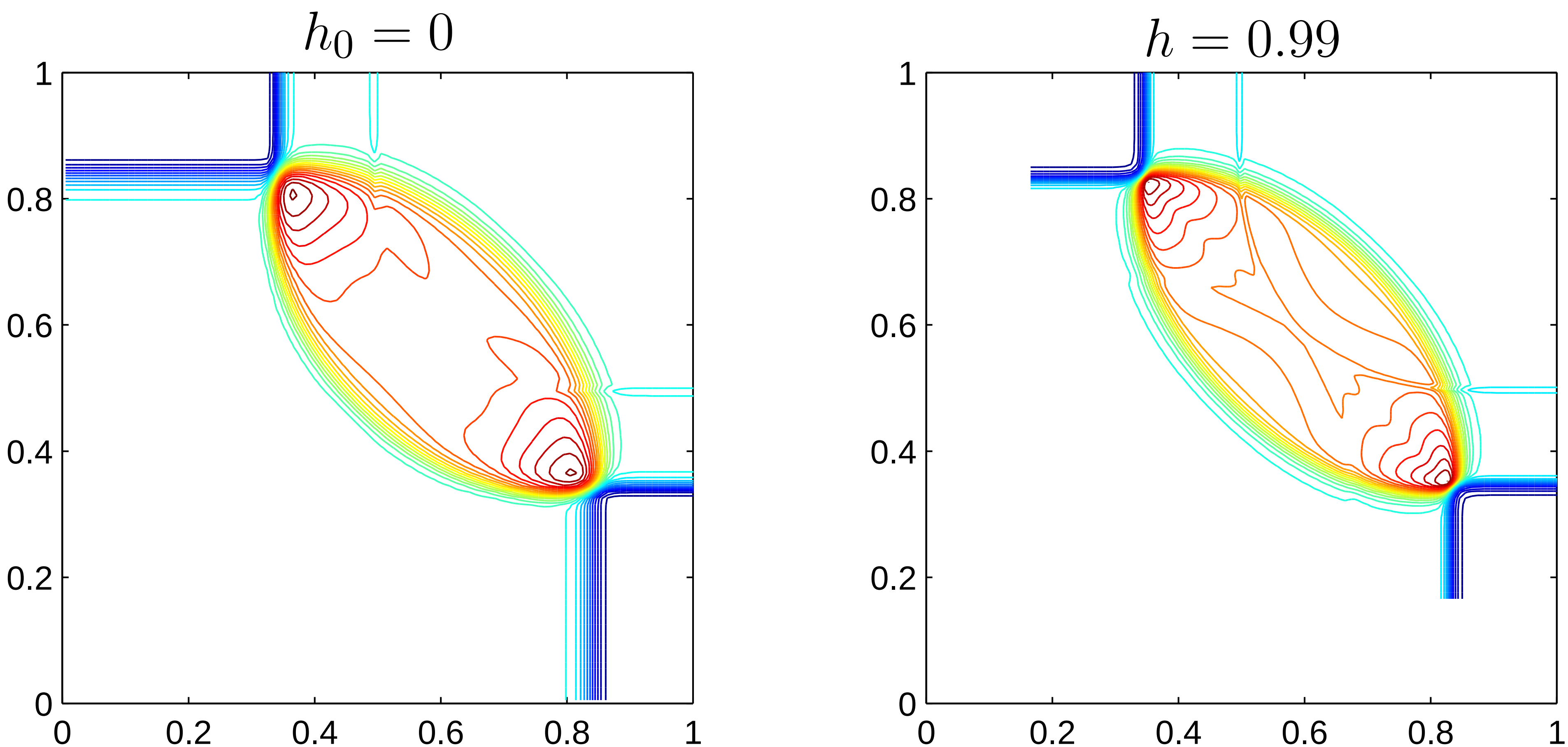
- Grid system with $h_0 = 0.99$





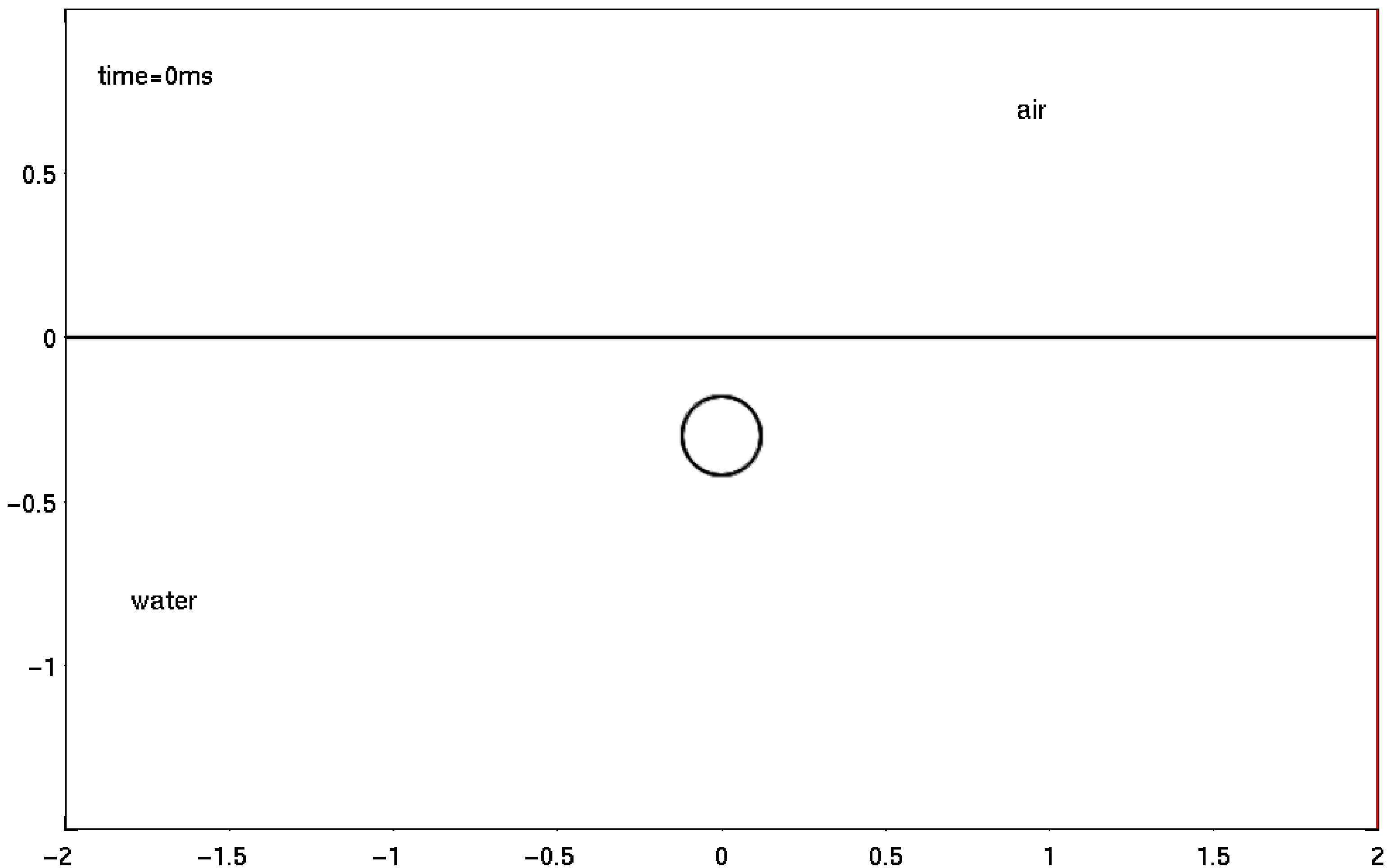
2D Riemann problem (Cont.)

- Eulerian ($h_0 = 0$) vs. generalized Lagrangian ($h_0 = 0.99$)



Underwater Explosions

- Numerical schlieren images $h_0 = 0.9$, 800×500 grid



Underwater Explosions

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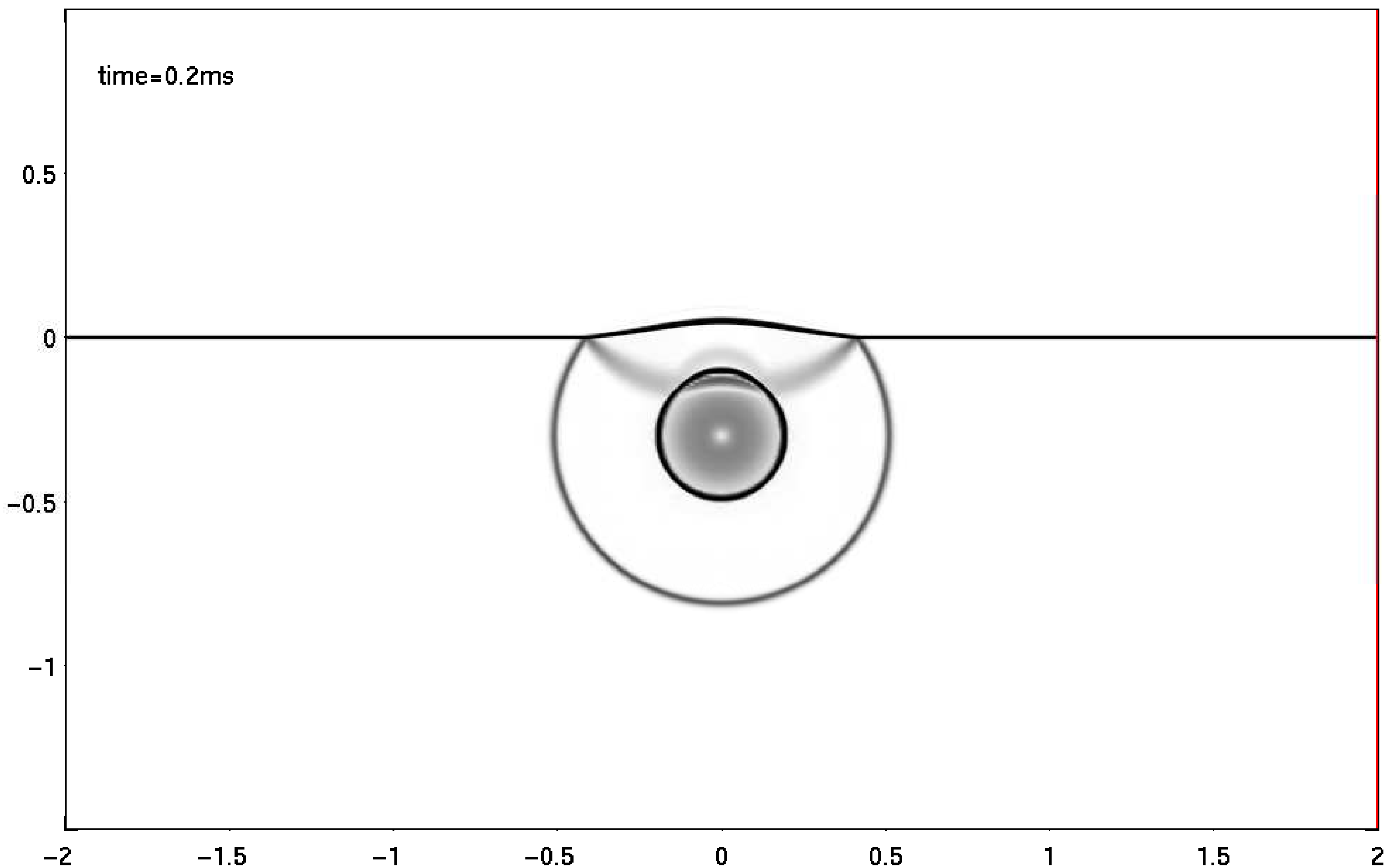

Underwater Explosions

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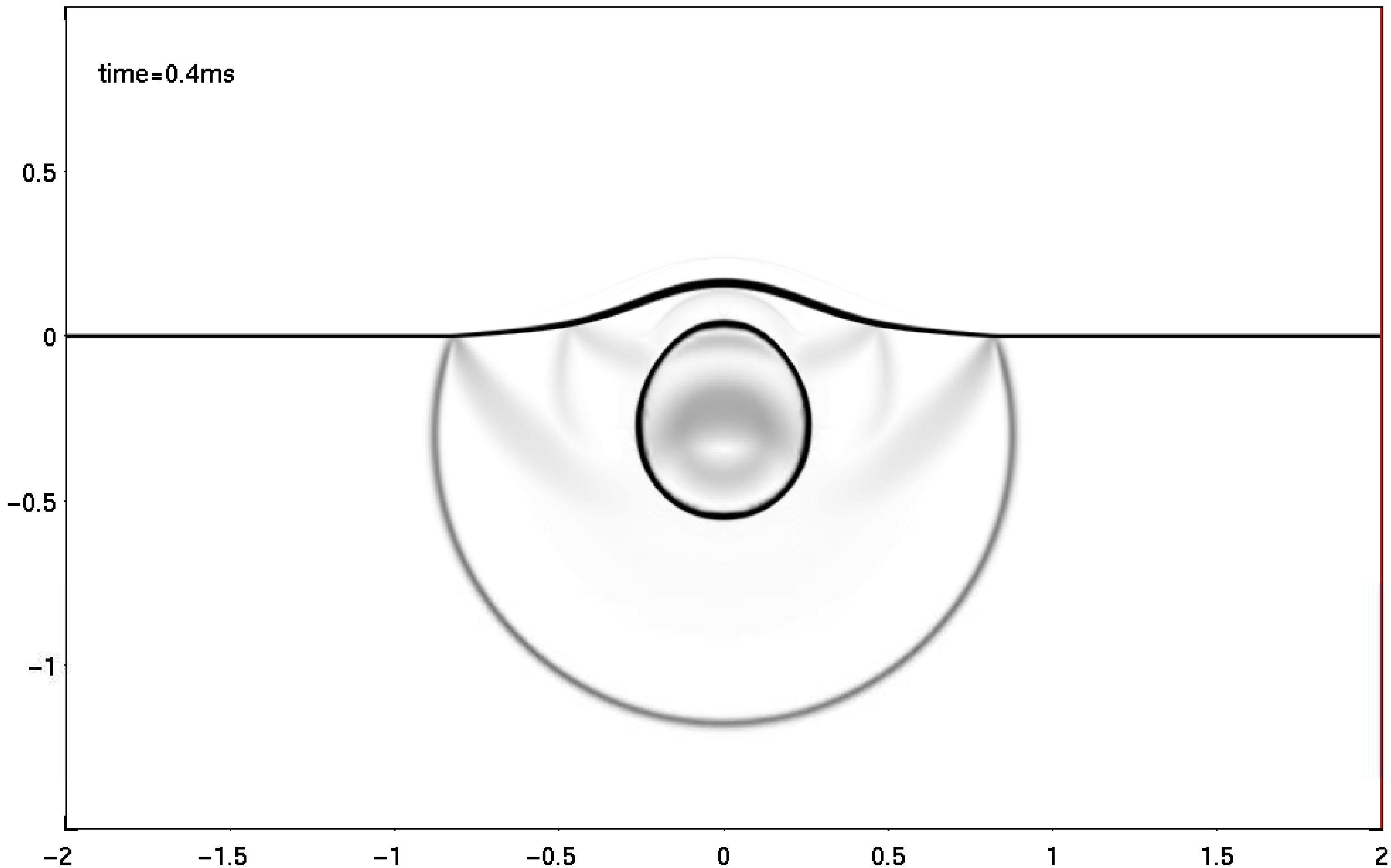
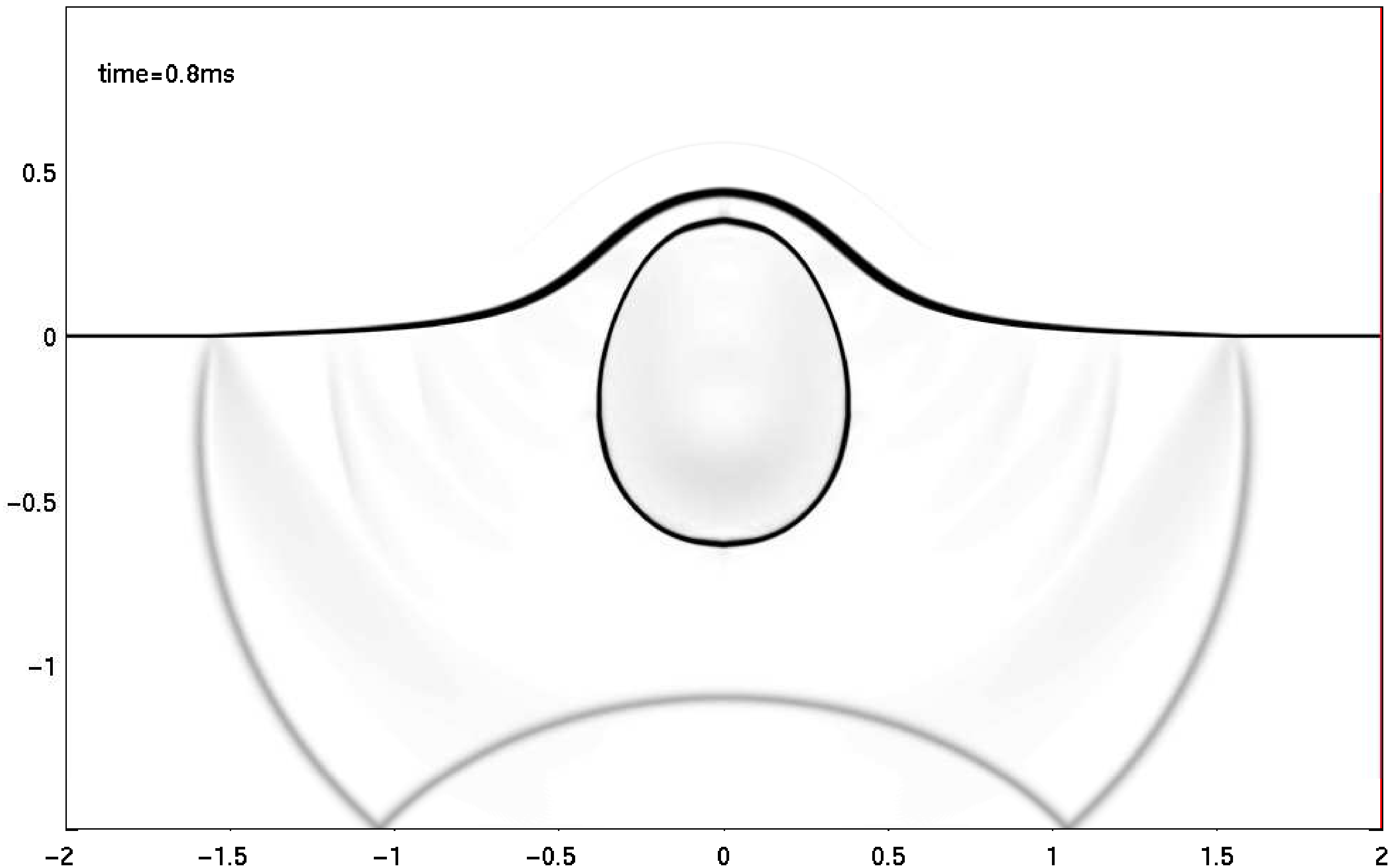


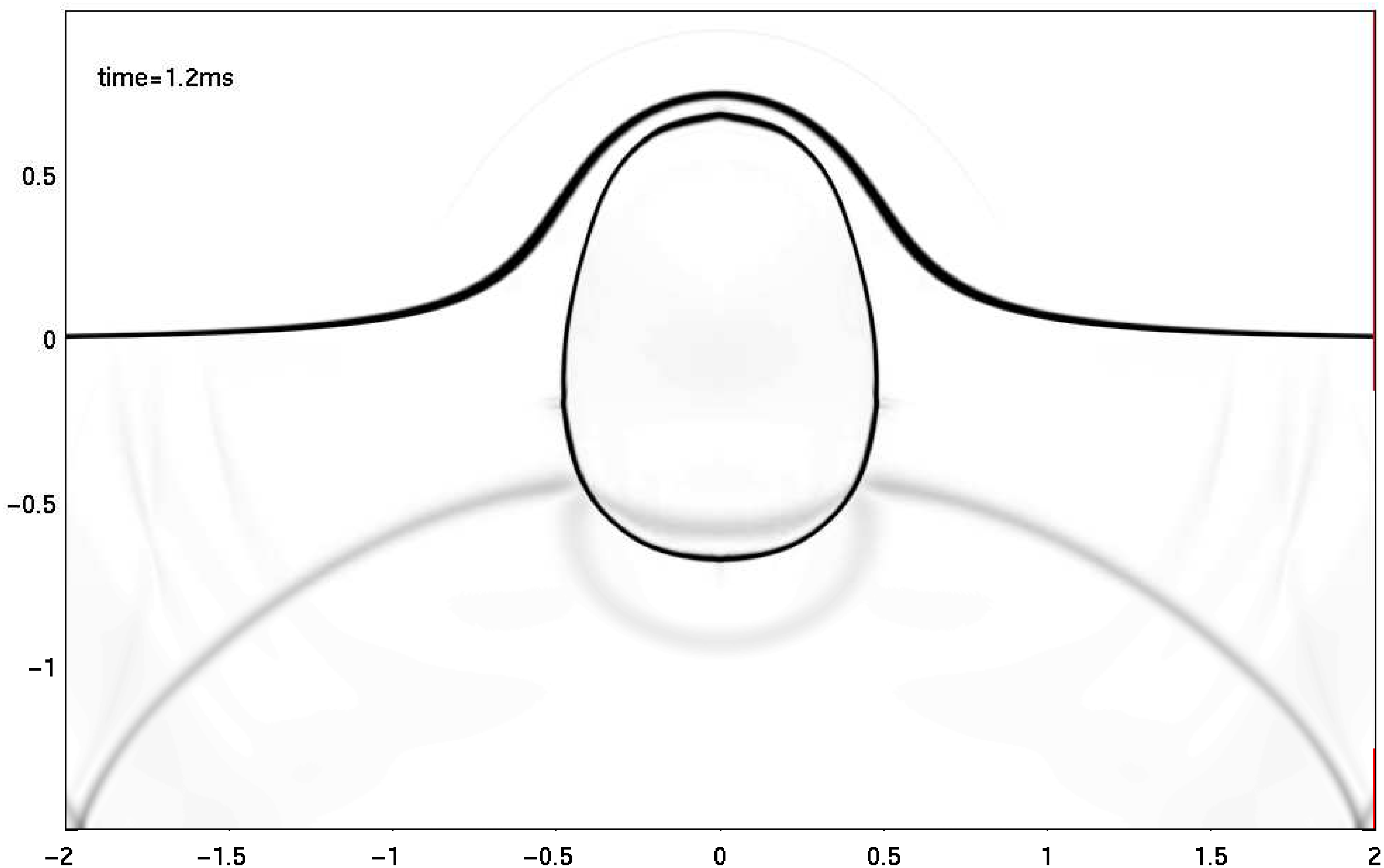
Underwater Explosions

- Numerical schlieren images $h_0 = 0.9$, 800×500 grid



Underwater Explosions

- Numerical schlieren images $h_0 = 0.9$, 800×500 grid

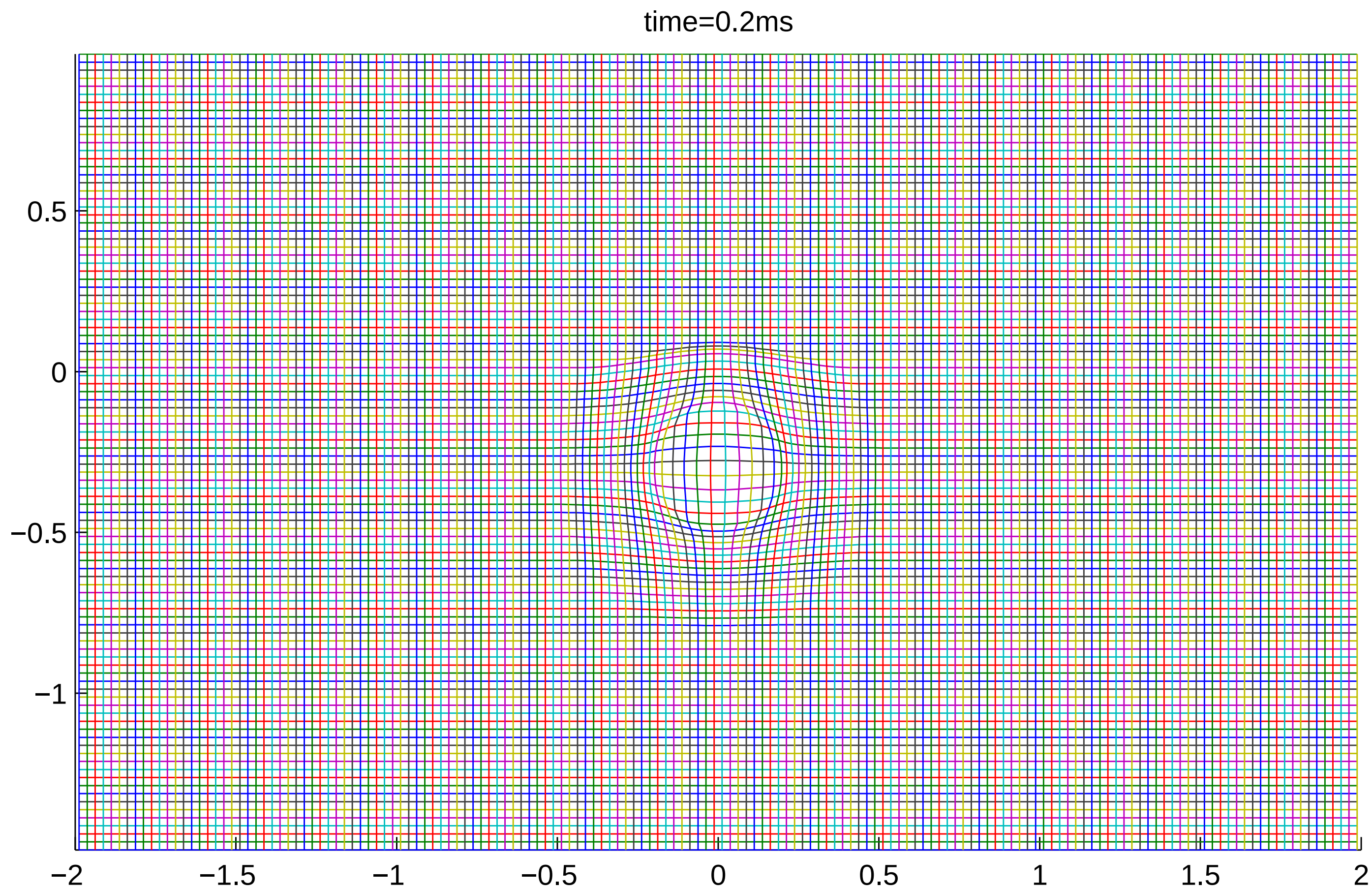


Underwater Explosions (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.9$

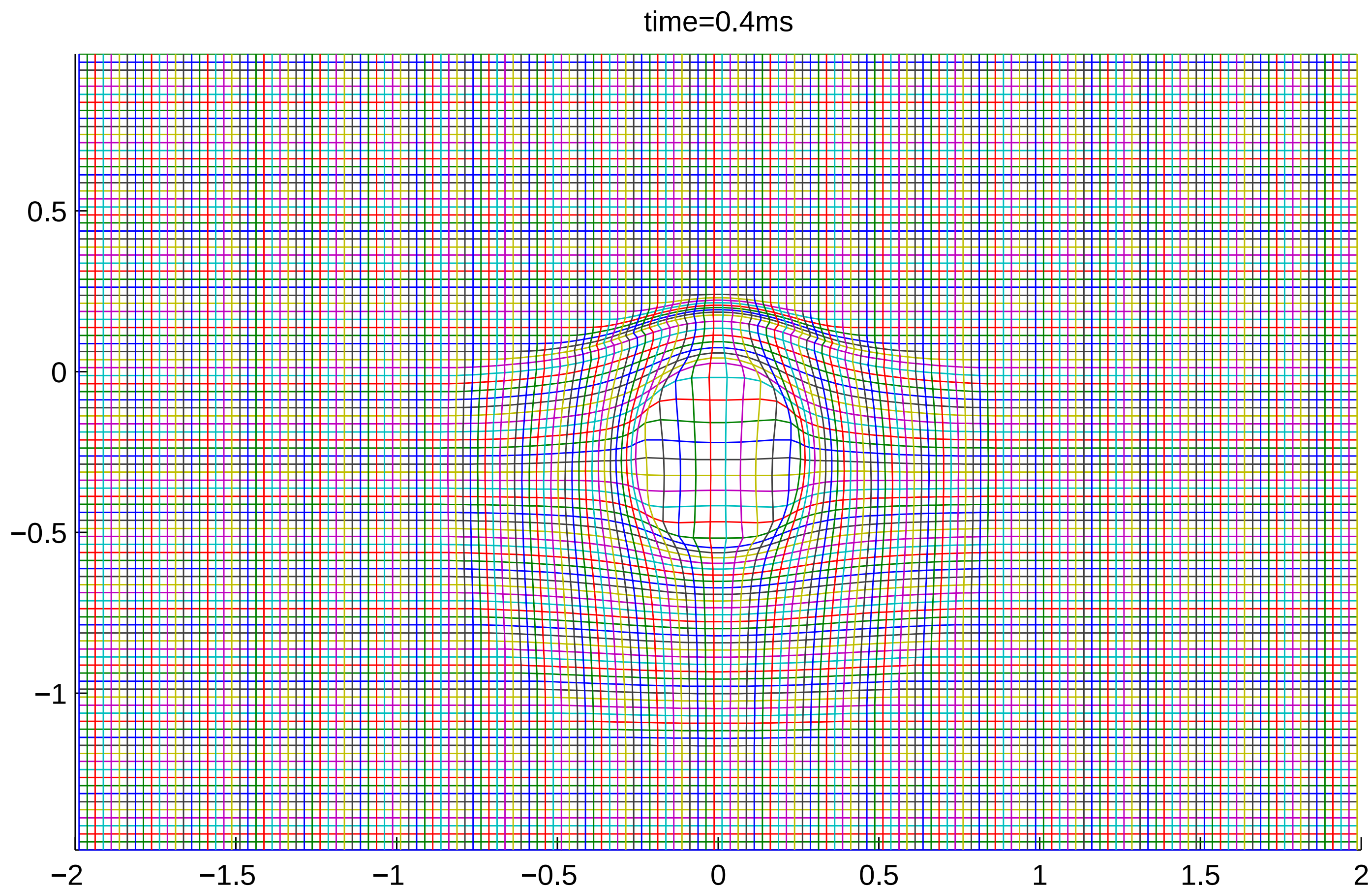
Underwater Explosions (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.9$



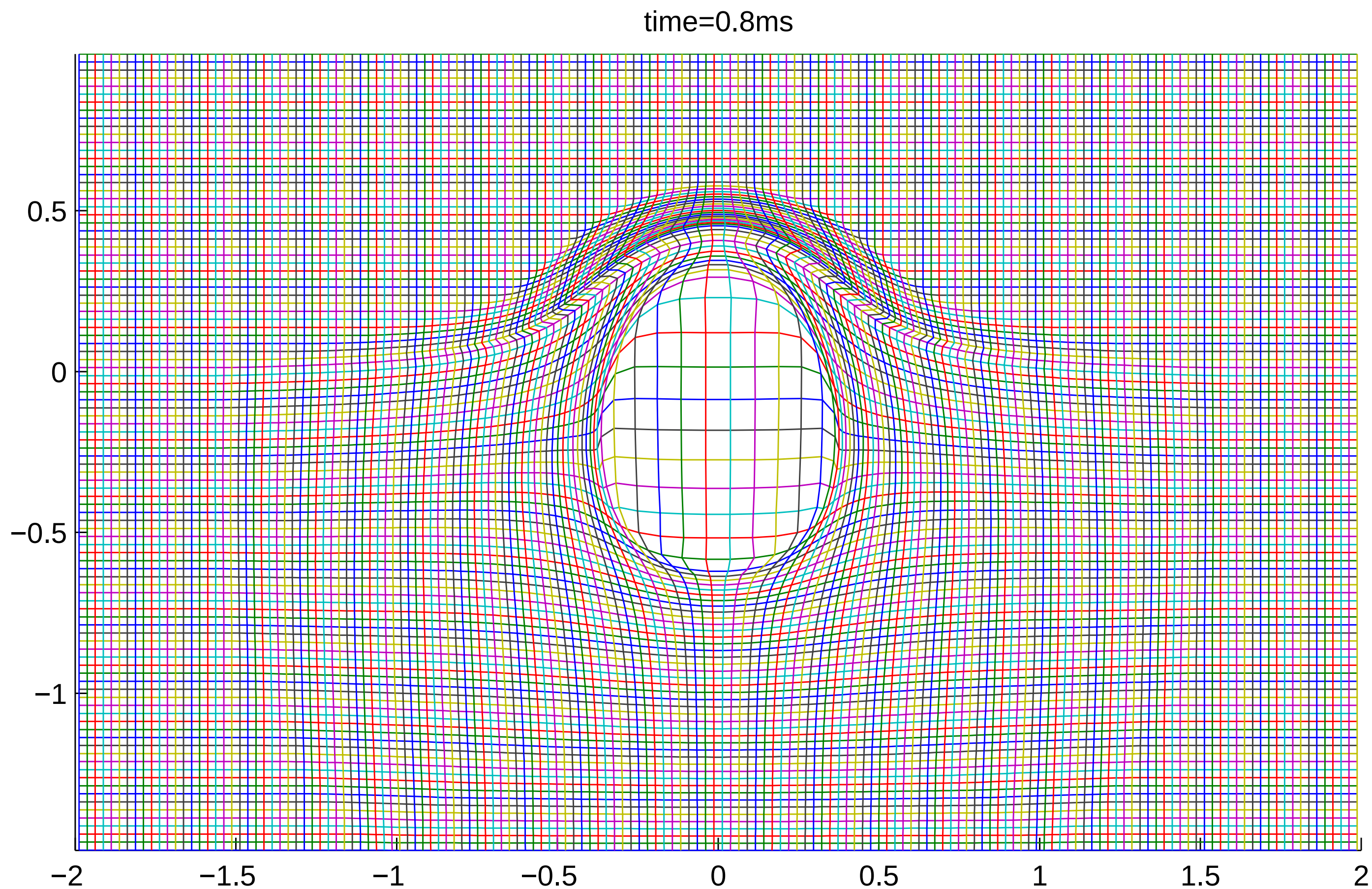
Underwater Explosions (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.9$



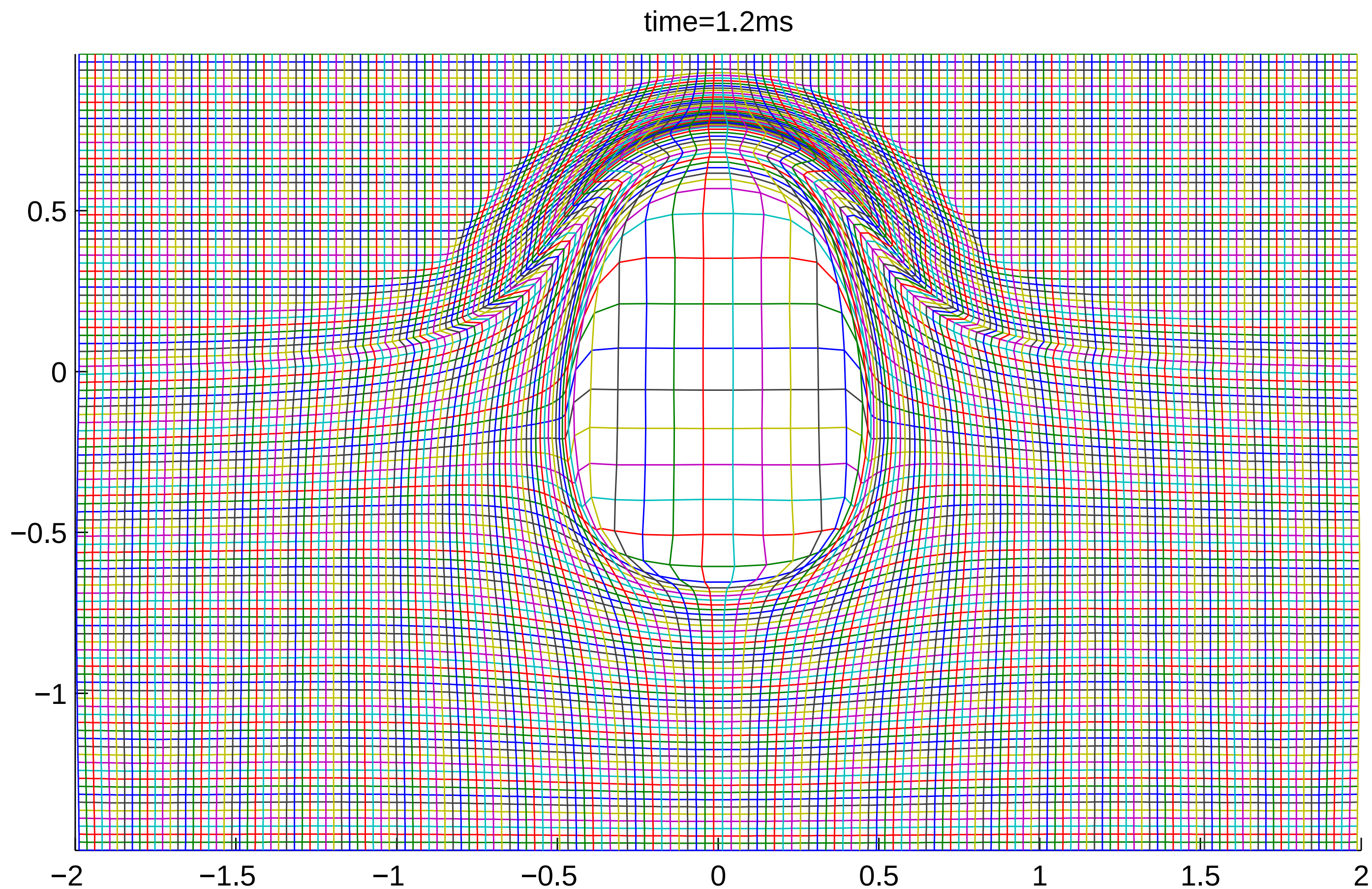
Underwater Explosions (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.9$



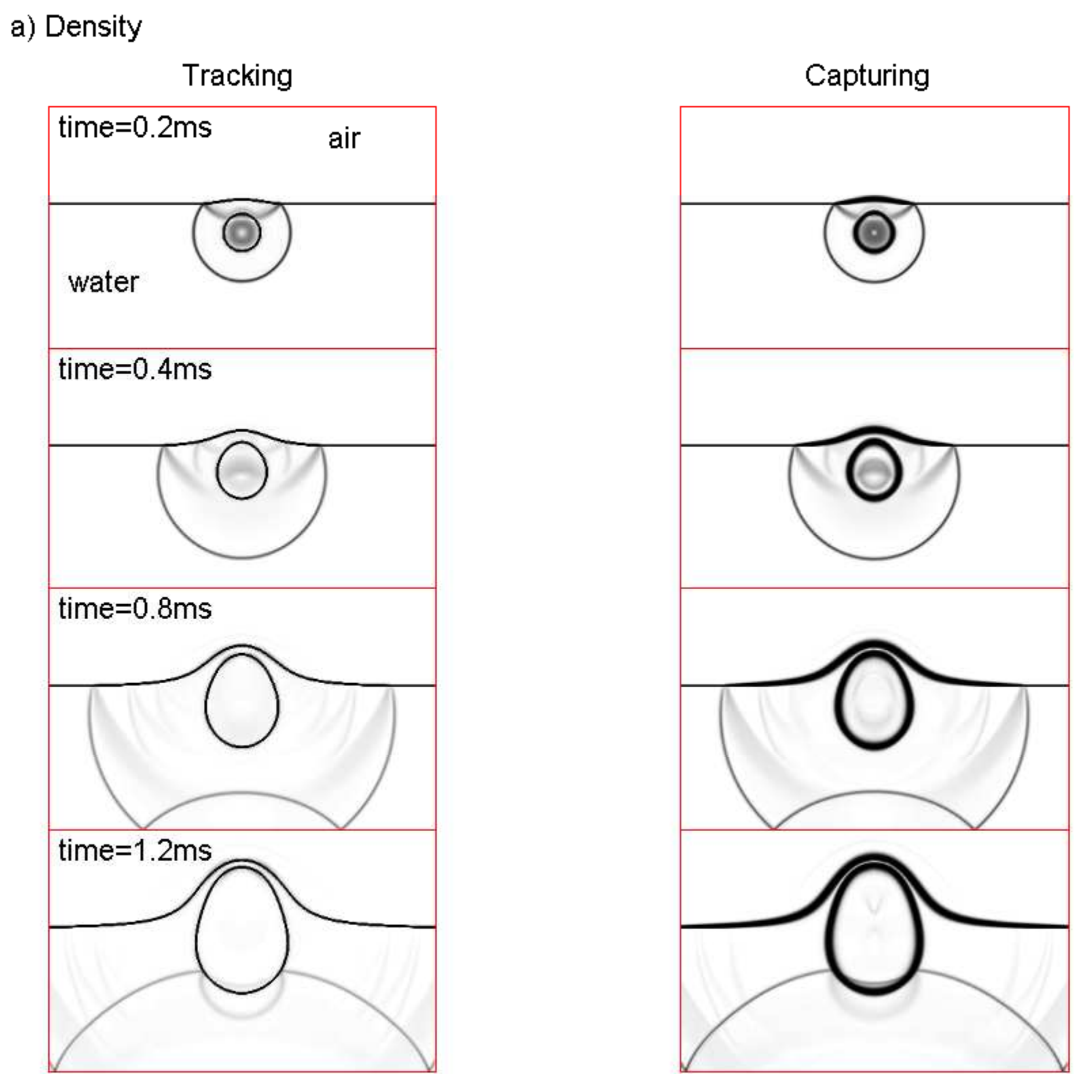
Underwater Explosions (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.9$



Underwater Explosions (Cont.)

- Volume tracking & interface capturing results

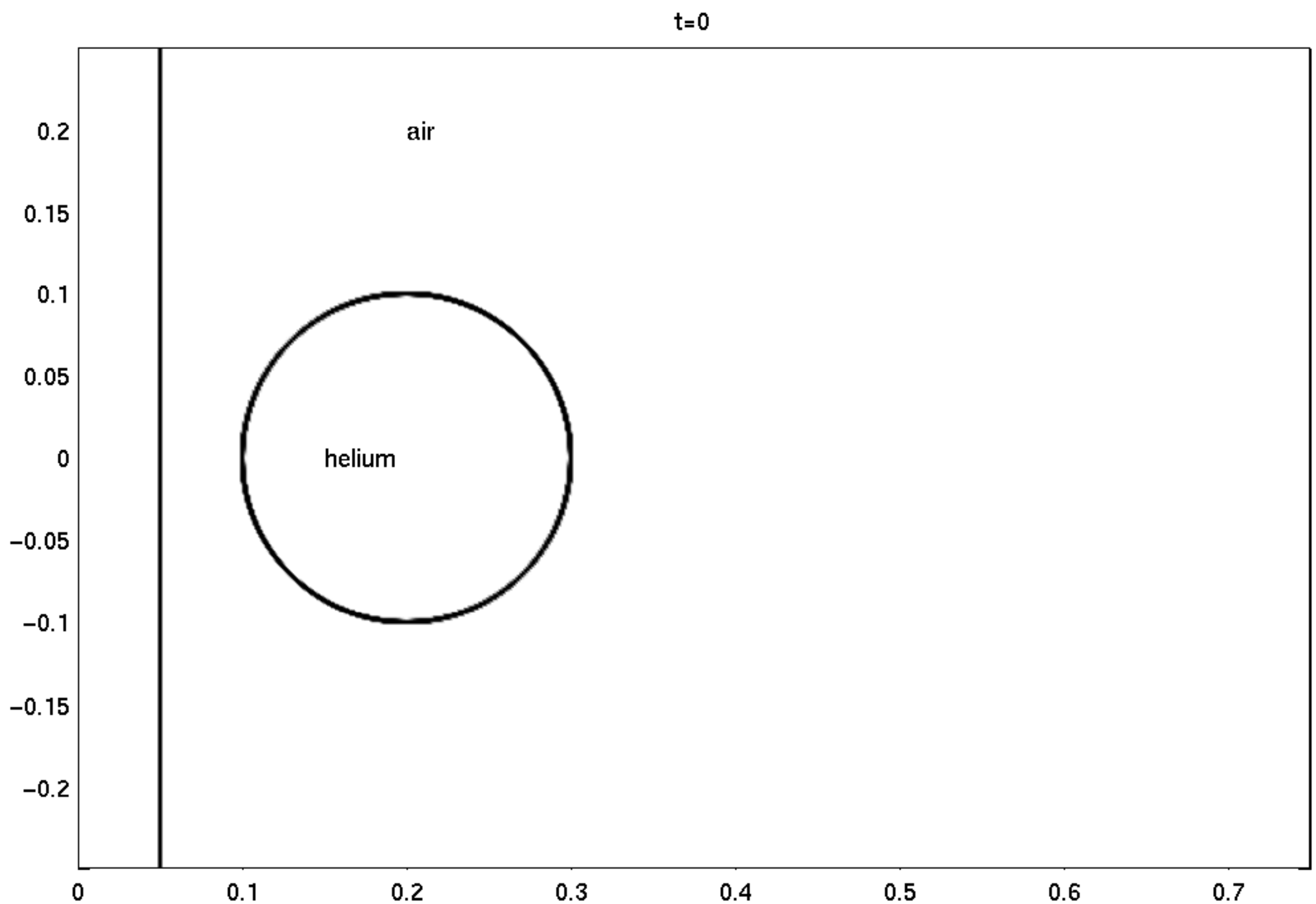


Underwater Explosions (Cont.)

- Generalized curvilinear grid: single bubble animation
- Cartesian grid: multiple bubble animation

Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



Shock-Bubble (Helium)

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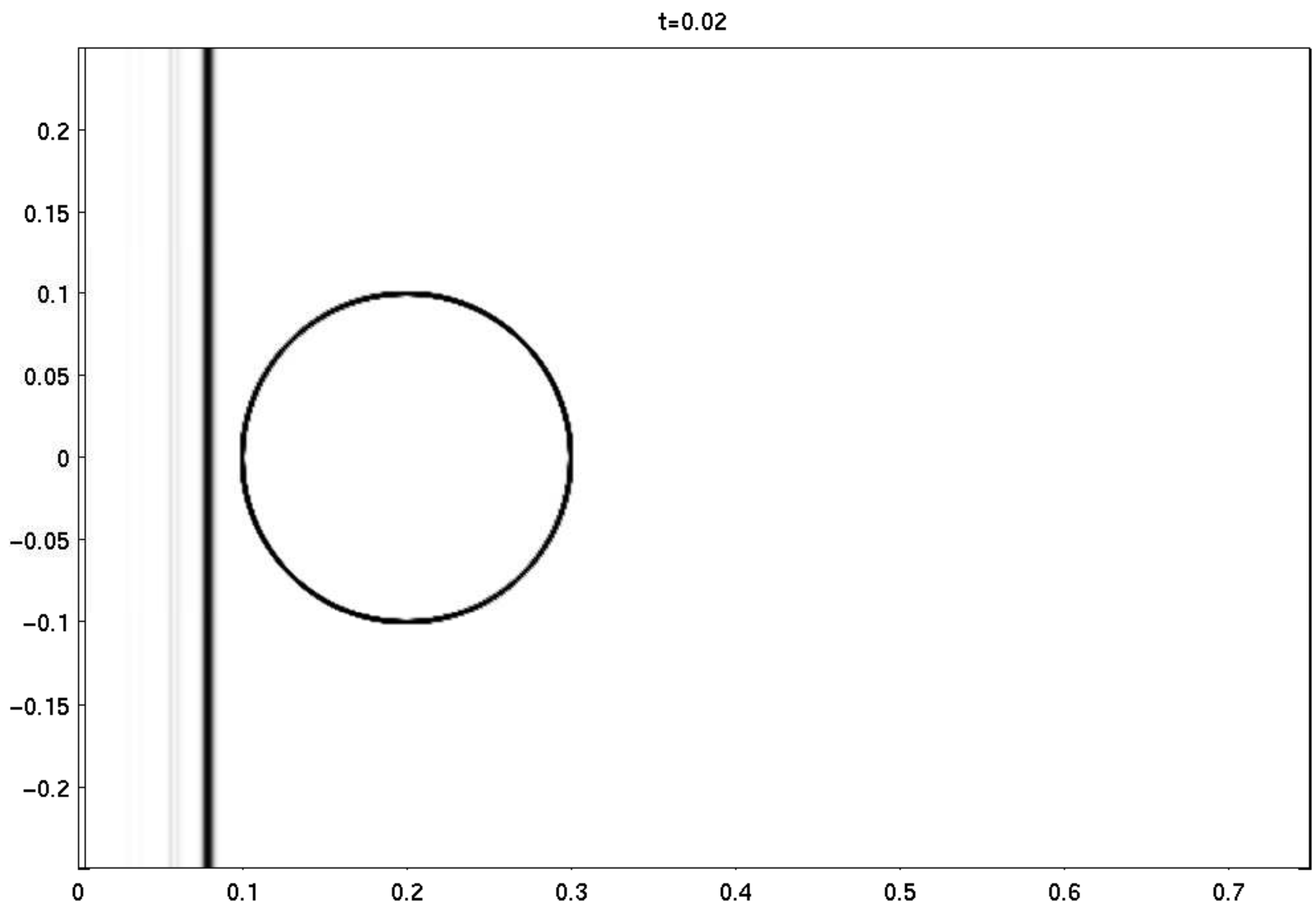

Shock-Bubble (Helium)

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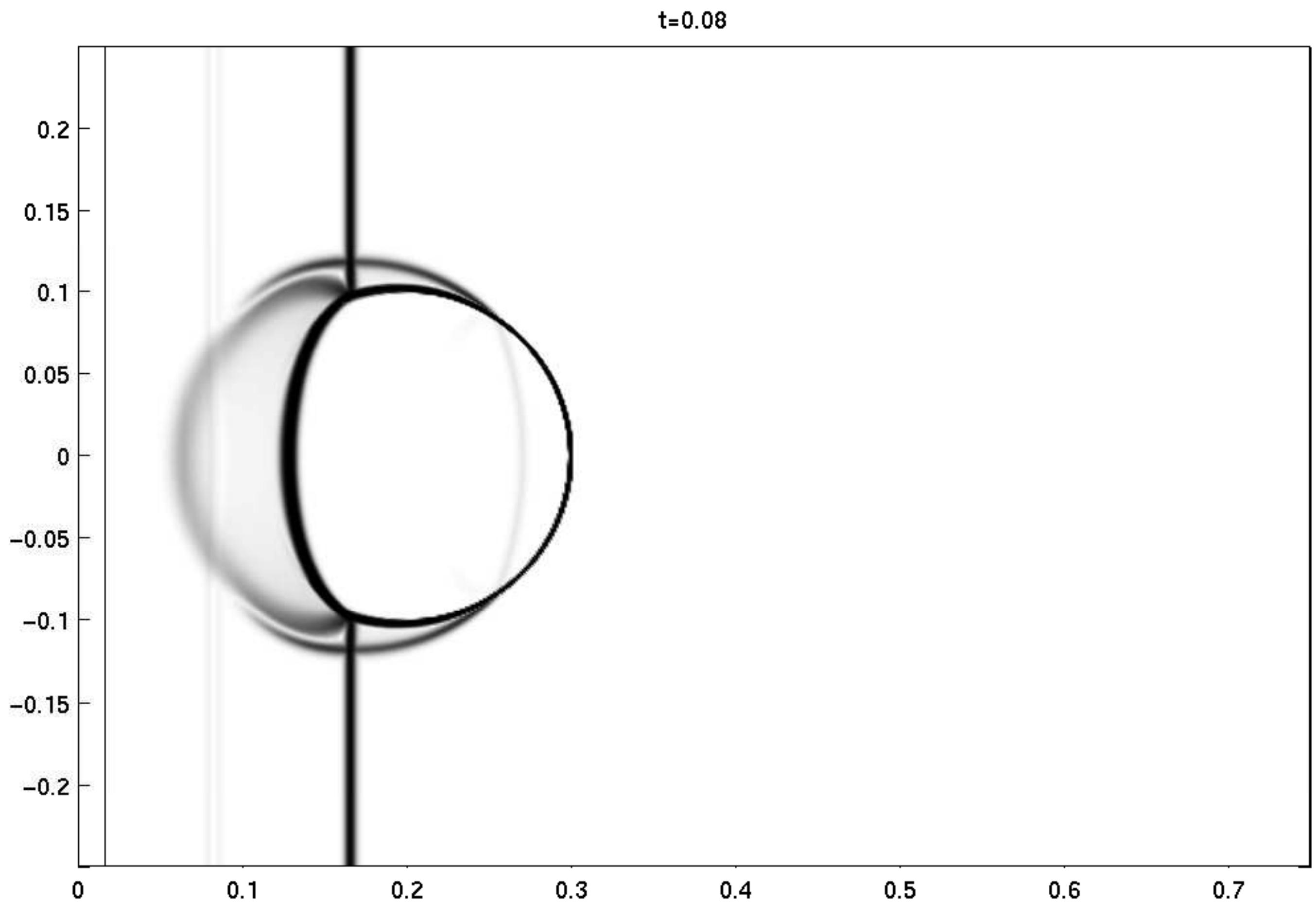

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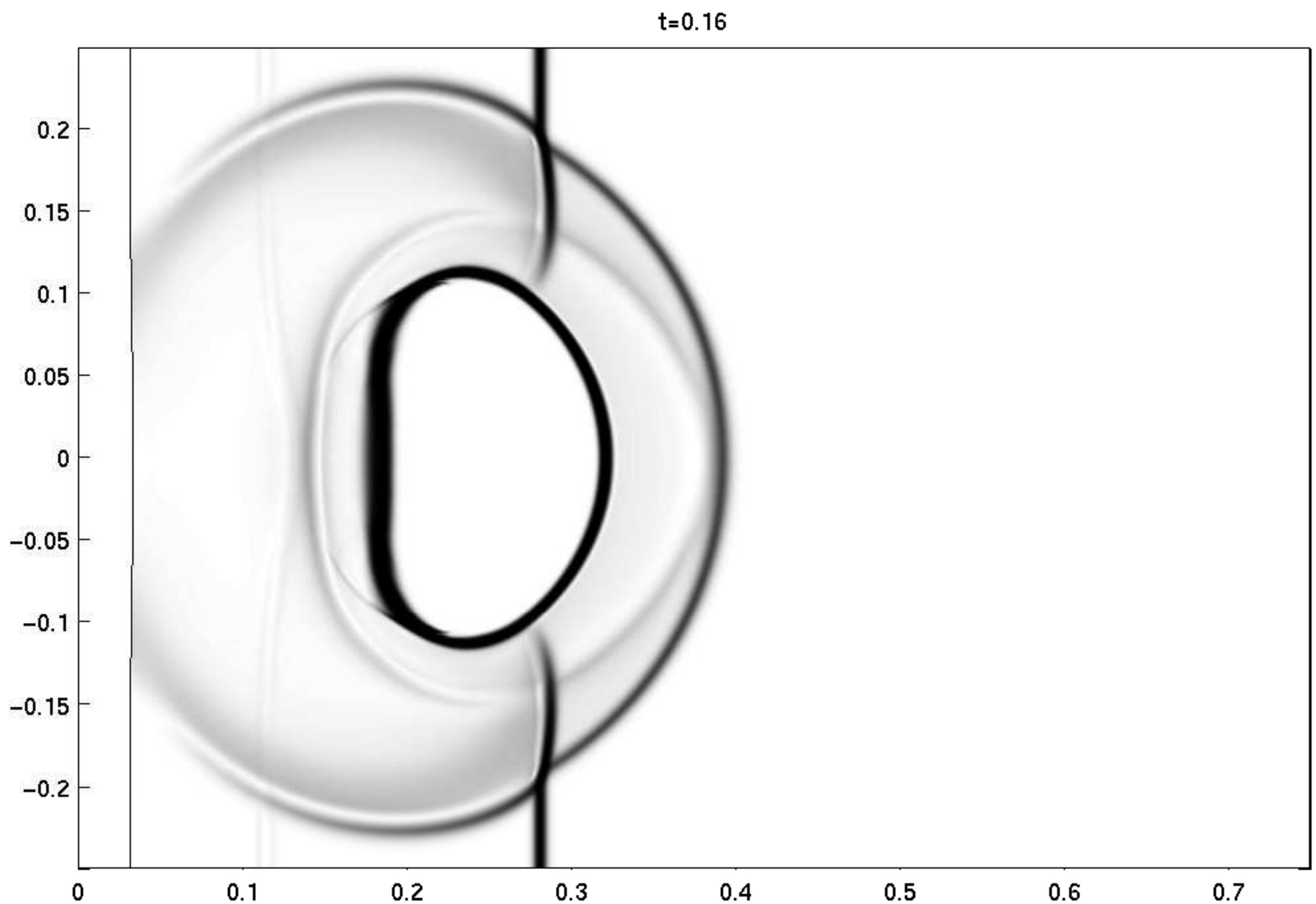
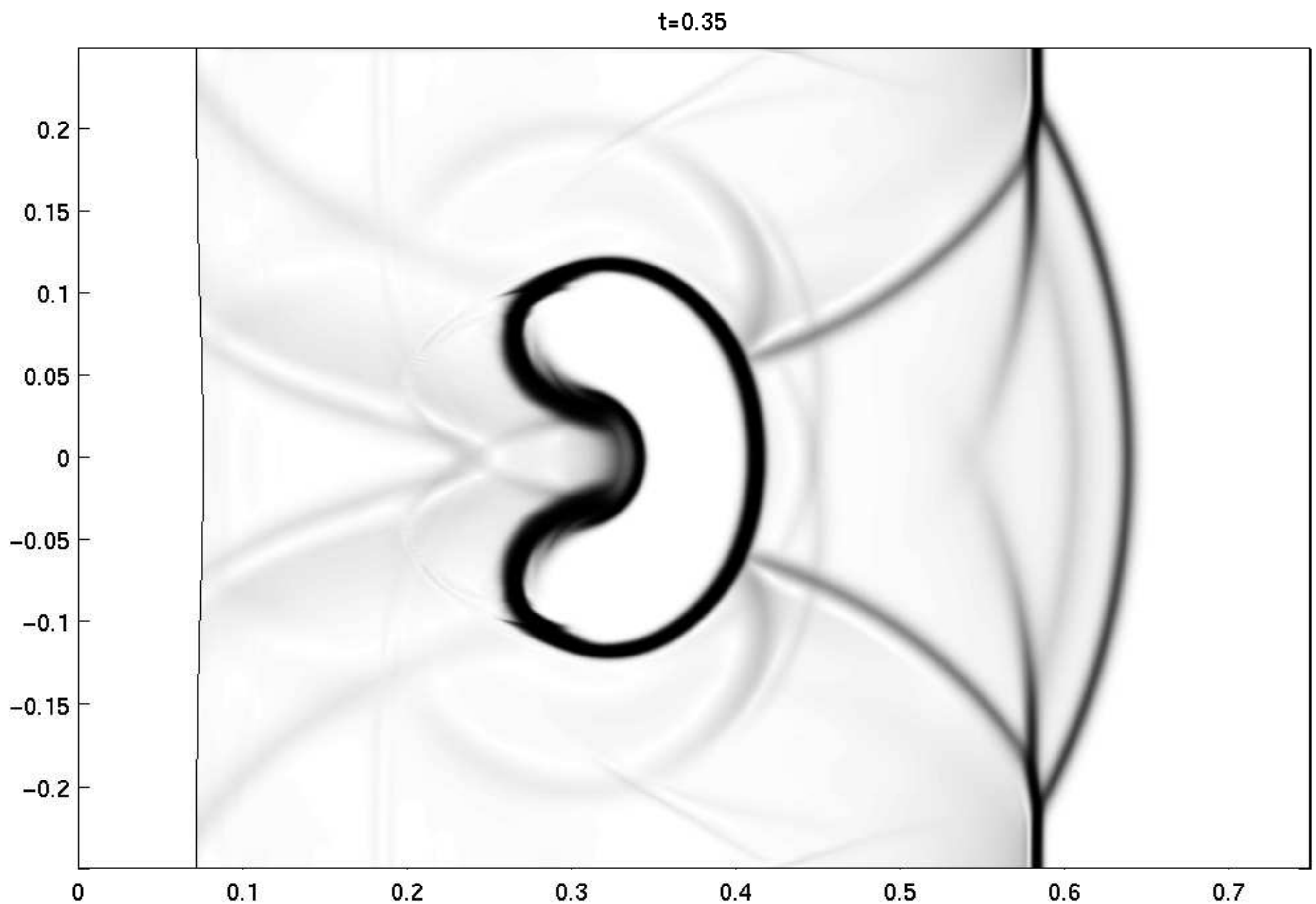


Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



Shock-Bubble (Helium) (Cont.)

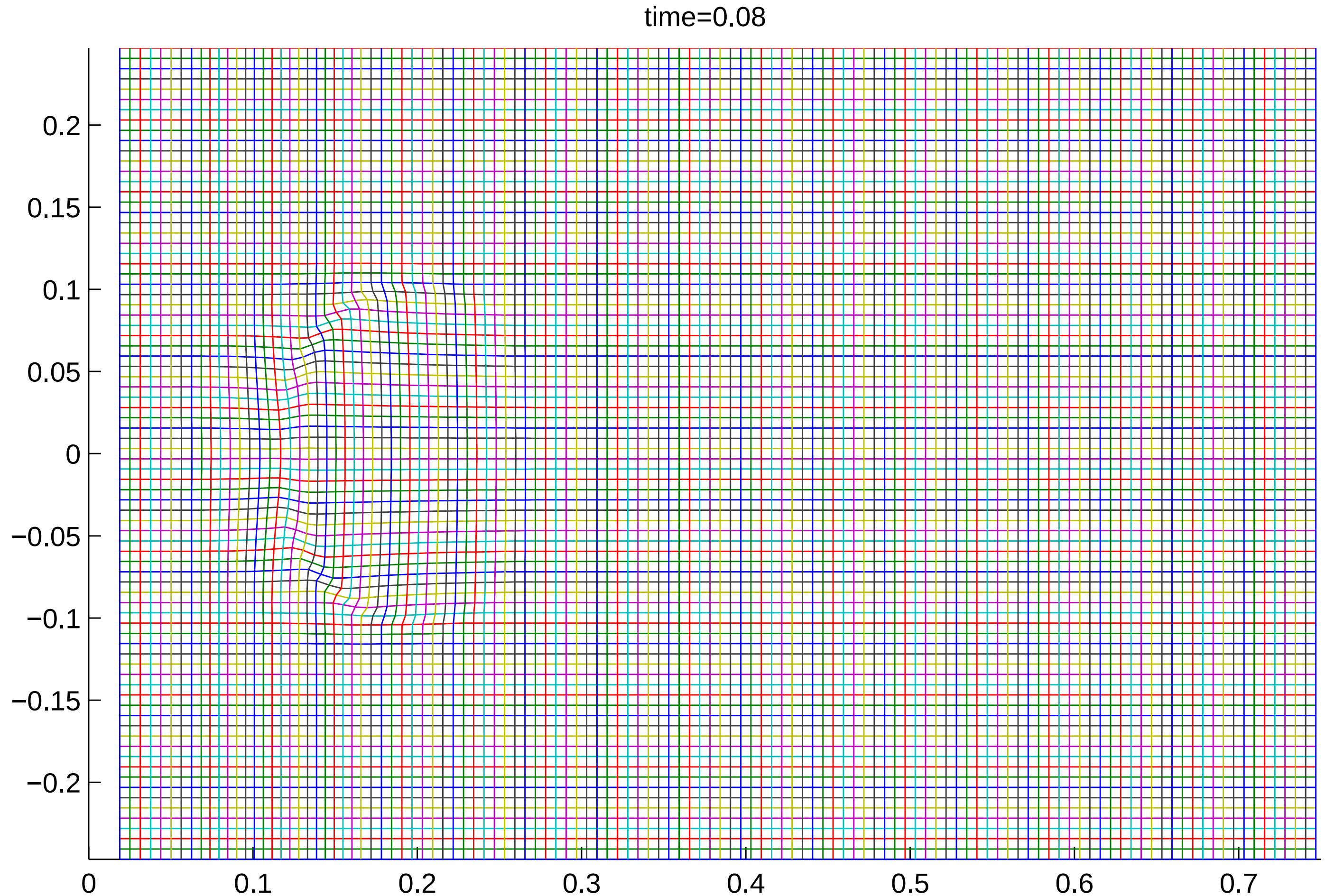
- Grid system (coarsen by factor 5) with $h_0 = 0.5$

Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$

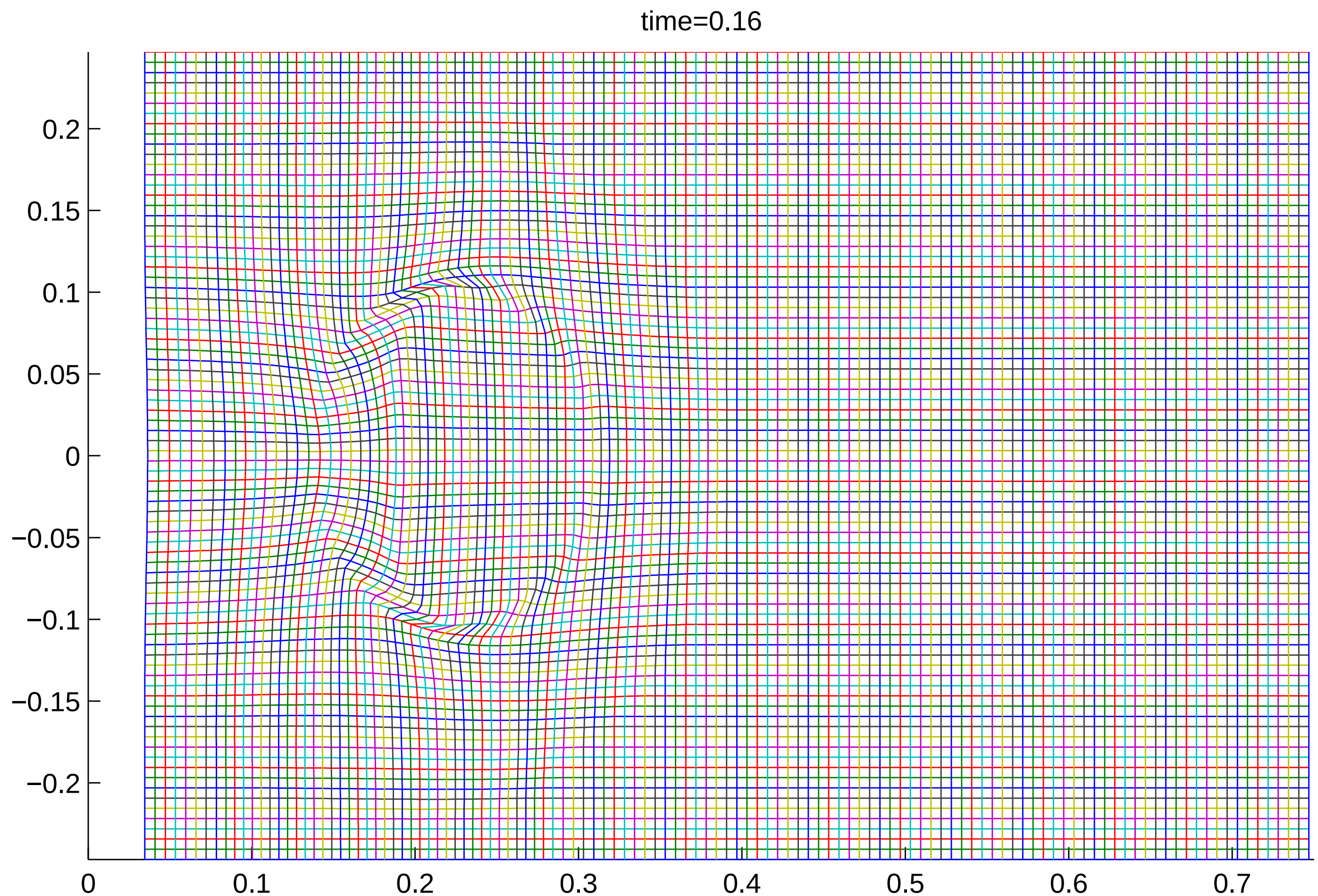
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



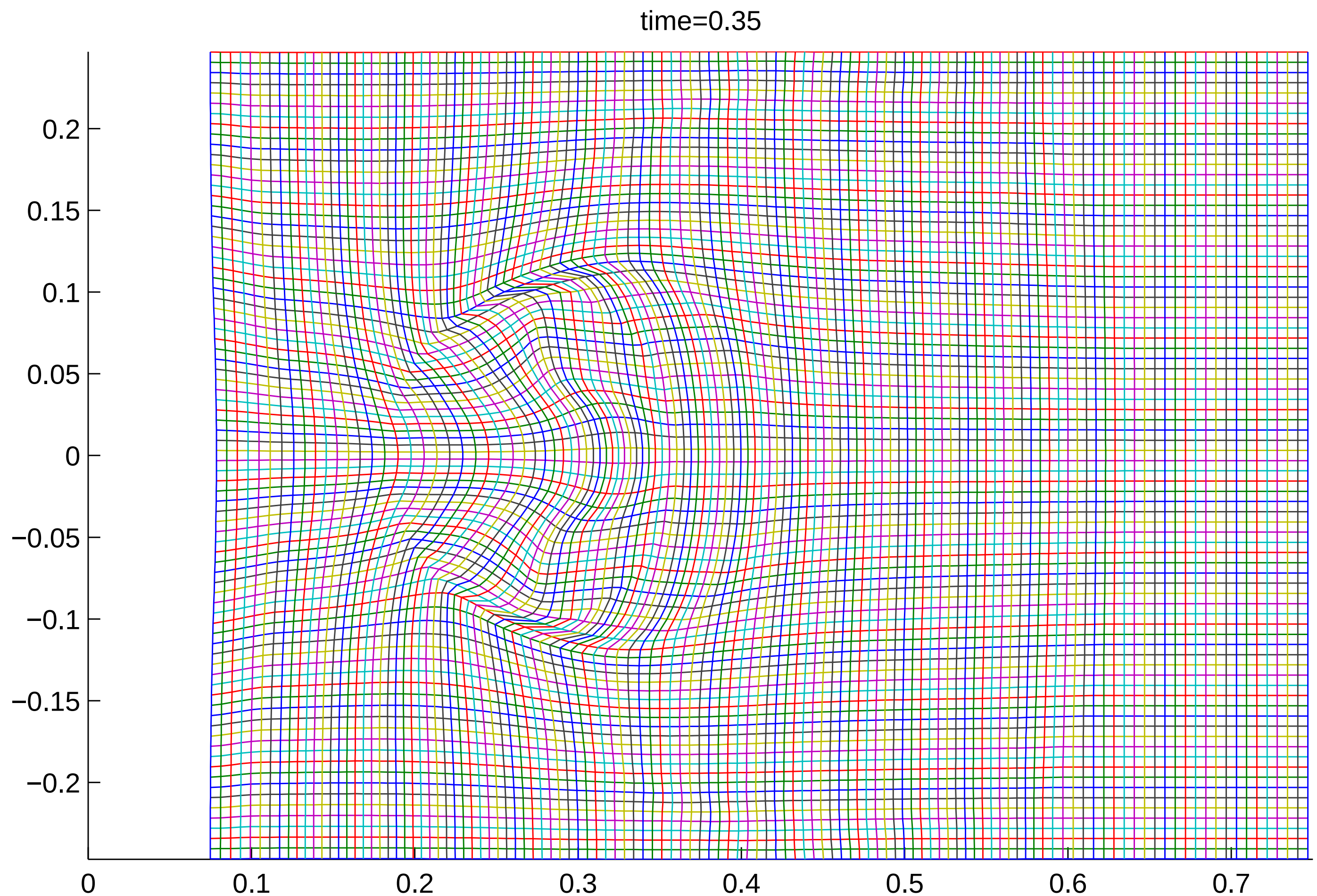
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



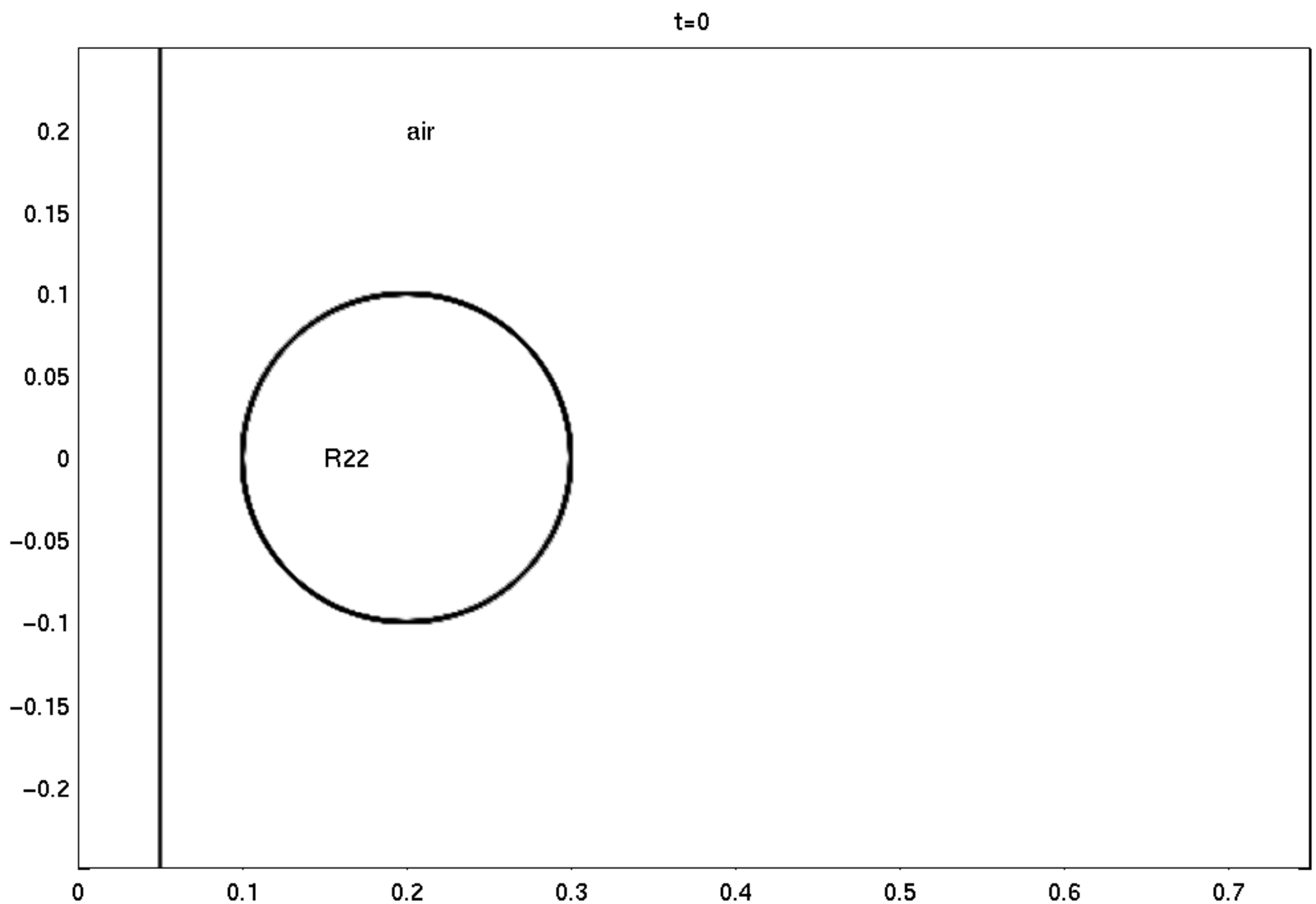
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



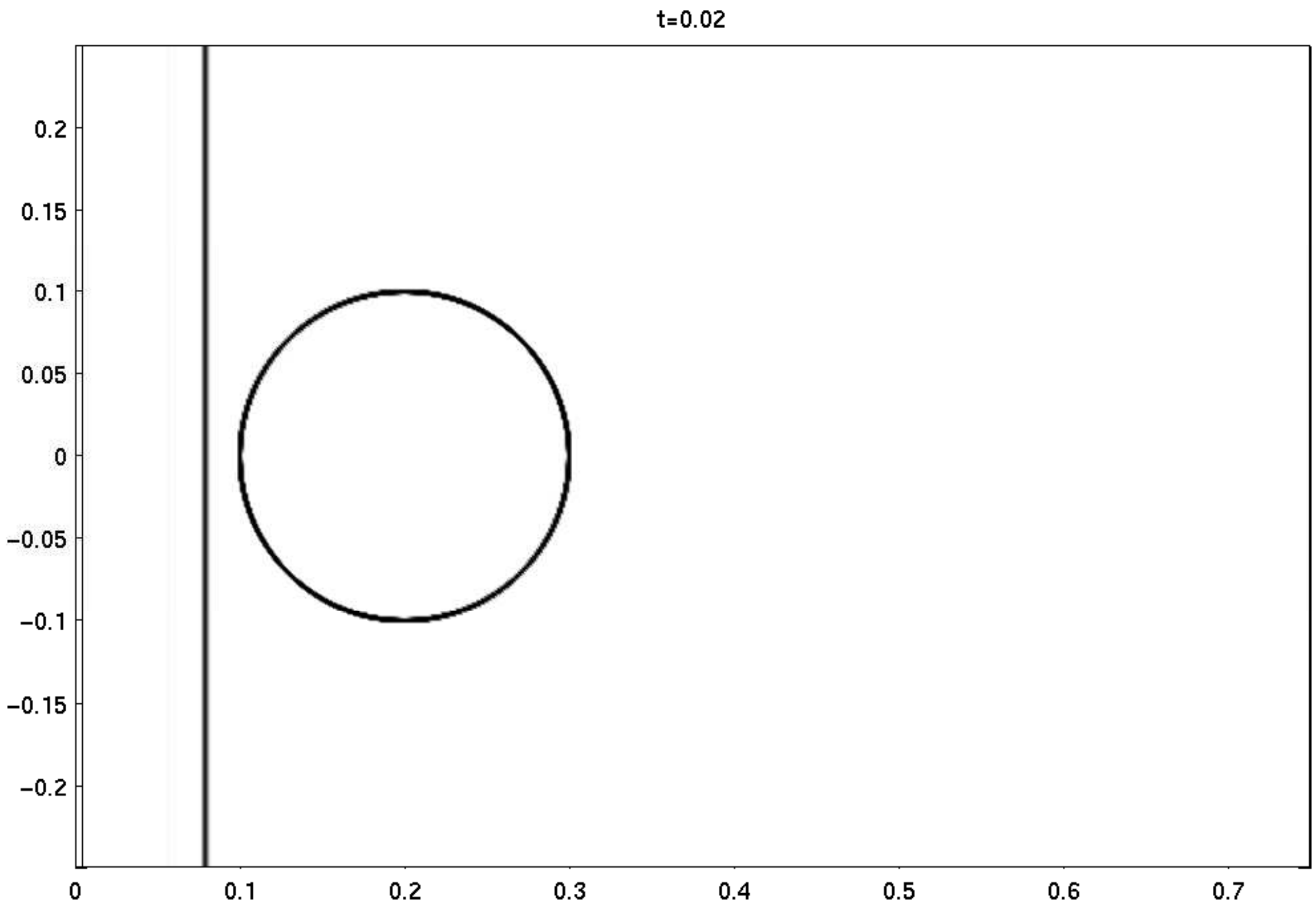
Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



Shock-Bubble (Refrigerant)

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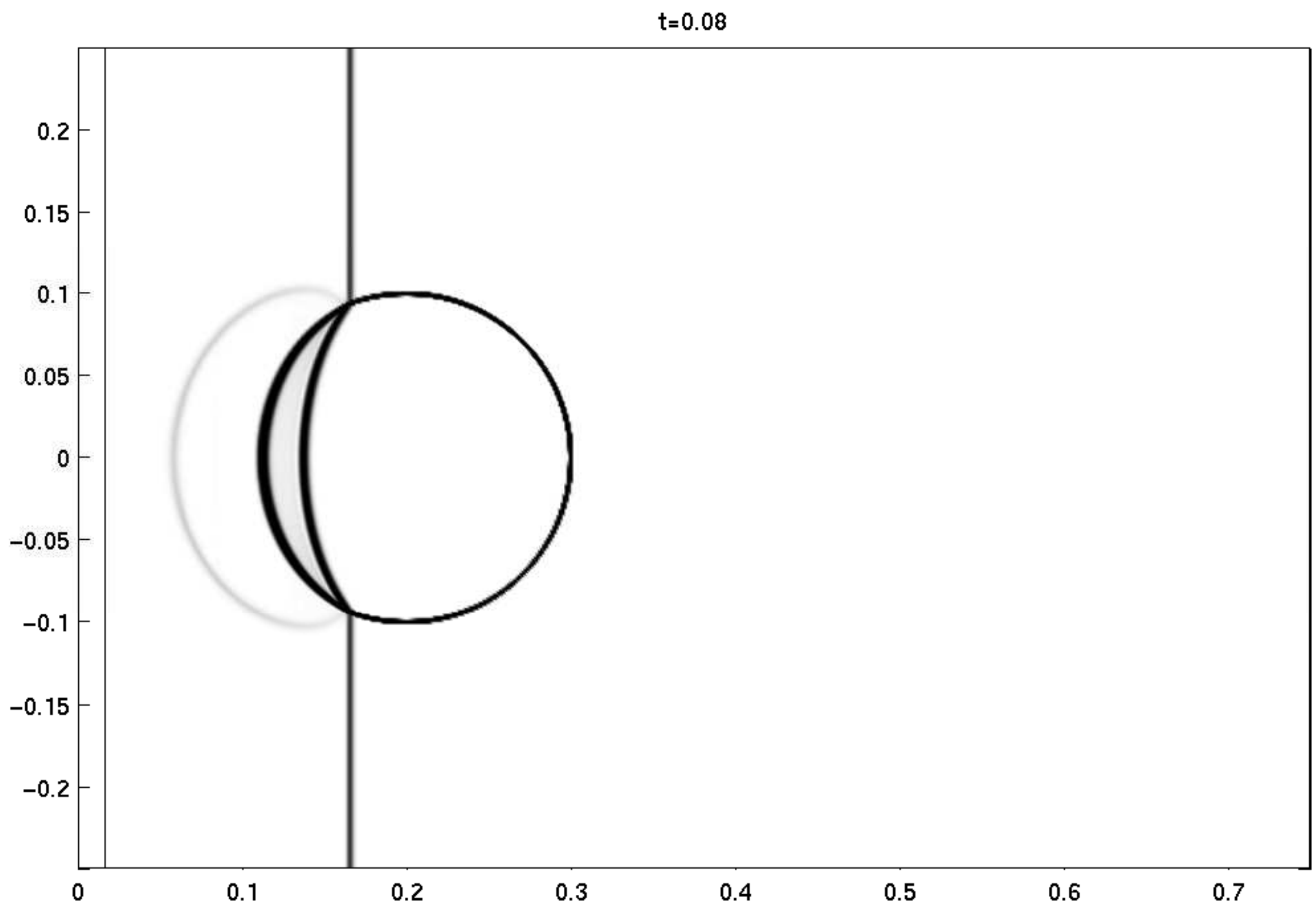
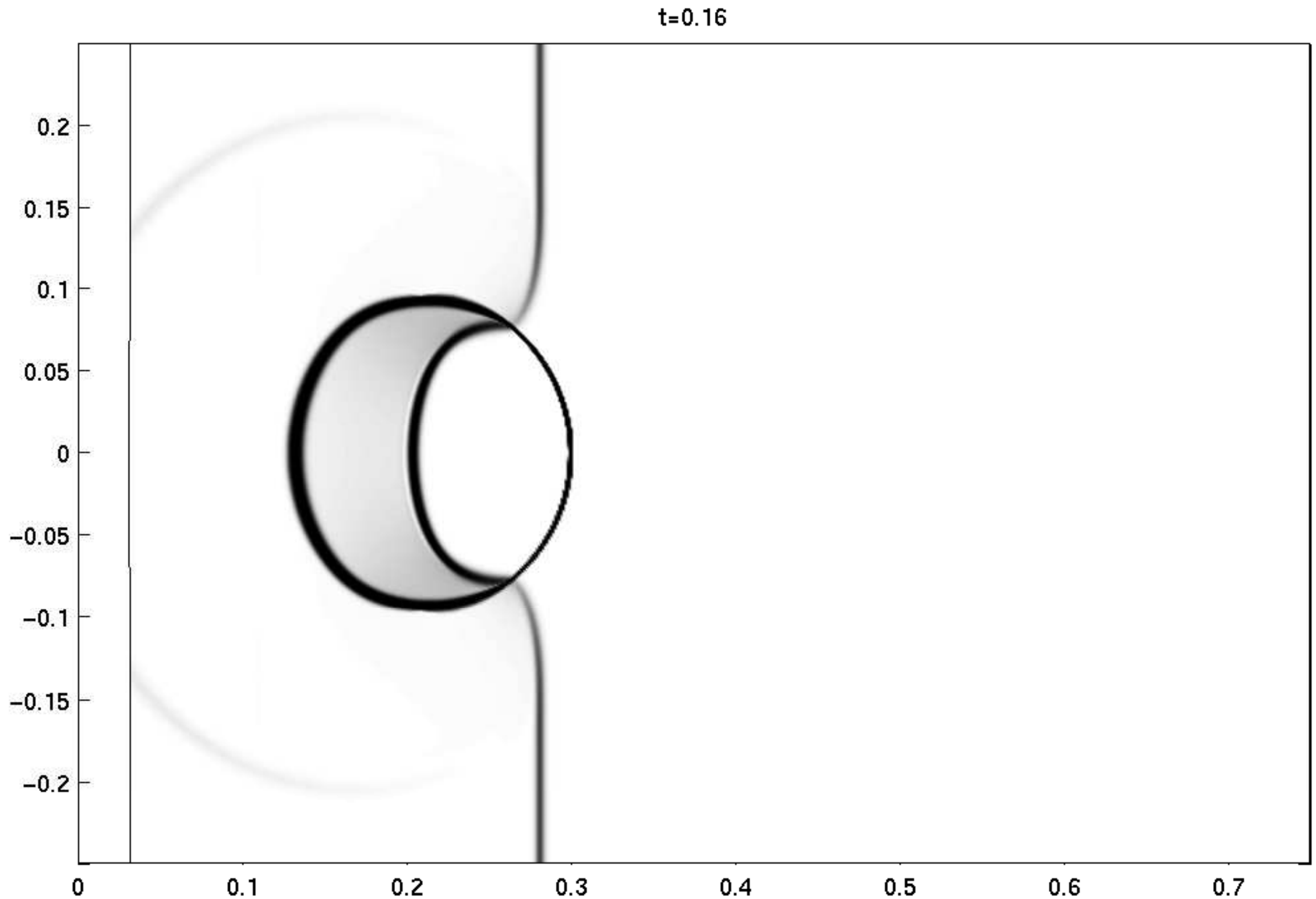


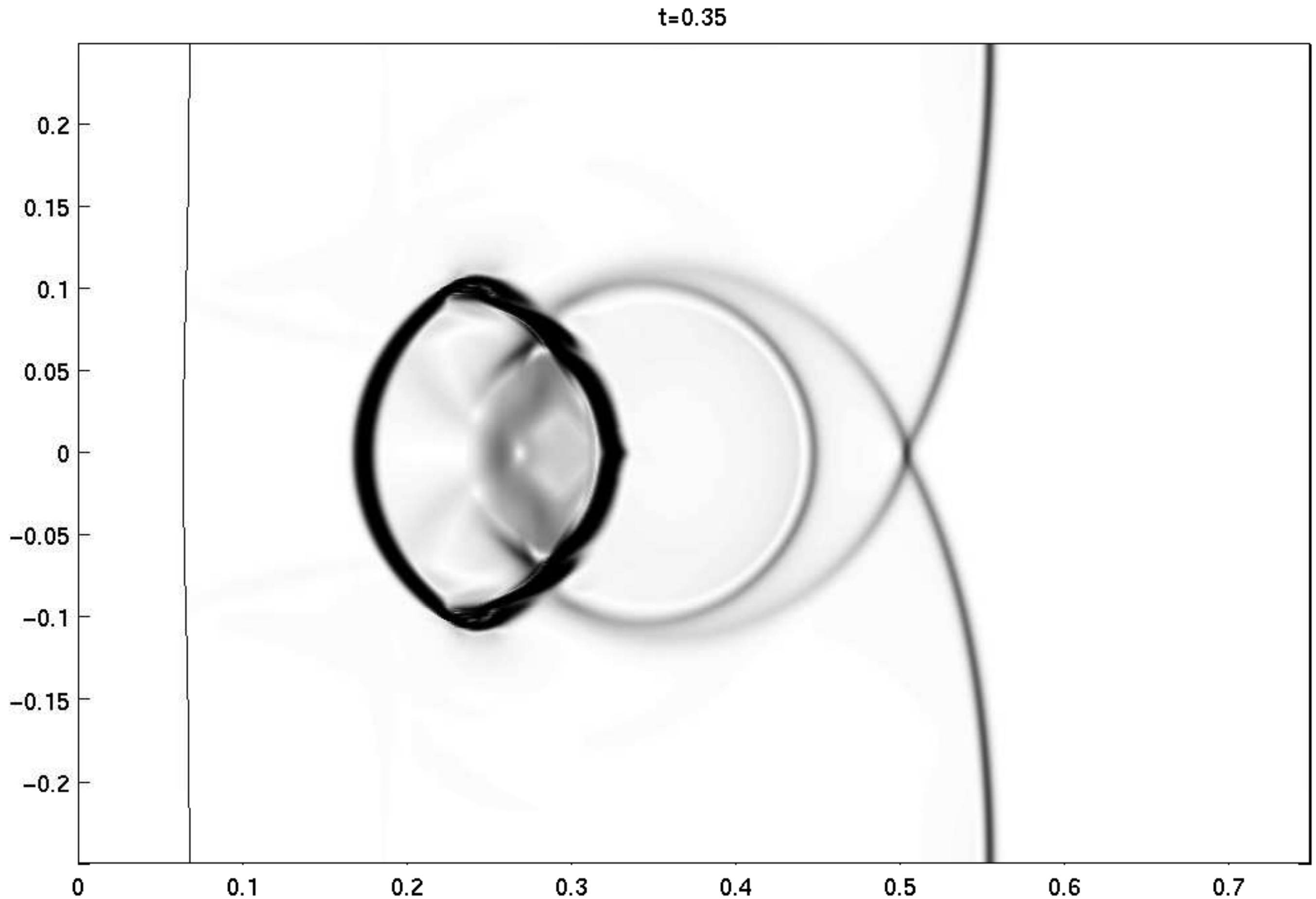
Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



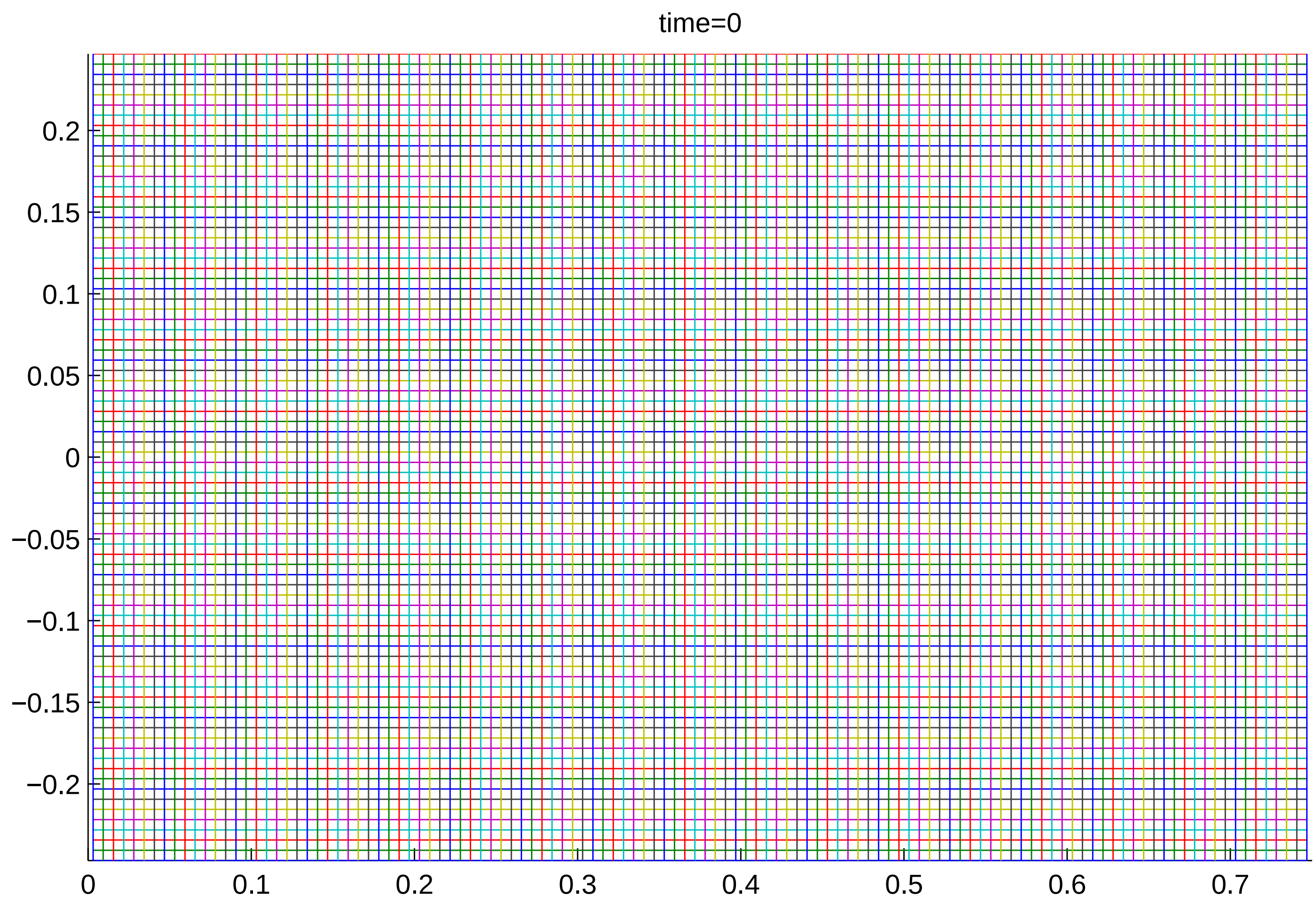
Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$

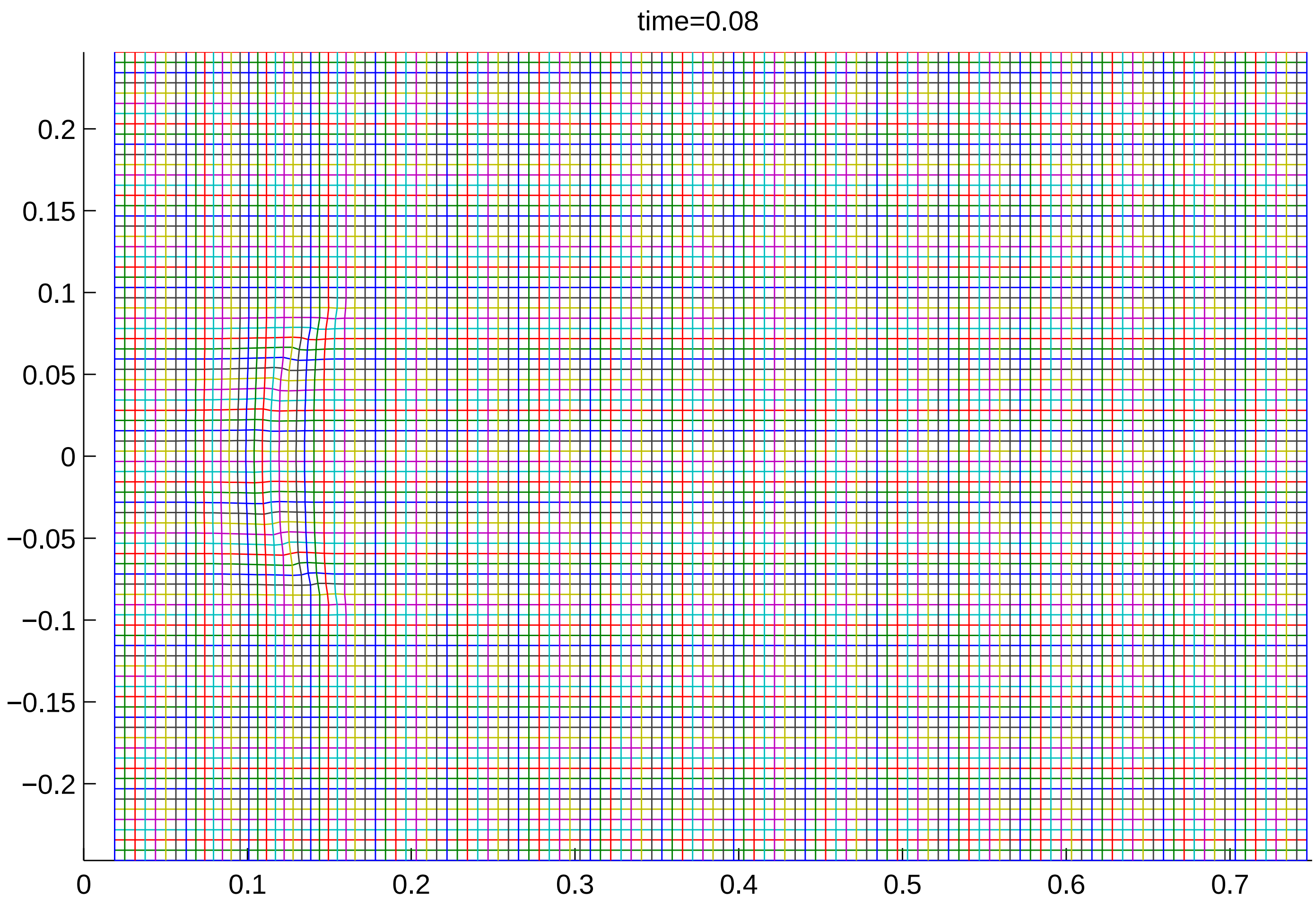


Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$

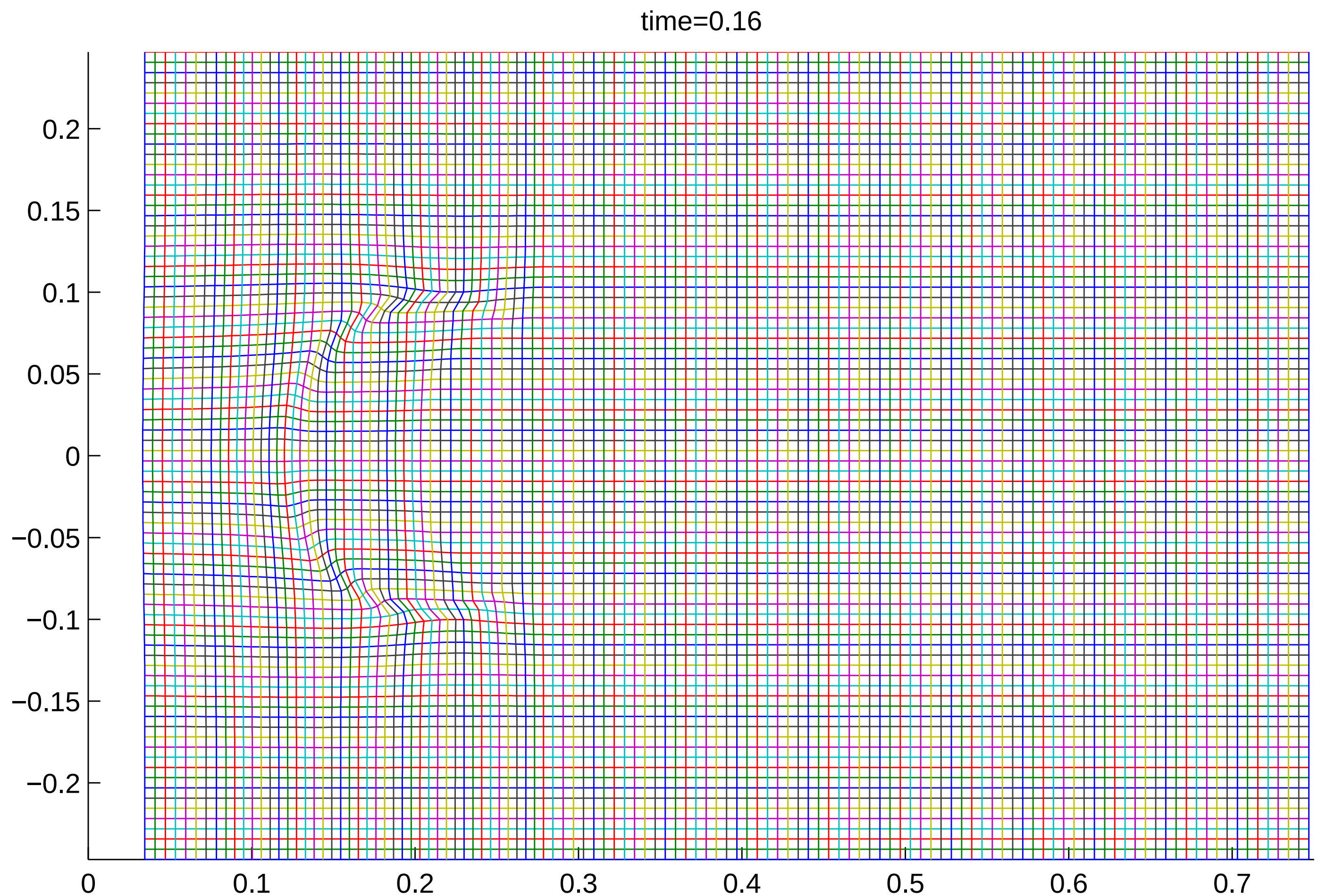
Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



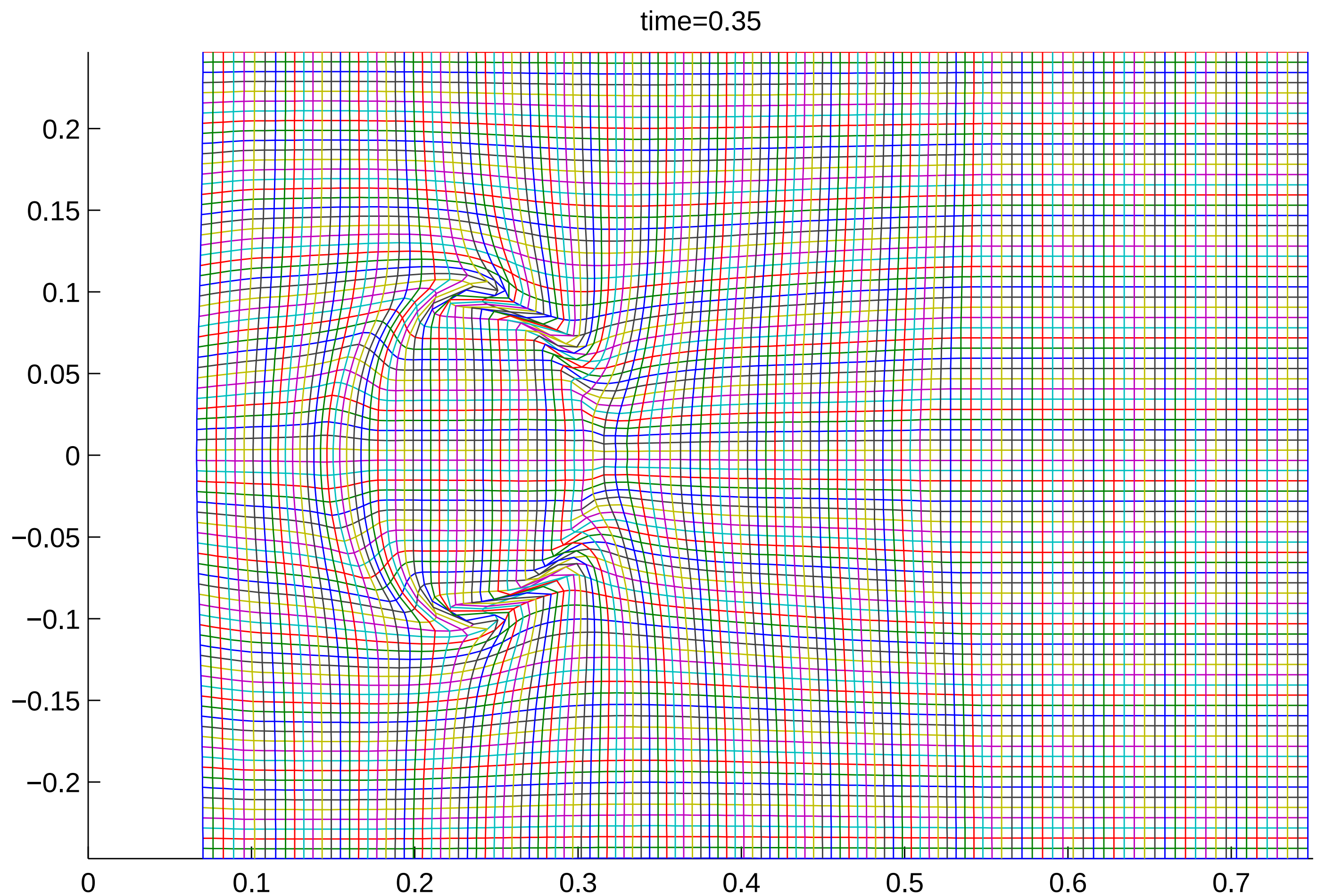
Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



Three Space Dimensions

Euler equations for inviscid compressible flow

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\\ \rho Eu+pu\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho vw\\ \rho Ev+pv\end{pmatrix}+\frac{\partial}{\partial z}\begin{pmatrix}\rho w\\ \rho uw\\ \rho vw\\ \rho w^2+p\\ \rho Ew+pw\end{pmatrix}=\psi$$

$E = e + (u^2+v^2+w^2)/2$, $\quad e(\rho, p)$: internal energy
ψ: source terms (geometrical, gravitational, & so on)

Three Space Dimensions (Cont.)

Introduce transformation $(t,\ x,\ y,\ z) \to (\tau,\ \xi,\ \eta,\ \zeta)$ via

$$\begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ U & A_1 & B_1 & C_1 \\ V & A_2 & B_2 & C_2 \\ W & A_3 & B_3 & C_3 \end{pmatrix} \begin{pmatrix} d\tau \\ d\xi \\ d\eta \\ d\zeta \end{pmatrix}$$

where
$\vec{Q} = (U,\ V,\ W)$: grid velocity

- $\vec{Q} = 0$ Eulerian case
- $\vec{Q} = (u,\ v,\ w)$ Lagrangian case

A_i, B_i, C_i: geometric variables, $i = 1, 2, 3$

Three Space Dimensions (Cont.)

Inverse transformation $(\tau,\ \xi,\ \eta,\ \zeta) \to (t,\ x,\ y,\ z)$ reads

$$\begin{pmatrix} d\tau \\ d\xi \\ d\eta \\ d\zeta \end{pmatrix} = \frac{1}{J} \begin{pmatrix} J & 0 & 0 & 0 \\ J_{01} & J_{11} & J_{21} & J_{31} \\ J_{02} & J_{12} & J_{22} & J_{32} \\ J_{03} & J_{13} & J_{23} & J_{33} \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}, \qquad J = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

where

$$J_{11} = B_2C_3 - B_3C_2, \quad J_{21} = C_1B_3 - B_1C_3, \quad J_{31} = B_1C_2 - C_1B_2$$

$$J_{12} = C_2A_3 - A_2C_3, \quad J_{22} = A_1C_3 - C_1A_3, \quad J_{32} = C_1A_2 - A_1C_2$$

$$J_{13} = A_2B_3 - B_2A_3, \quad J_{23} = B_1A_3 - A_1B_3, \quad J_{33} = A_1B_2 - B_1A_2$$

$$J_{01} = -\left(UJ_{11} + VJ_{21} + WJ_{31}\right), \quad J_{02} = -\left(UJ_{12} + VJ_{22} + WJ_{32}\right)$$

$$J_{03} = -\left(UJ_{13} + VJ_{23} + WJ_{33}\right)$$

Three Space Dimensions (Cont.)

Euler equations in generalized curvilinear coordinates

$$\frac{\partial}{\partial \tau}\begin{pmatrix} \rho J \\ \rho J u \\ \rho J v \\ \rho J w \\ \rho J E \end{pmatrix} + \frac{\partial}{\partial \xi}\begin{pmatrix} \rho \mathcal{U} \\ \rho u \mathcal{U} + p J_{11} \\ \rho v \mathcal{U} + p J_{21} \\ \rho w \mathcal{U} + p J_{31} \\ \rho E \mathcal{U} + p \mathcal{X} \end{pmatrix} + \frac{\partial}{\partial \eta}\begin{pmatrix} \rho \mathcal{V} \\ \rho u \mathcal{V} + p J_{12} \\ \rho v \mathcal{V} + p J_{22} \\ \rho w \mathcal{V} + p J_{32} \\ \rho E \mathcal{V} + p \mathcal{Y} \end{pmatrix} + \frac{\partial}{\partial \zeta}\begin{pmatrix} \rho \mathcal{W} \\ \rho u \mathcal{W} + p J_{13} \\ \rho v \mathcal{W} + p J_{23} \\ \rho w \mathcal{W} + p J_{33} \\ \rho E \mathcal{W} + p \mathcal{Z} \end{pmatrix} = \psi$$

where

$$\begin{aligned} \mathcal{U} &= (u-U)J_{11} + (v-V)J_{21} + (w-W)J_{31}, & \mathcal{X} &= uJ_{11} + vJ_{21} + wJ_{31} \\ \mathcal{V} &= (u-U)J_{12} + (v-V)J_{22} + (w-W)J_{32}, & \mathcal{Y} &= uJ_{12} + vJ_{22} + wJ_{32} \\ \mathcal{W} &= (u-U)J_{13} + (v-V)J_{23} + (w-W)J_{33}, & \mathcal{Z} &= uJ_{13} + vJ_{23} + wJ_{33} \end{aligned}$$

Three Space Dimensions (Cont.)

Geometrical conservation laws

$$\frac{\partial}{\partial \tau}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ B_1 \\ B_2 \\ B_3 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix} + \frac{\partial}{\partial \xi}\begin{pmatrix} -U \\ -V \\ -W \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{\partial}{\partial \eta}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -U \\ -V \\ -W \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{\partial}{\partial \zeta}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -U \\ -V \\ -W \end{pmatrix} = 0$$

Grid-Velocity Selection

- Pseudo-Lagrangian like

$$(U,\ V,\ W) = h_0(u,\ v,\ w), \qquad h_0 \in (0,1)$$

- Mesh-volume preserving: $\partial J/\partial t = 0$
- Grid-angle preserving
- Other novel approach

Three Space Dimensions (Cont.)

In summary, Euler equations in generalized coord. takes

$$\frac{\partial q}{\partial t} + \frac{\partial f(q,\Xi)}{\partial \xi} + \frac{\partial g(q,\Xi)}{\partial \eta} + \frac{\partial h(q,\Xi)}{\partial \zeta} = \psi$$

where

$$q = (\rho J,\ \rho J u,\ \rho J v,\ \rho J w,\ \rho J E,\ A_i,\ B_i,\ C_i)$$

$$f(q,\Xi) = (\rho\mathcal{U},\ \rho u\mathcal{U} + pJ_{11},\ \rho v\mathcal{U} + pJ_{21},\ \rho w\mathcal{U} + pJ_{31},\ \rho E\mathcal{U} + p\mathcal{X}, \cdots)$$

$$g(q,\Xi) = (\rho\mathcal{V},\ \rho u\mathcal{V} + pJ_{12},\ \rho v\mathcal{V} + pJ_{22},\ \rho w\mathcal{V} + pJ_{32},\ \rho E\mathcal{V} + p\mathcal{Y}, \cdots)$$

$$h(q,\Xi) = (\rho\mathcal{W},\ \rho u\mathcal{W} + pJ_{13},\ \rho v\mathcal{W} + pJ_{23},\ \rho w\mathcal{W} + pJ_{33},\ \rho E\mathcal{W} + p\mathcal{Z}, \cdot$$

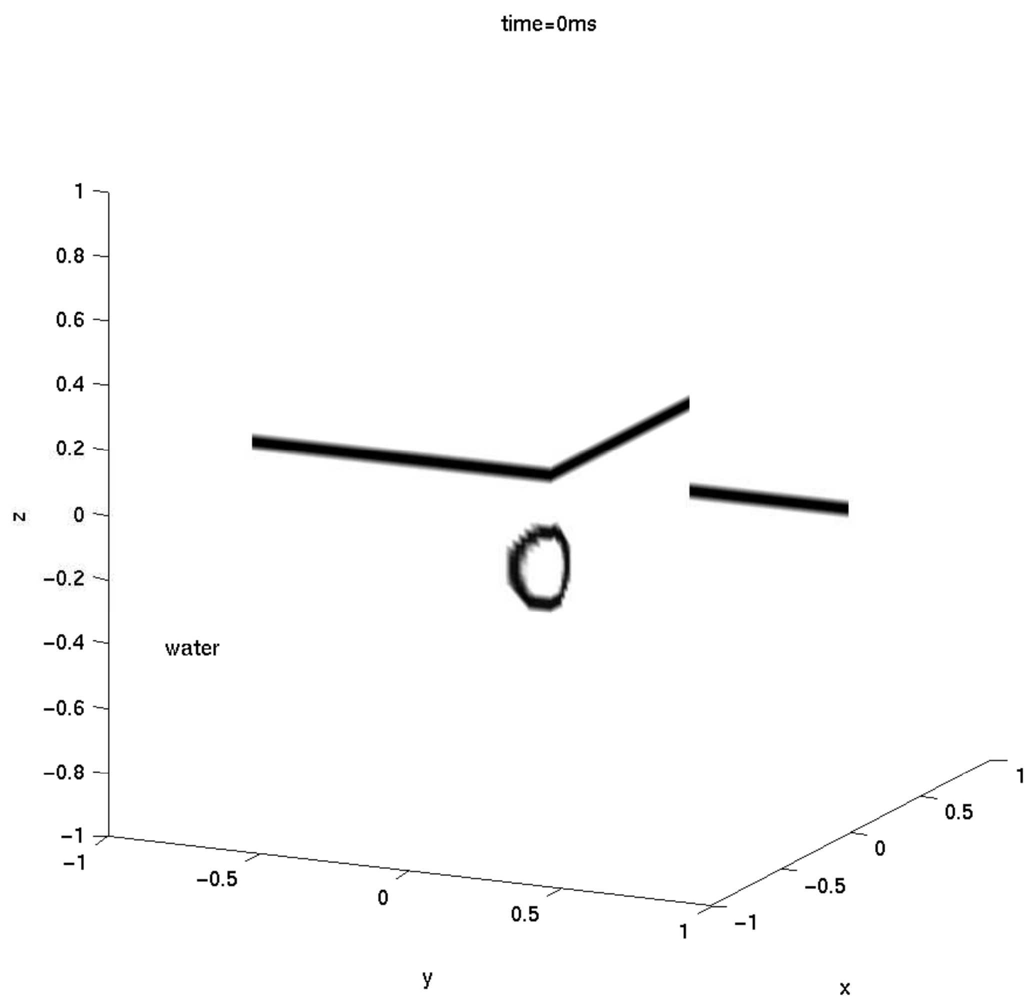
with Ξ : grid metrics & equation of state $p = p(\rho,\ e)$

Numerical Examples: 3D

- Underwater explosion
- Shock-bubble interaction
  - Helium bubble case
  - Refrigerant bubble case

Underwater Explosions

- Numerical schlieren images $h_0 = 0.6$, 100^3 grid



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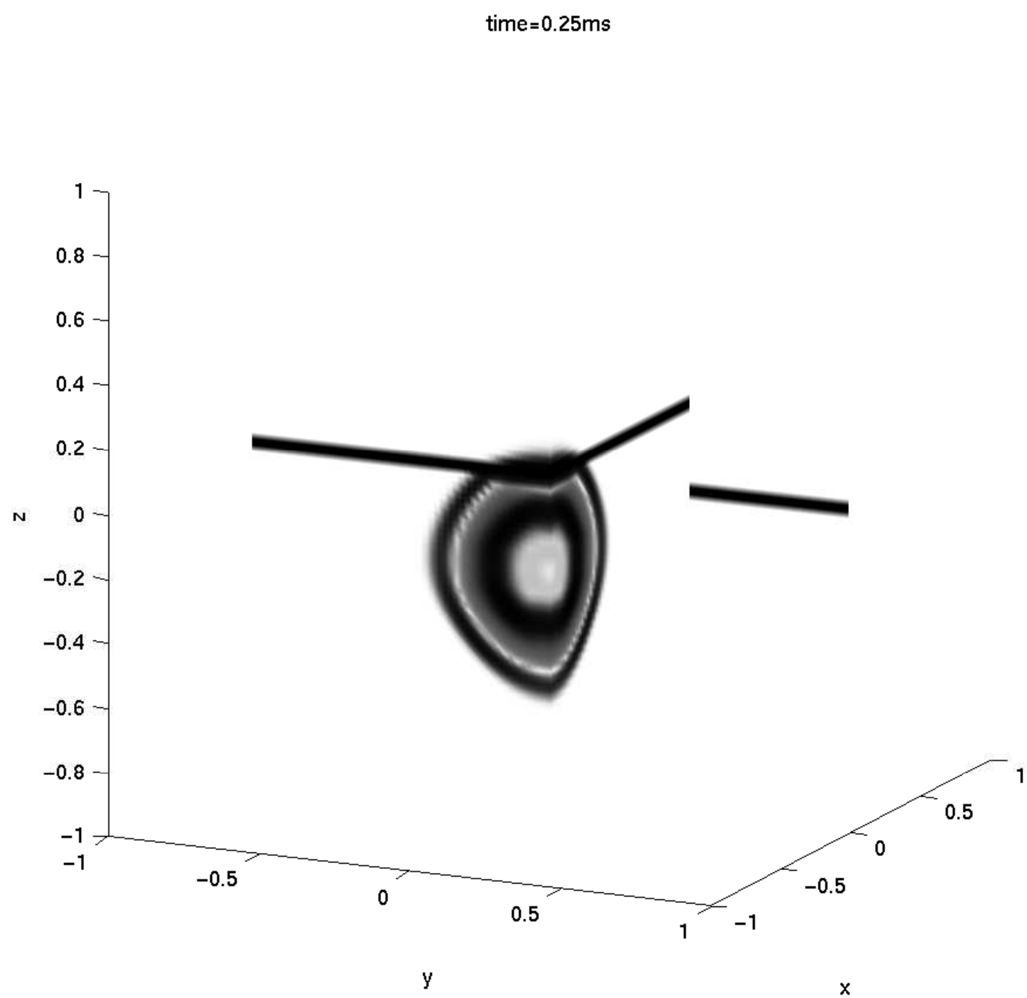

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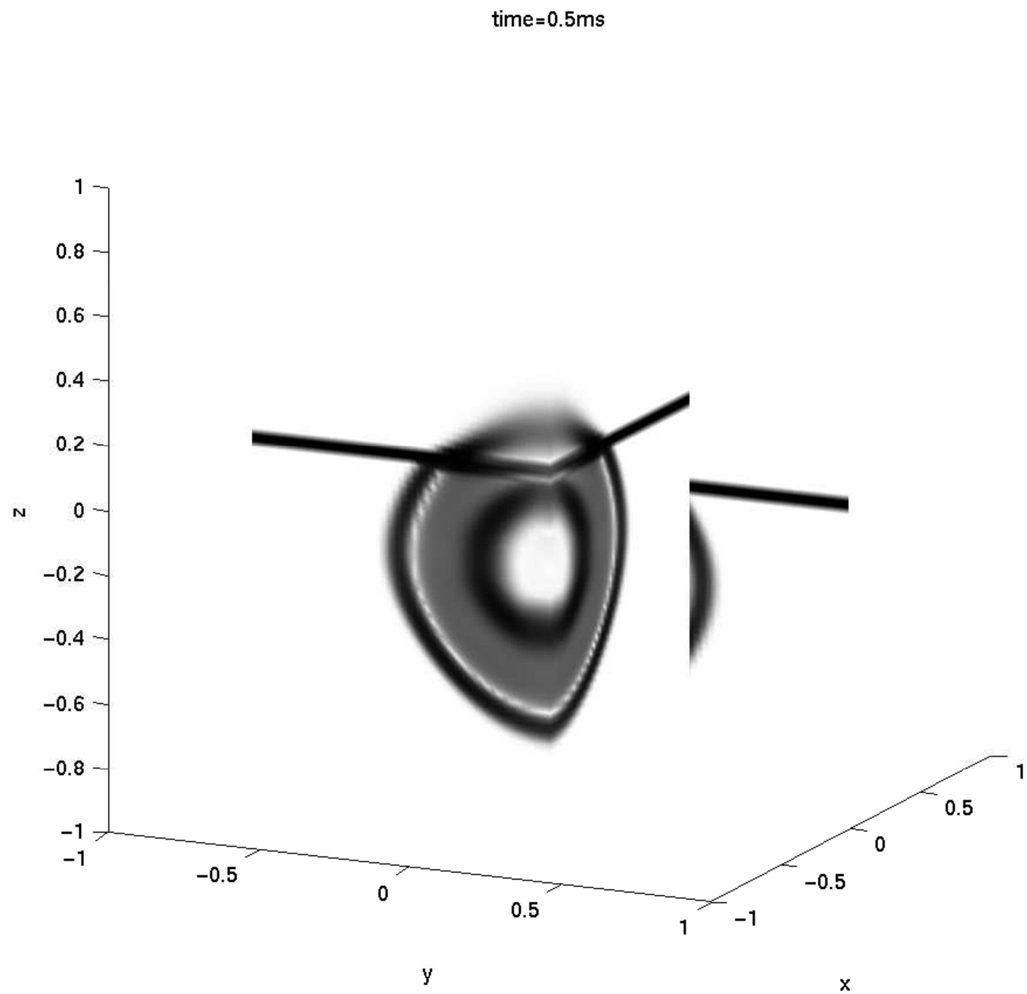

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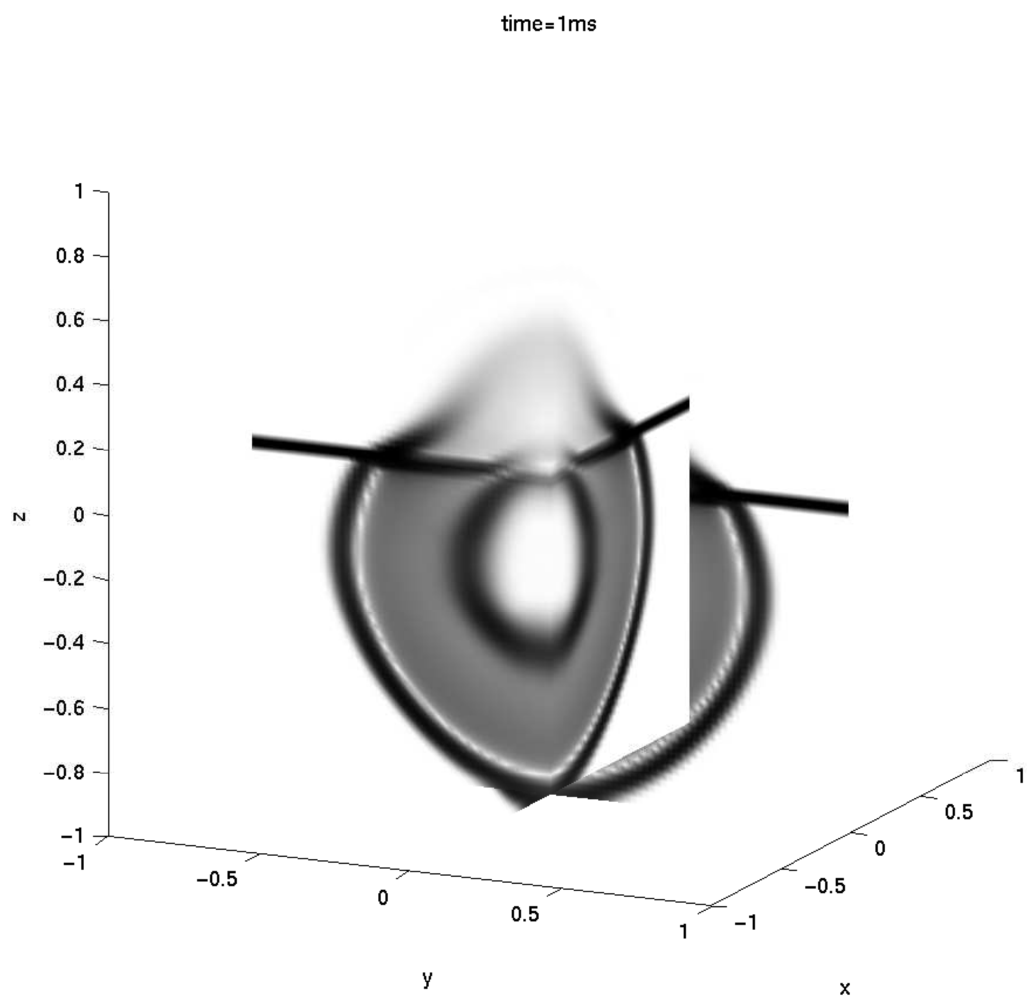
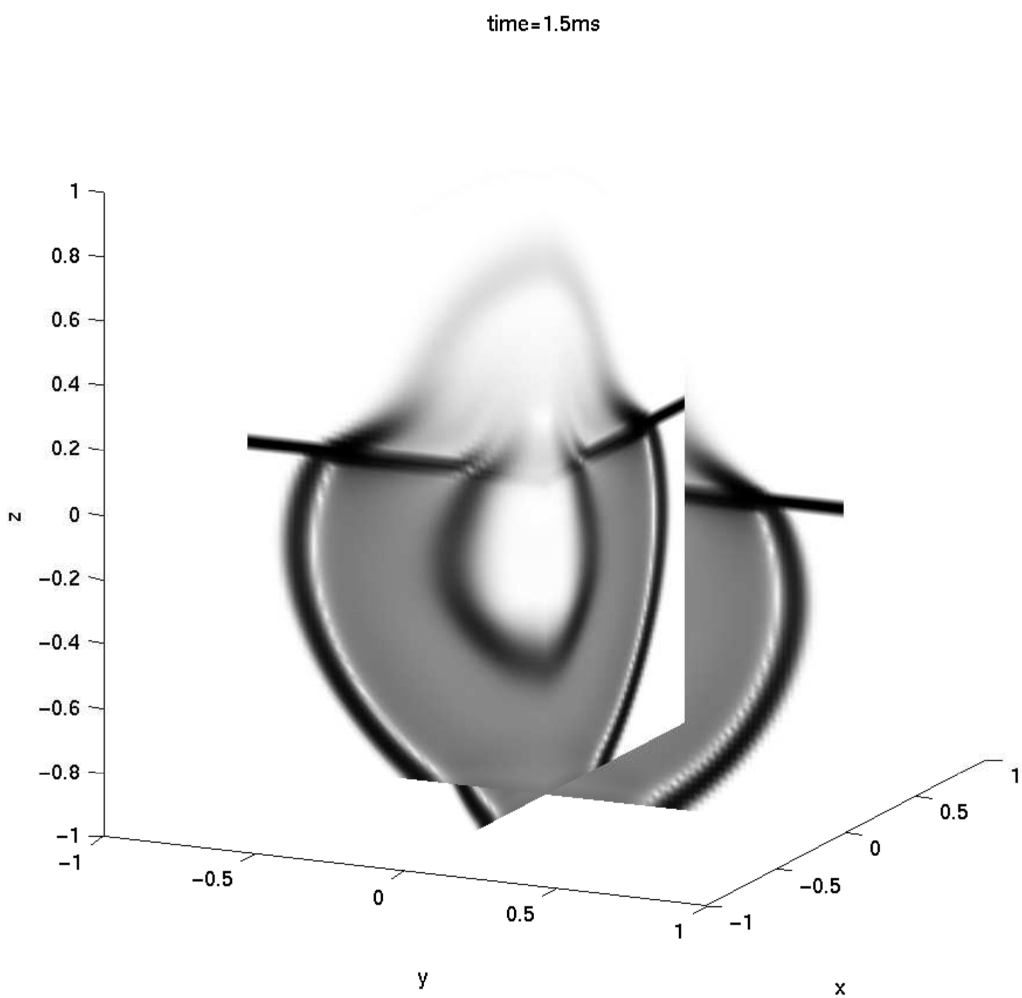


Underwater Explosions

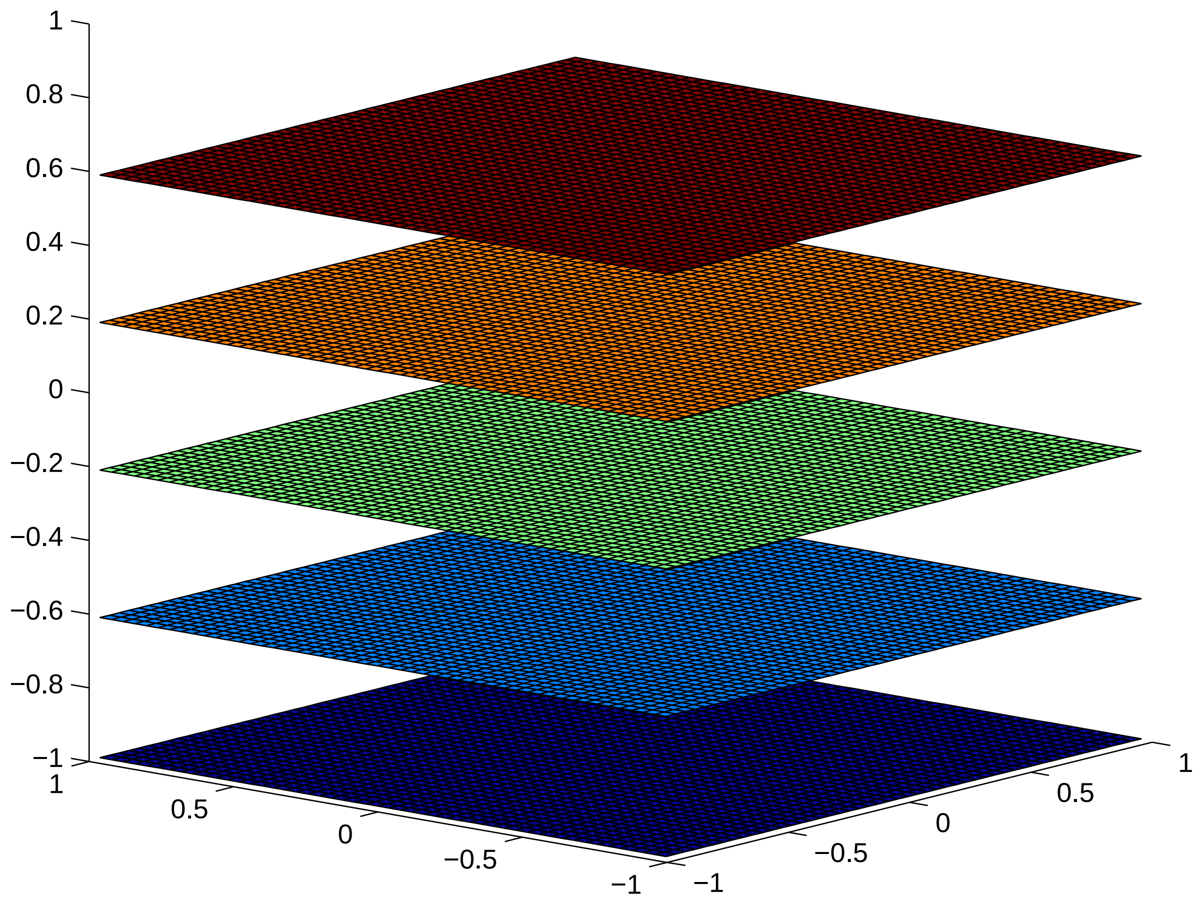
- Numerical schlieren images $h_0 = 0.6$, 100^3 grid



3D Underwater Explosions (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$

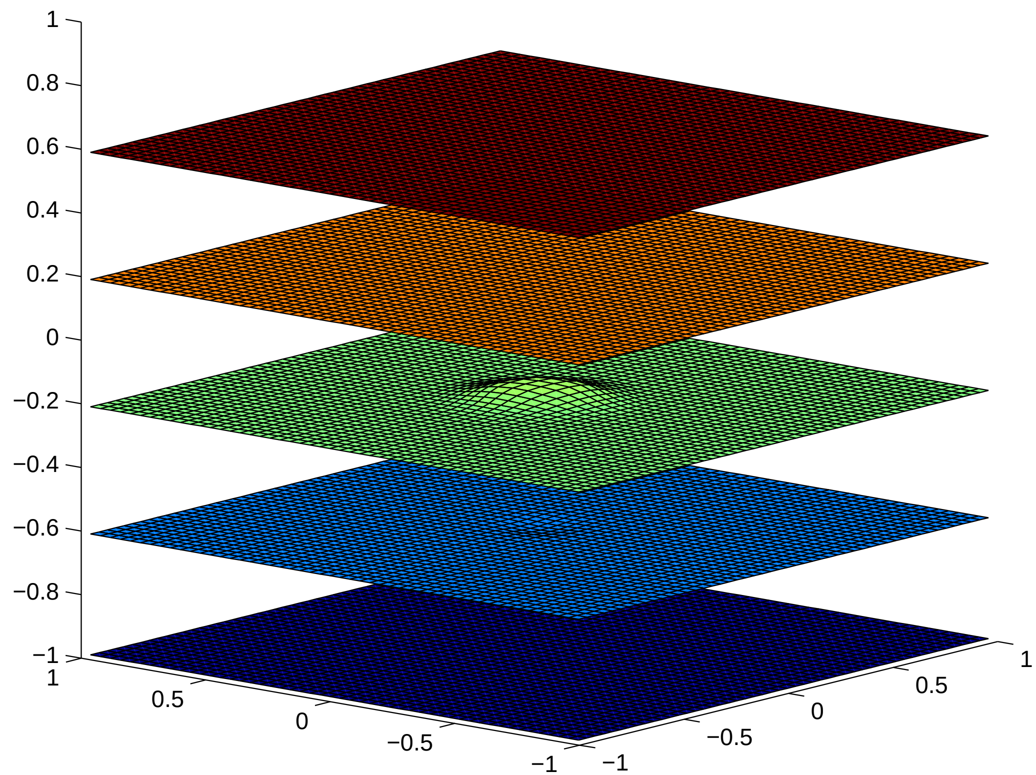
time $= 0$



3D Underwater Explosions (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$

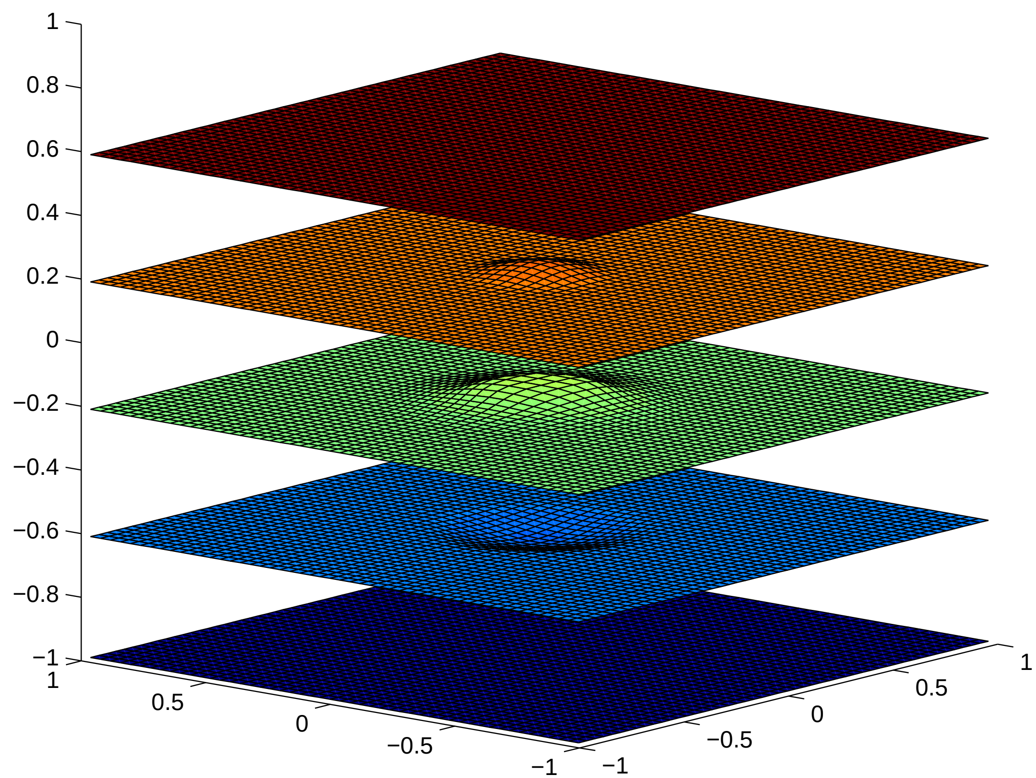
time $= 0.25$ms



3D Underwater Explosions (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$

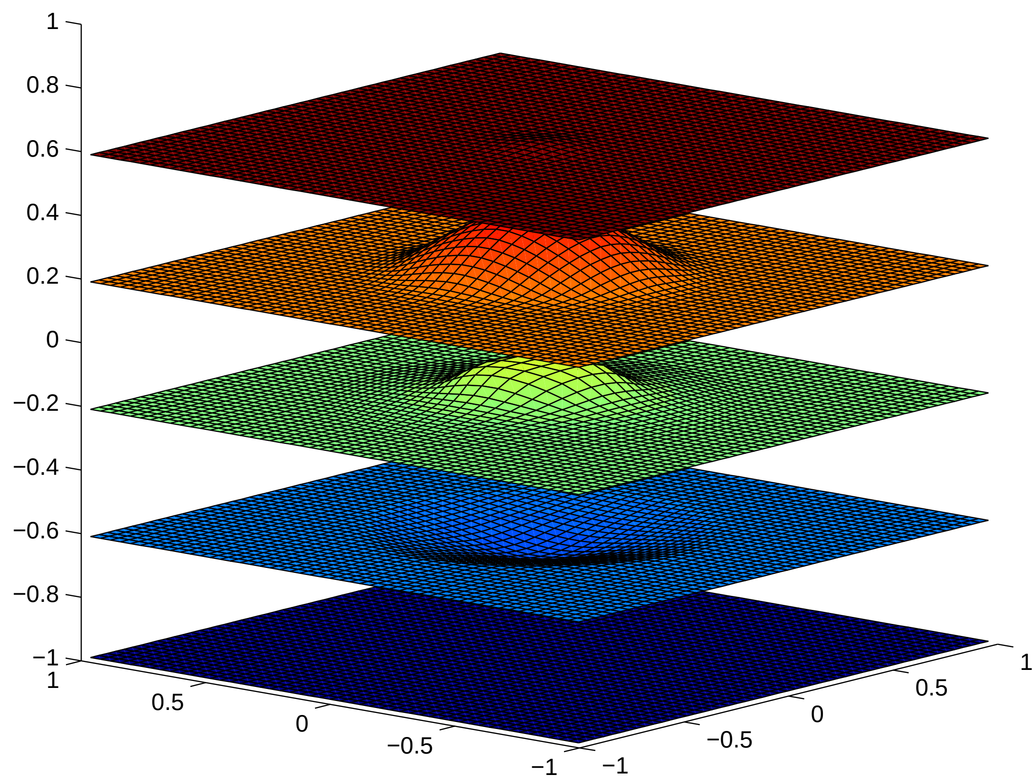
time $= 0.5$ms



3D Underwater Explosions (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$

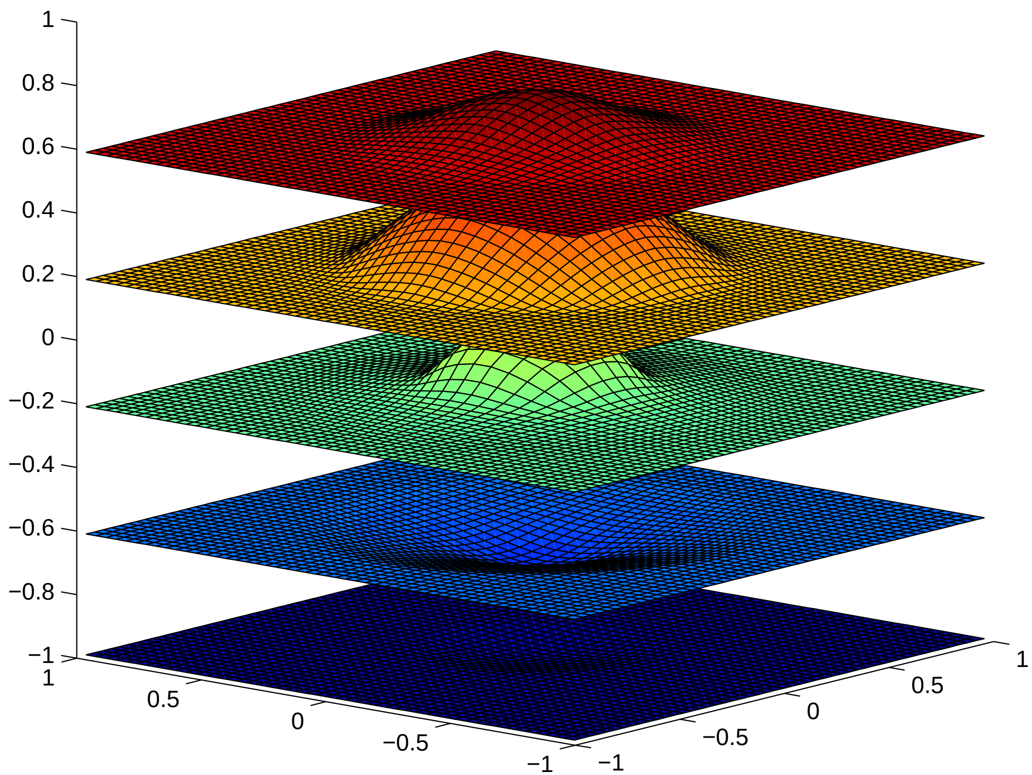
time $= 1.0$ms



3D Underwater Explosions (Cont.)

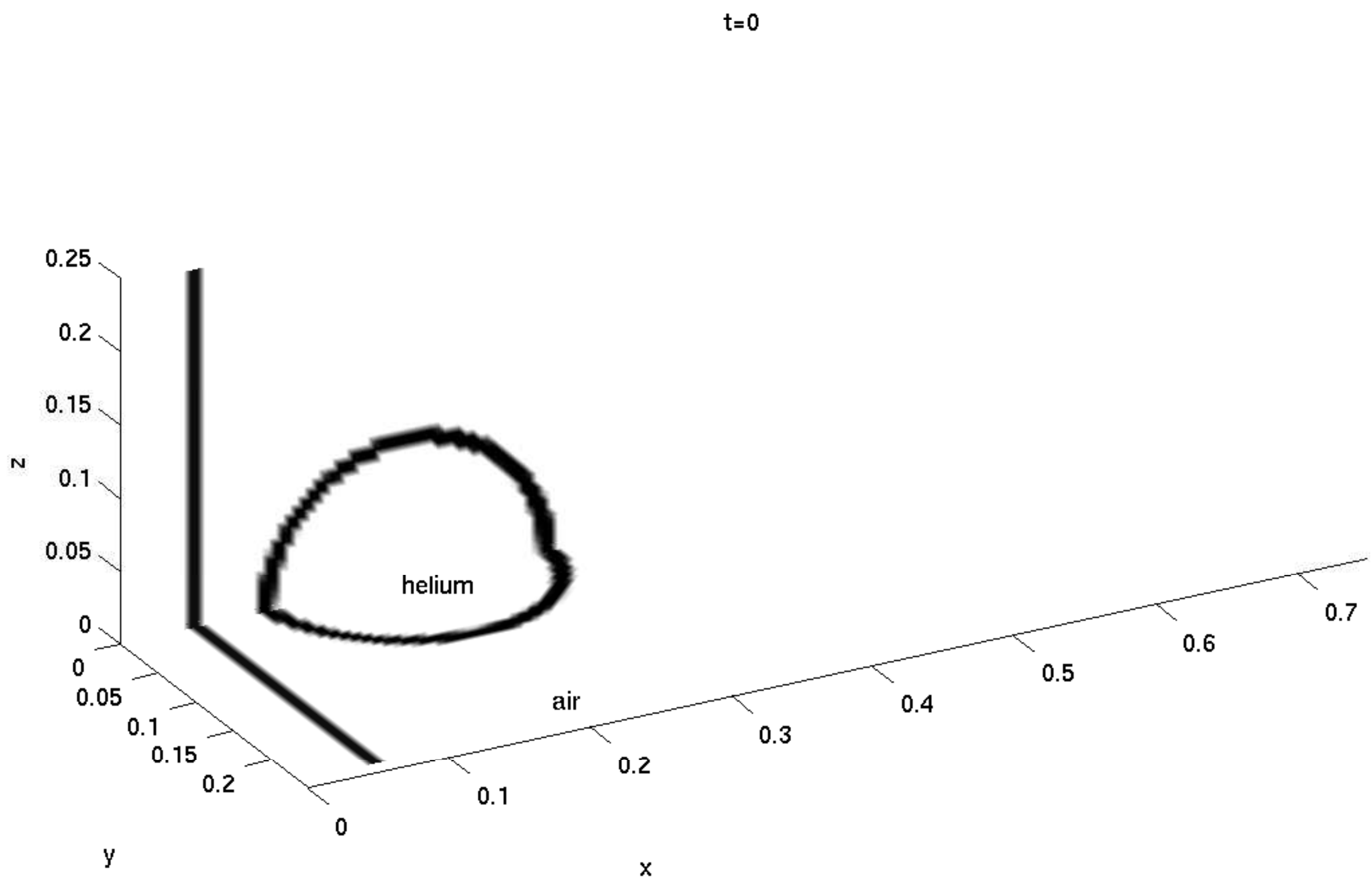
- Grid system (coarsen by factor 2) with $h_0 = 0.6$

time $= 1.5$ms



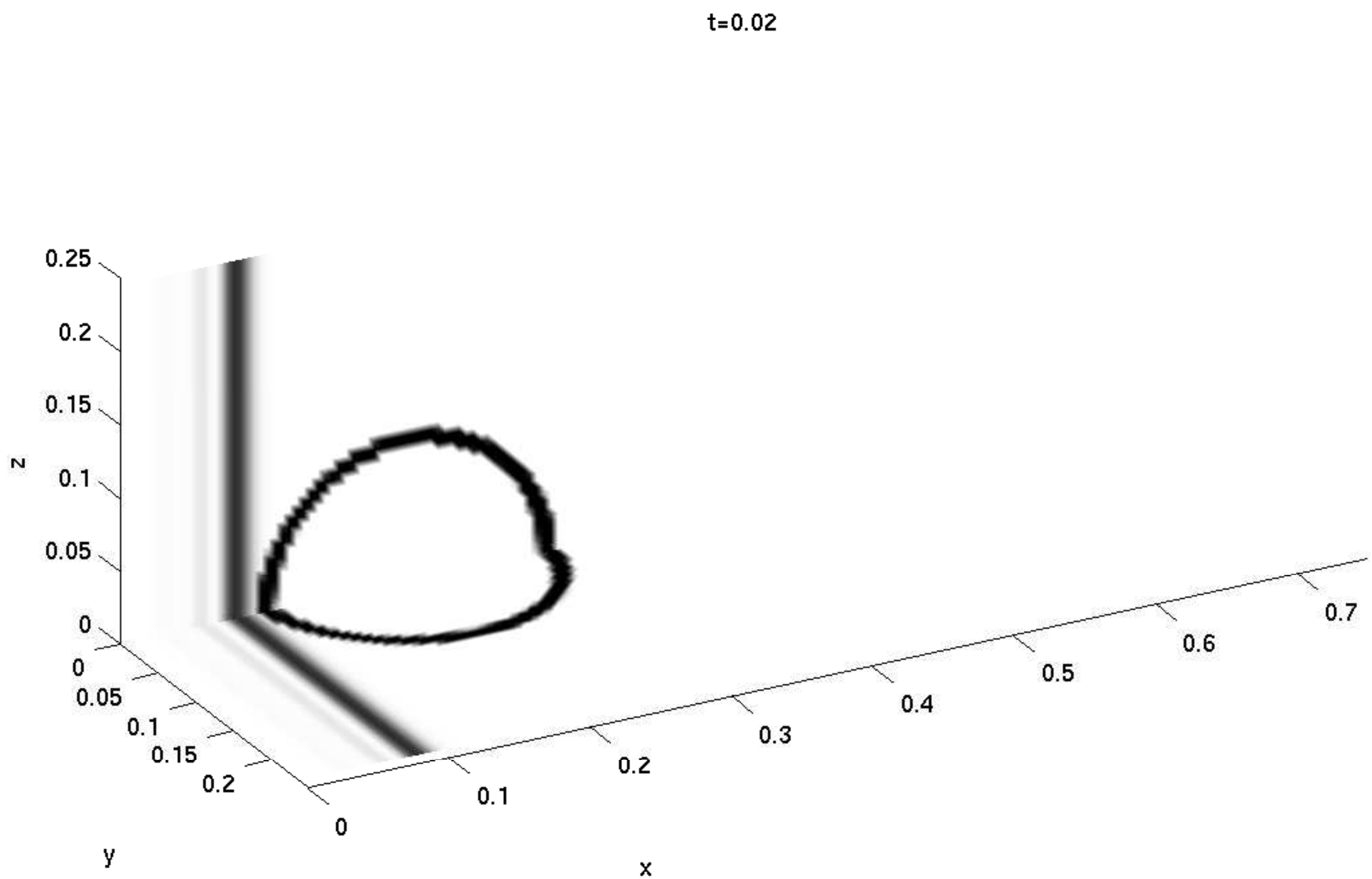
3D Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



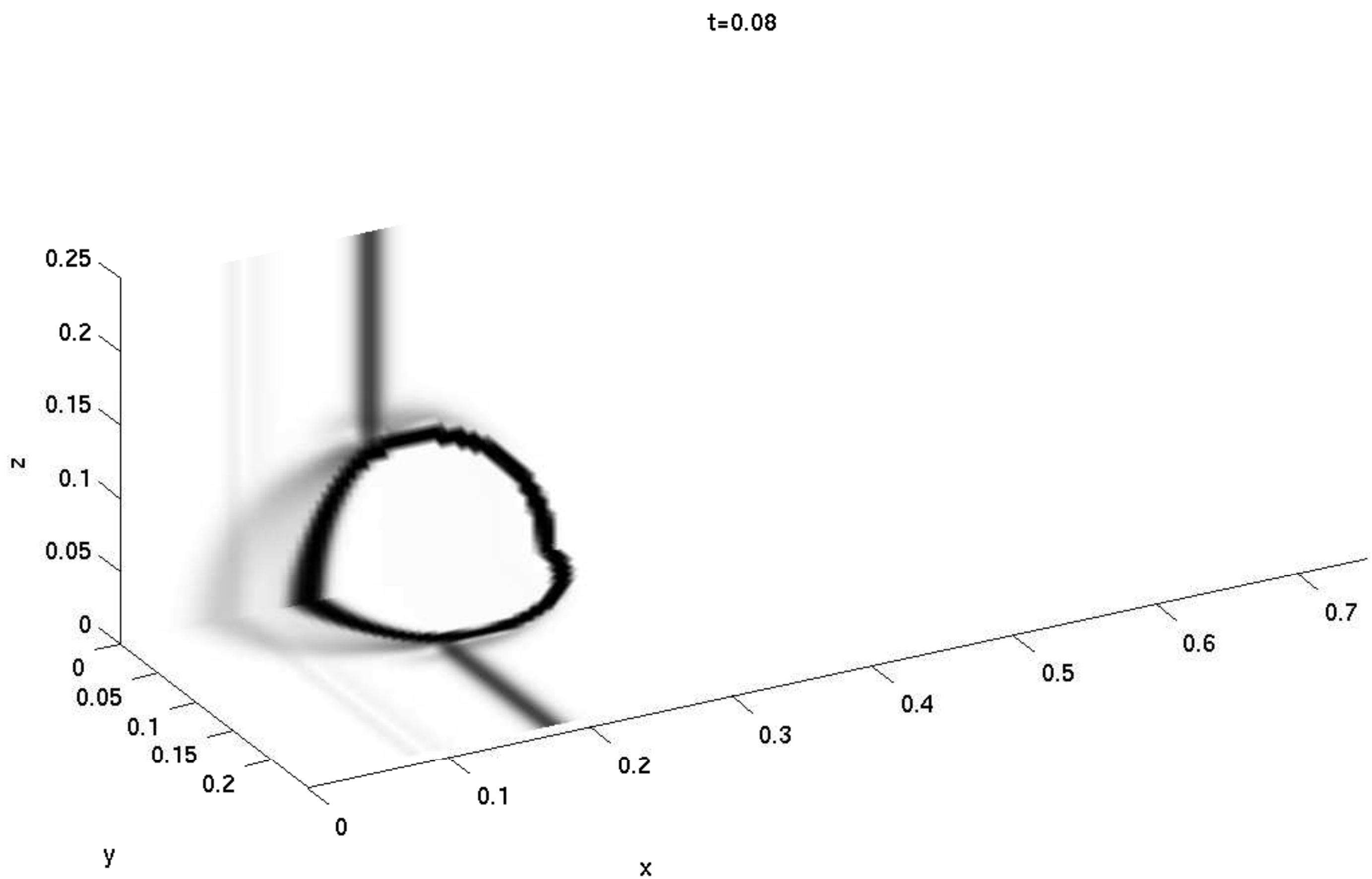
3D Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



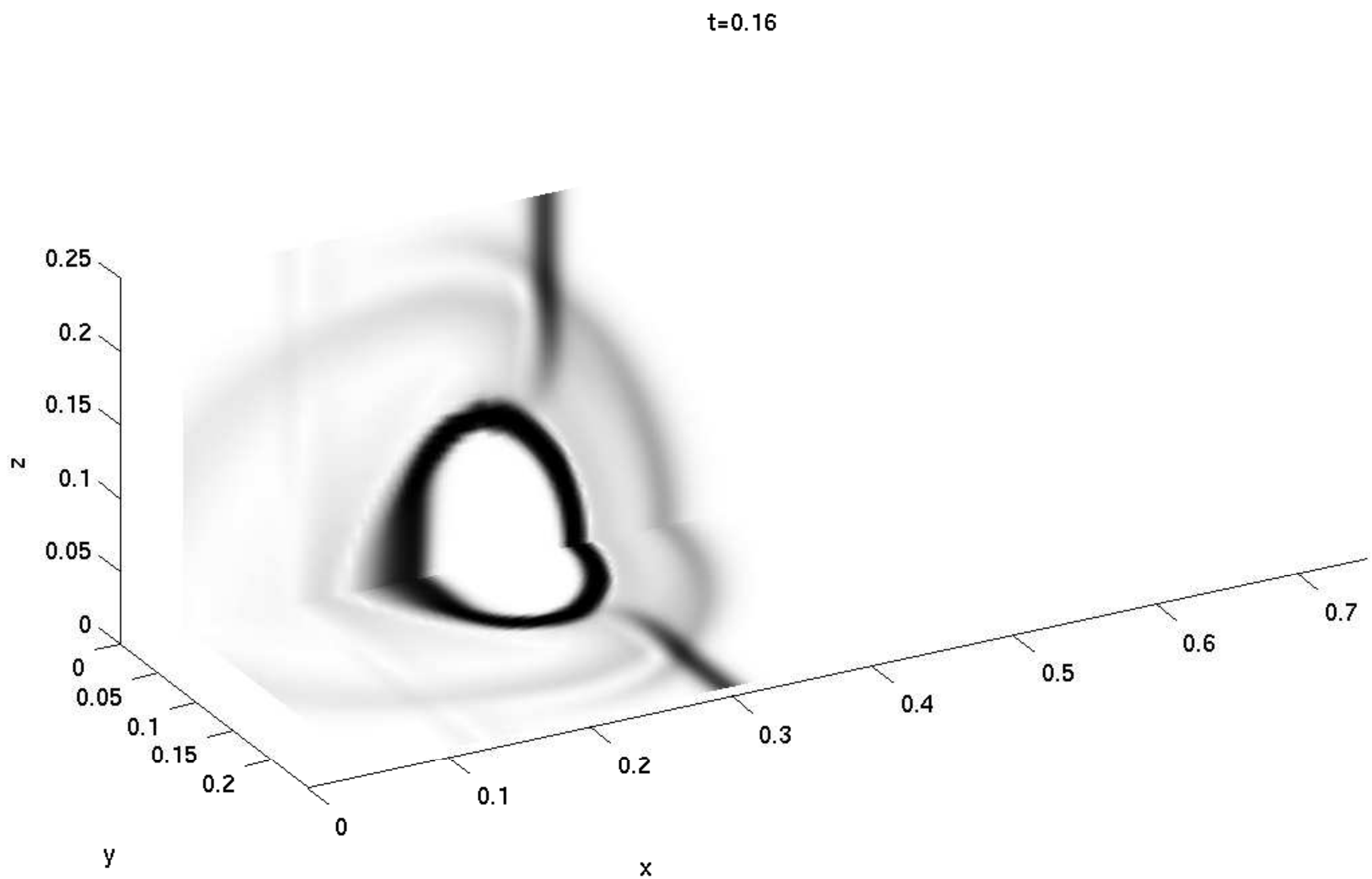
3D Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



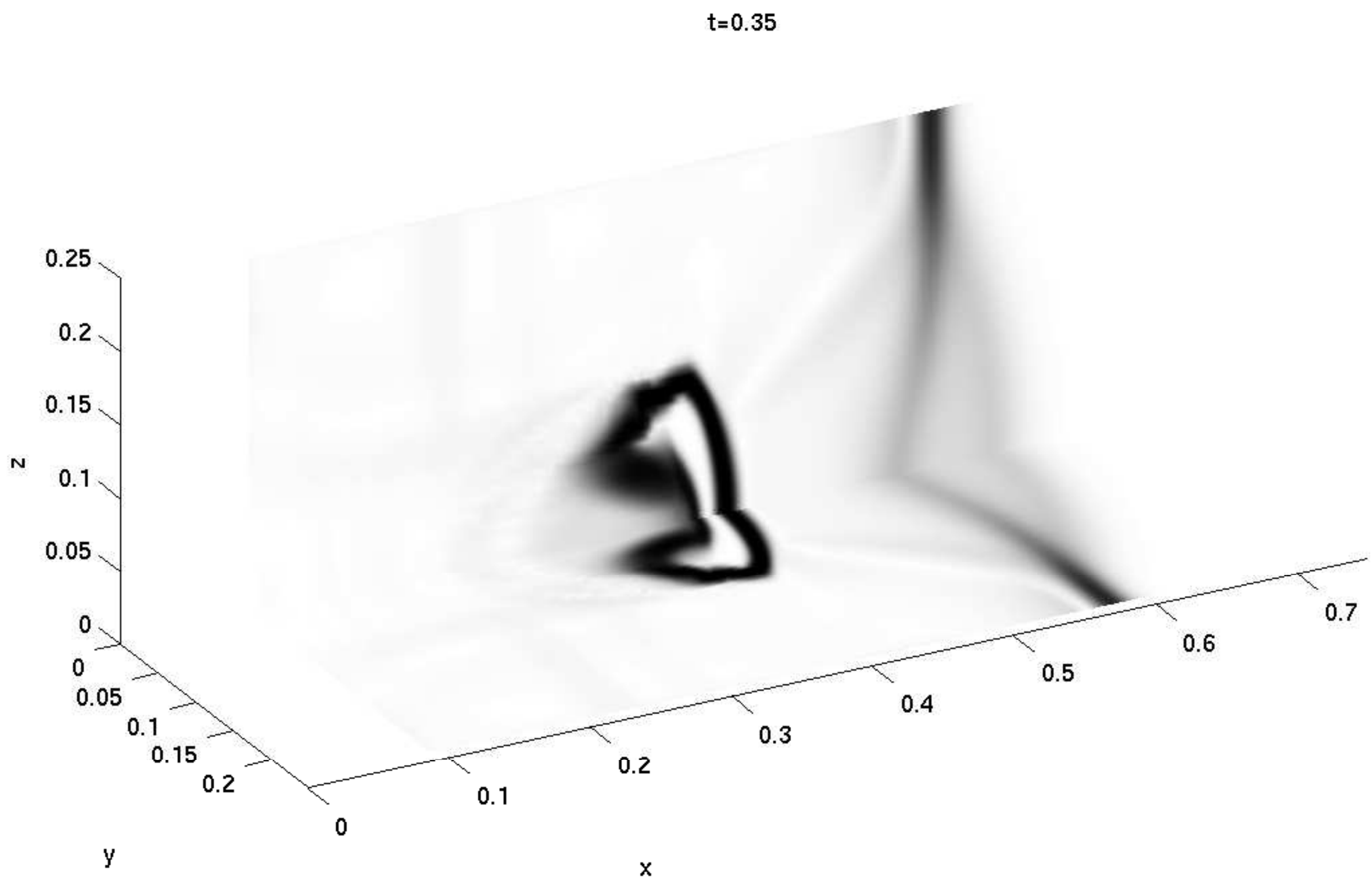
3D Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



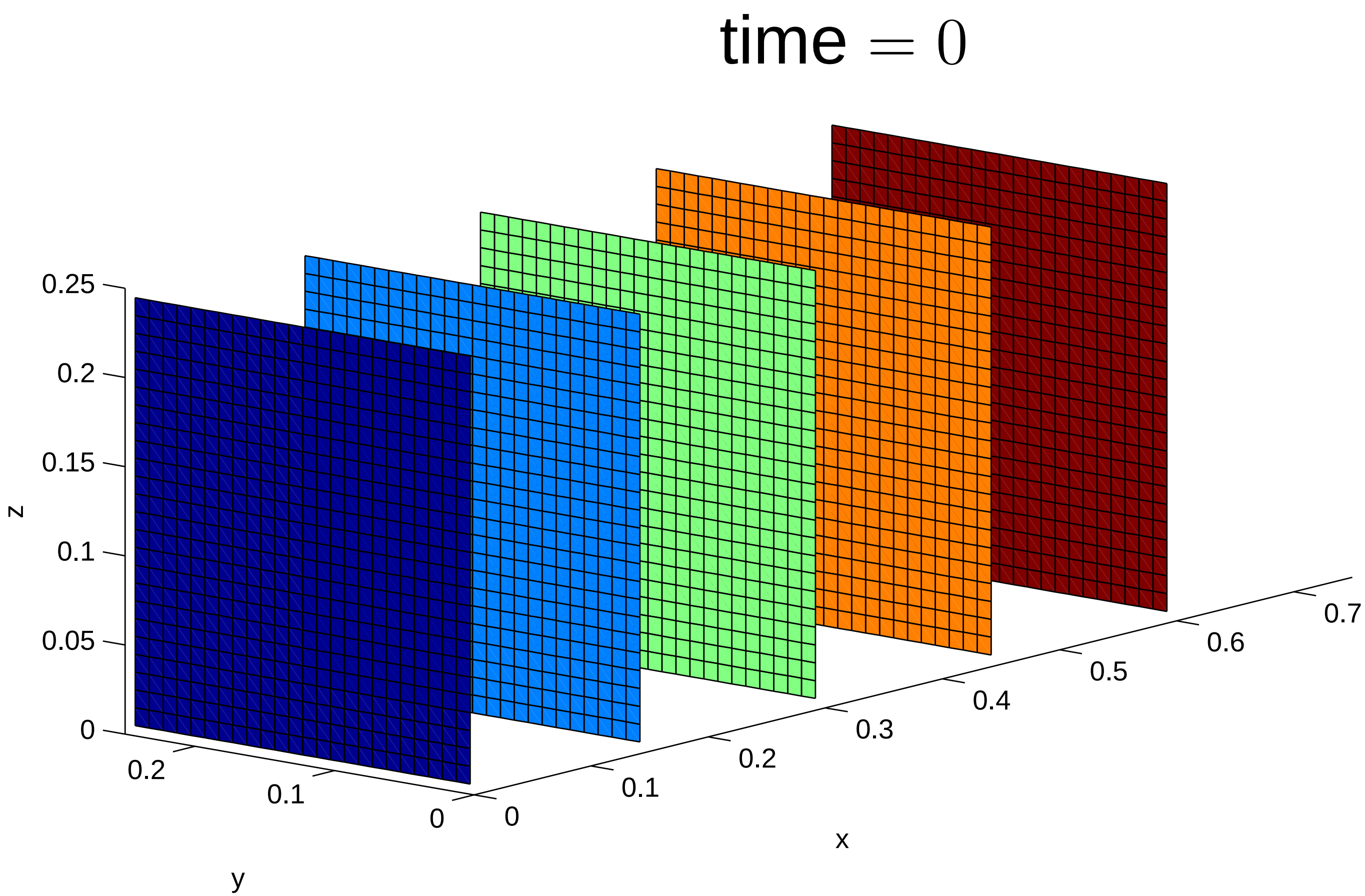
3D Shock-Bubble (Helium)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



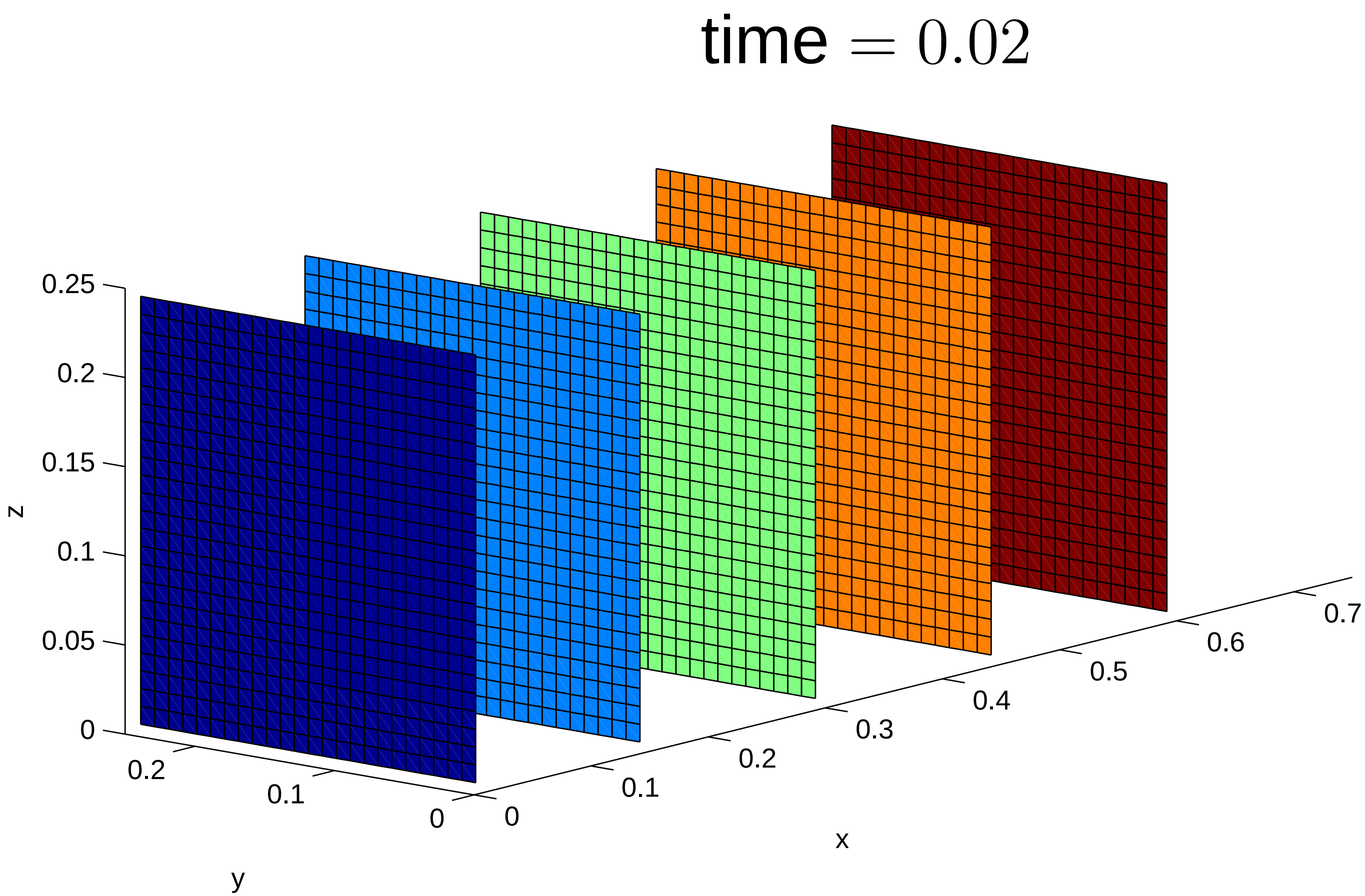
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



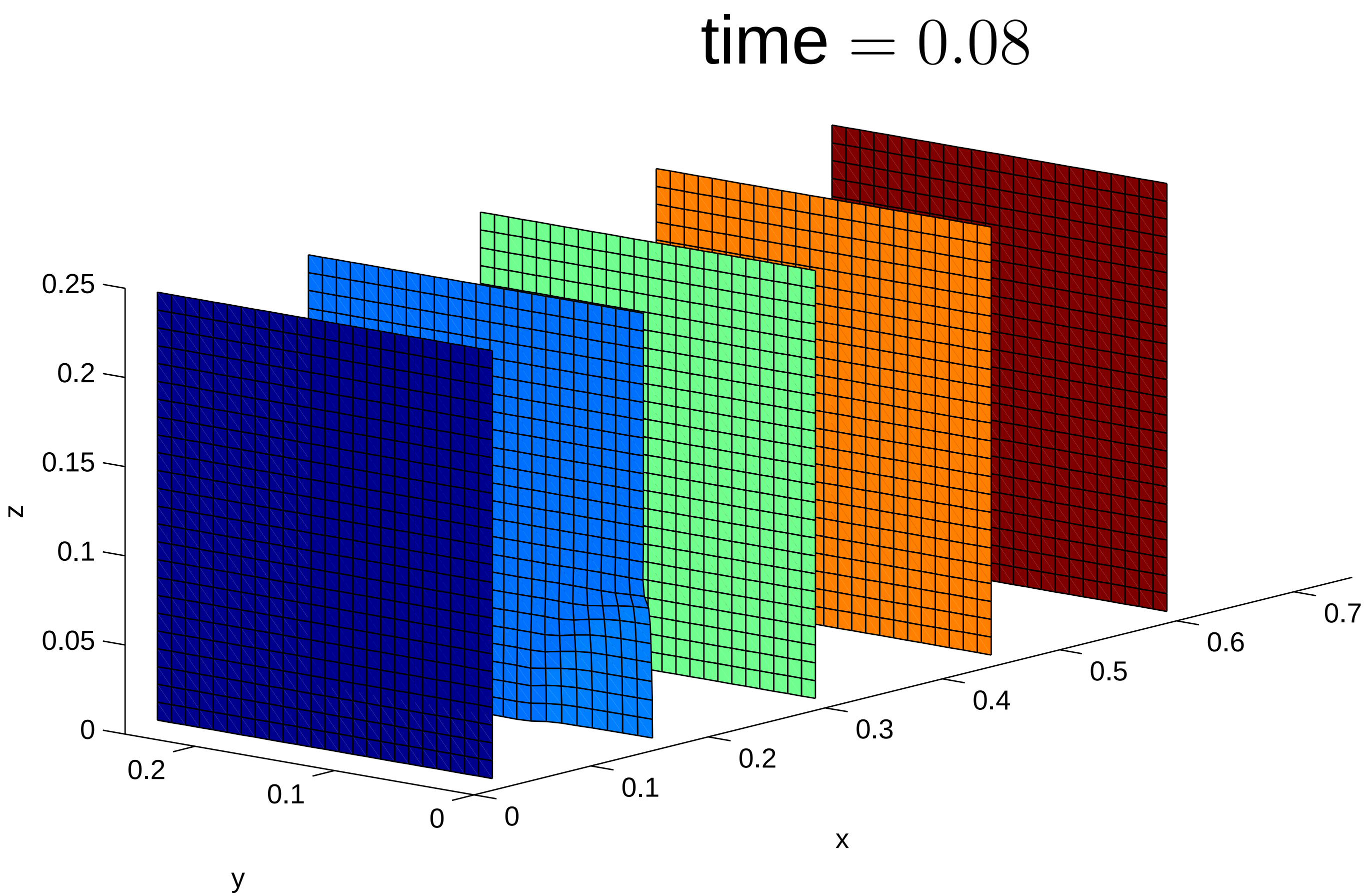
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



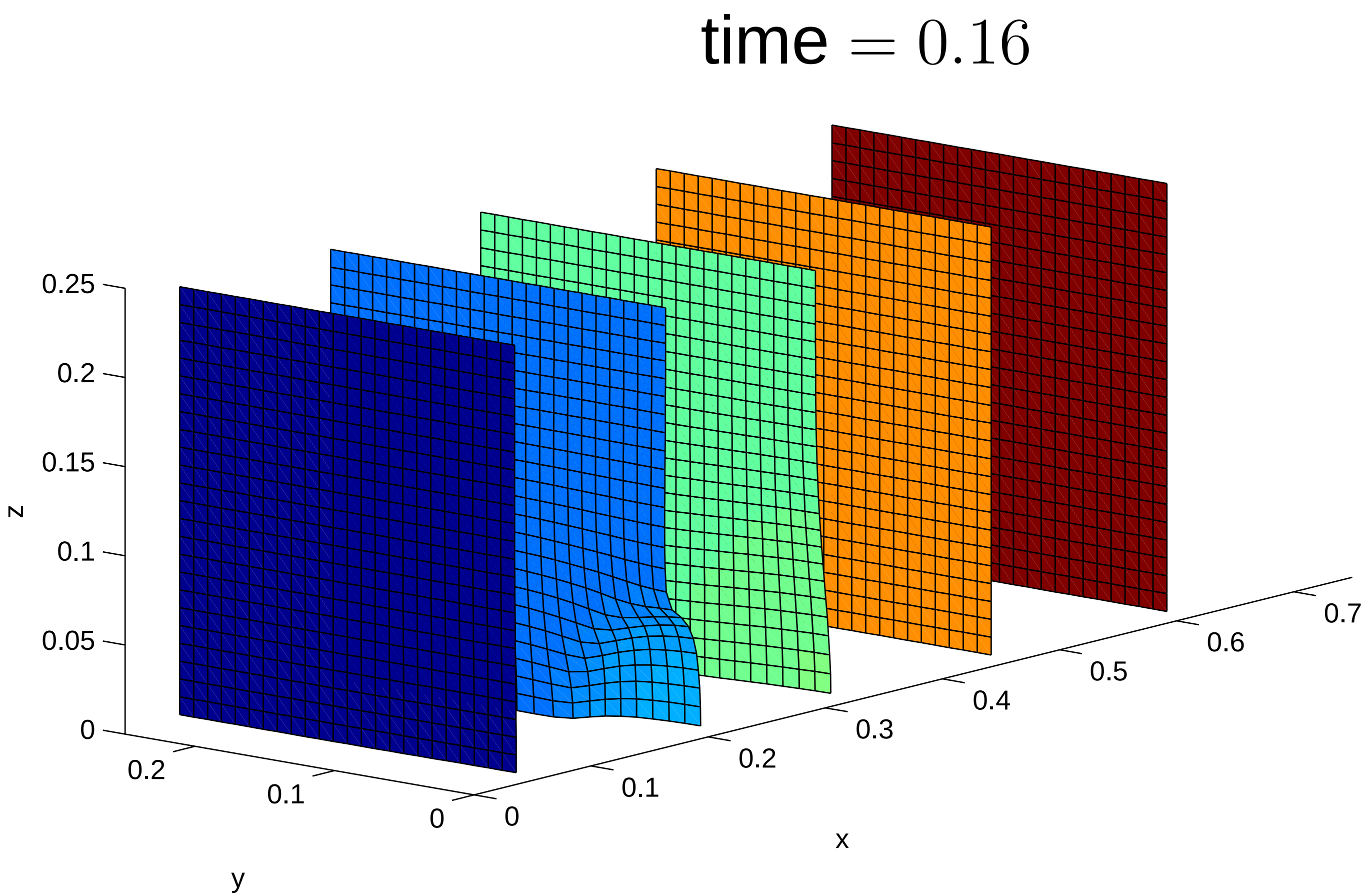
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



Shock-Bubble (Helium) (Cont.)

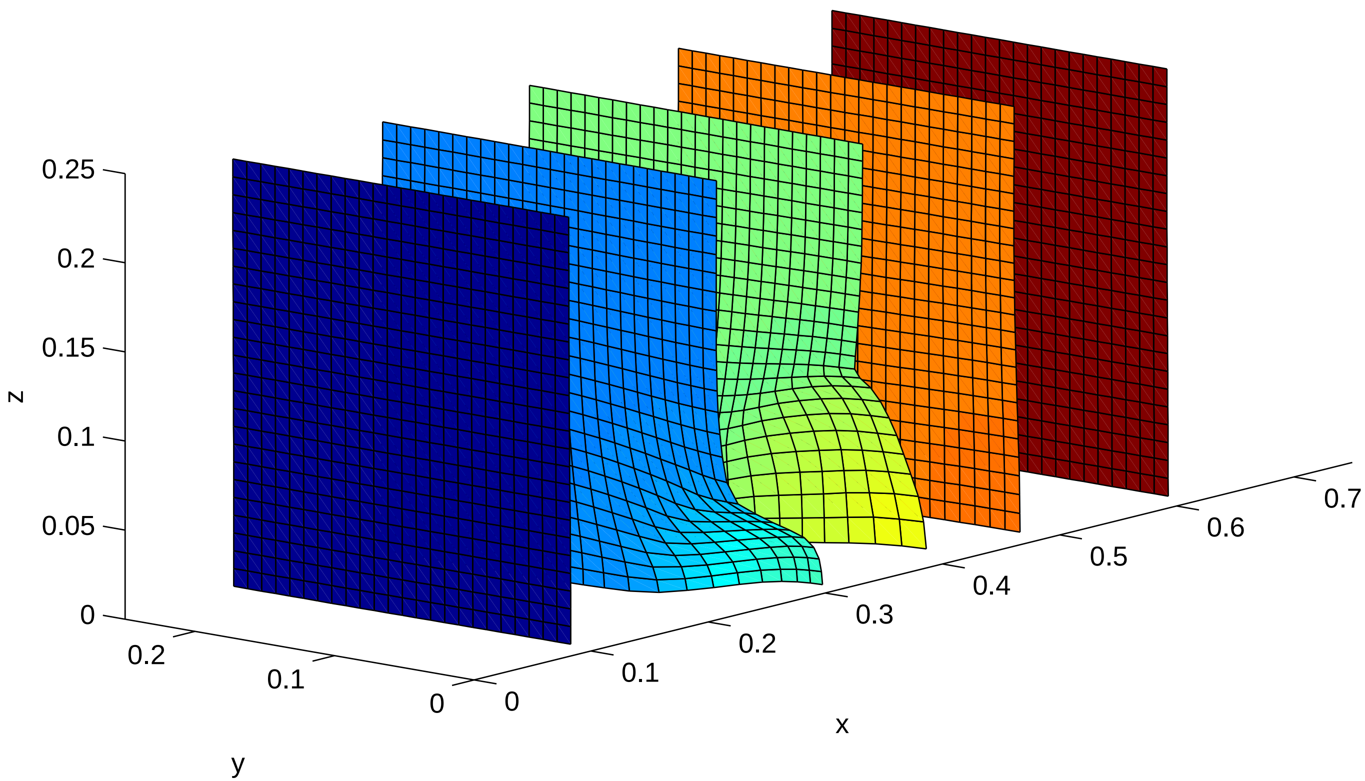
- Grid system (coarsen by factor 2) with $h_0 = 0.6$



Shock-Bubble (Helium) (Cont.)

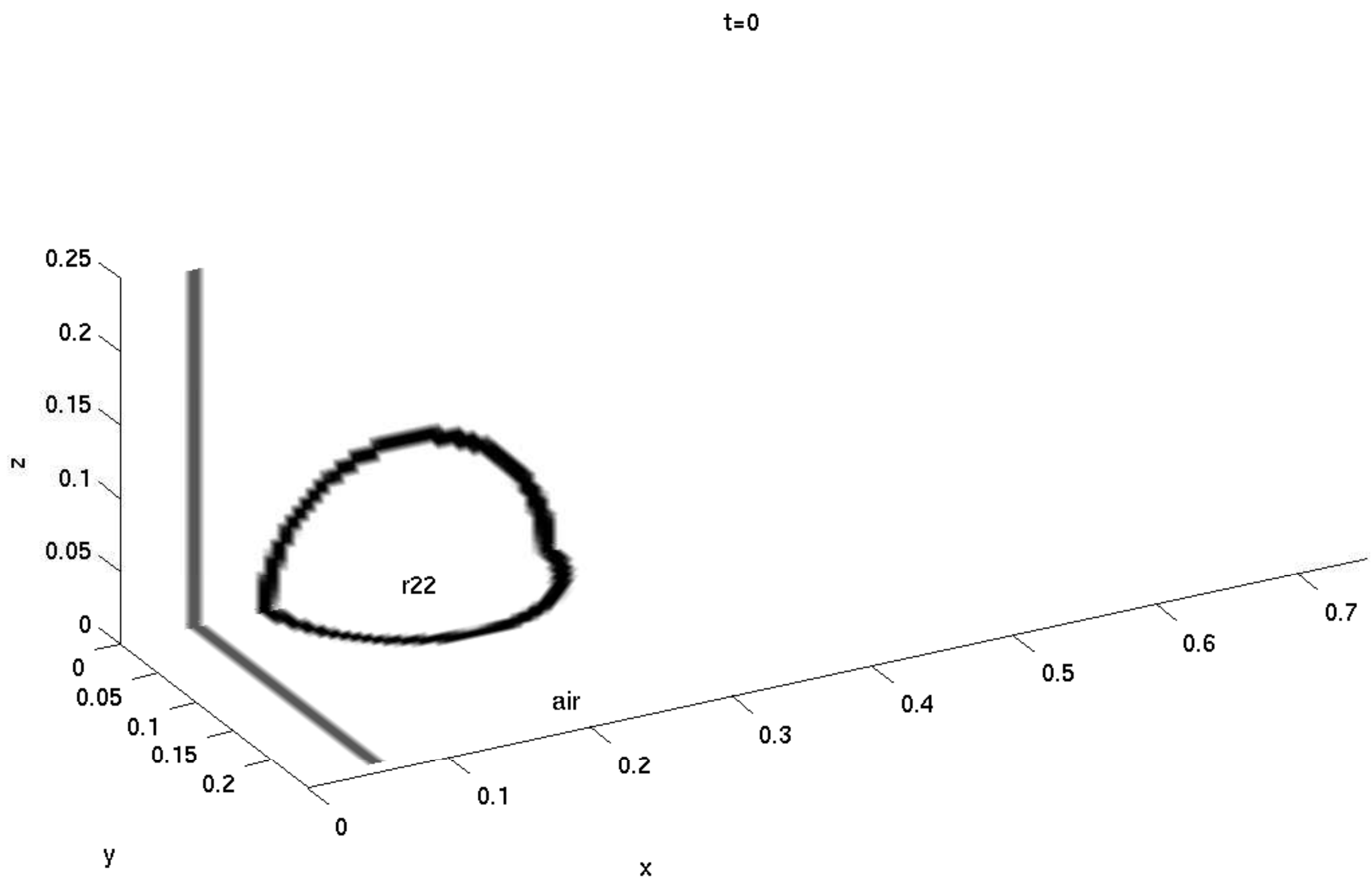
- Grid system (coarsen by factor 2) with $h_0 = 0.6$

time $= 0.35$



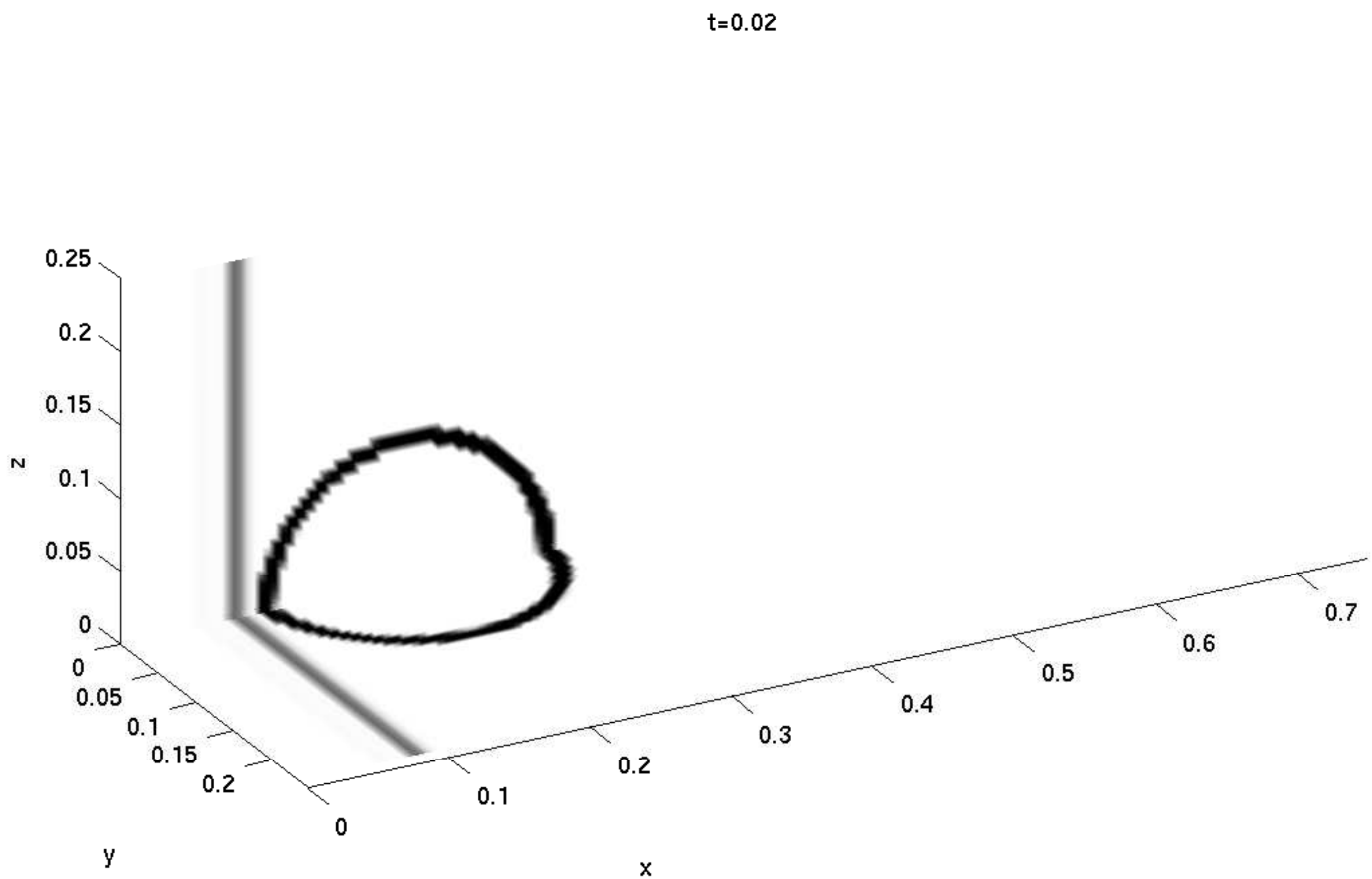
3D Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



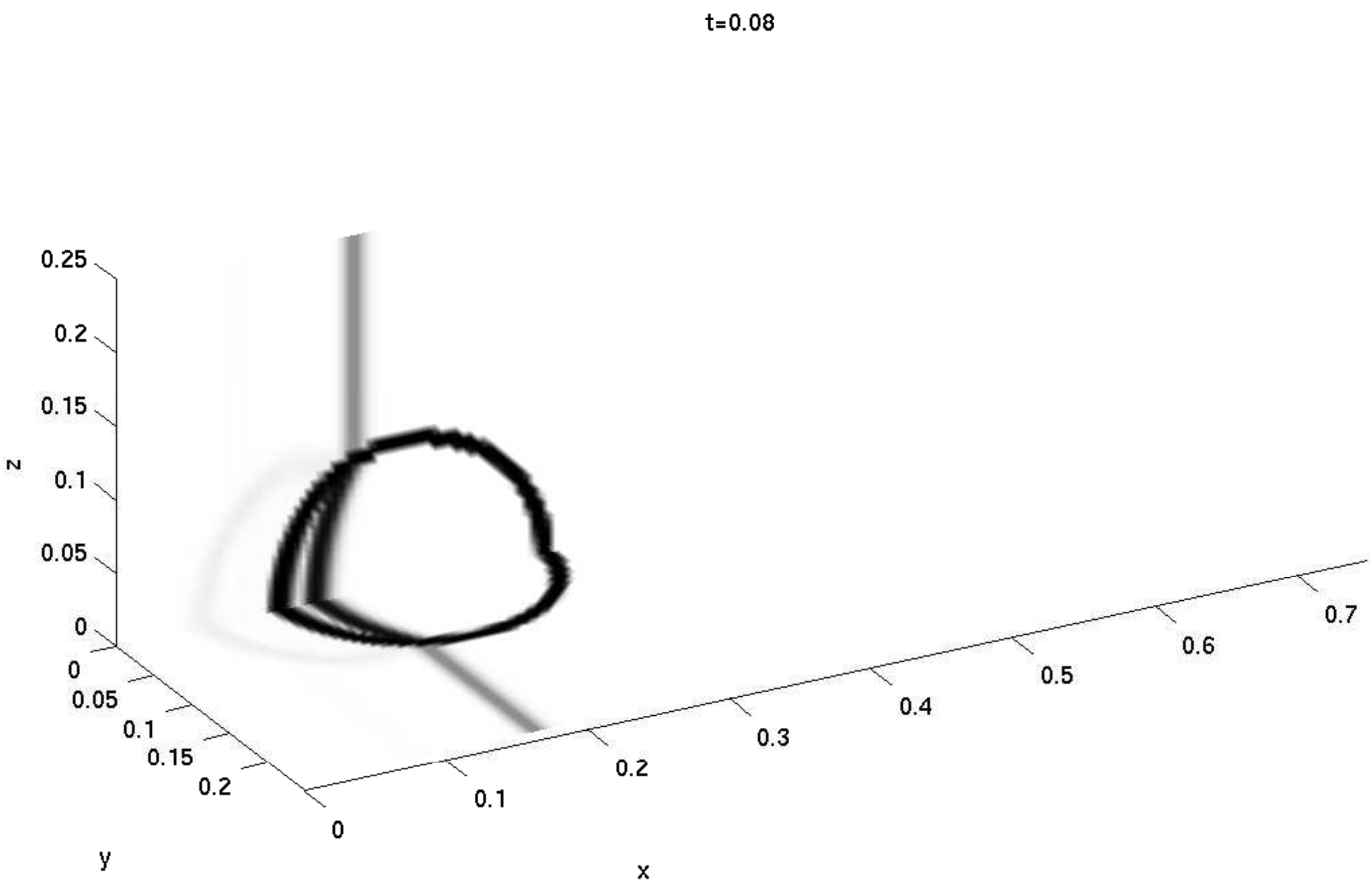
3D Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



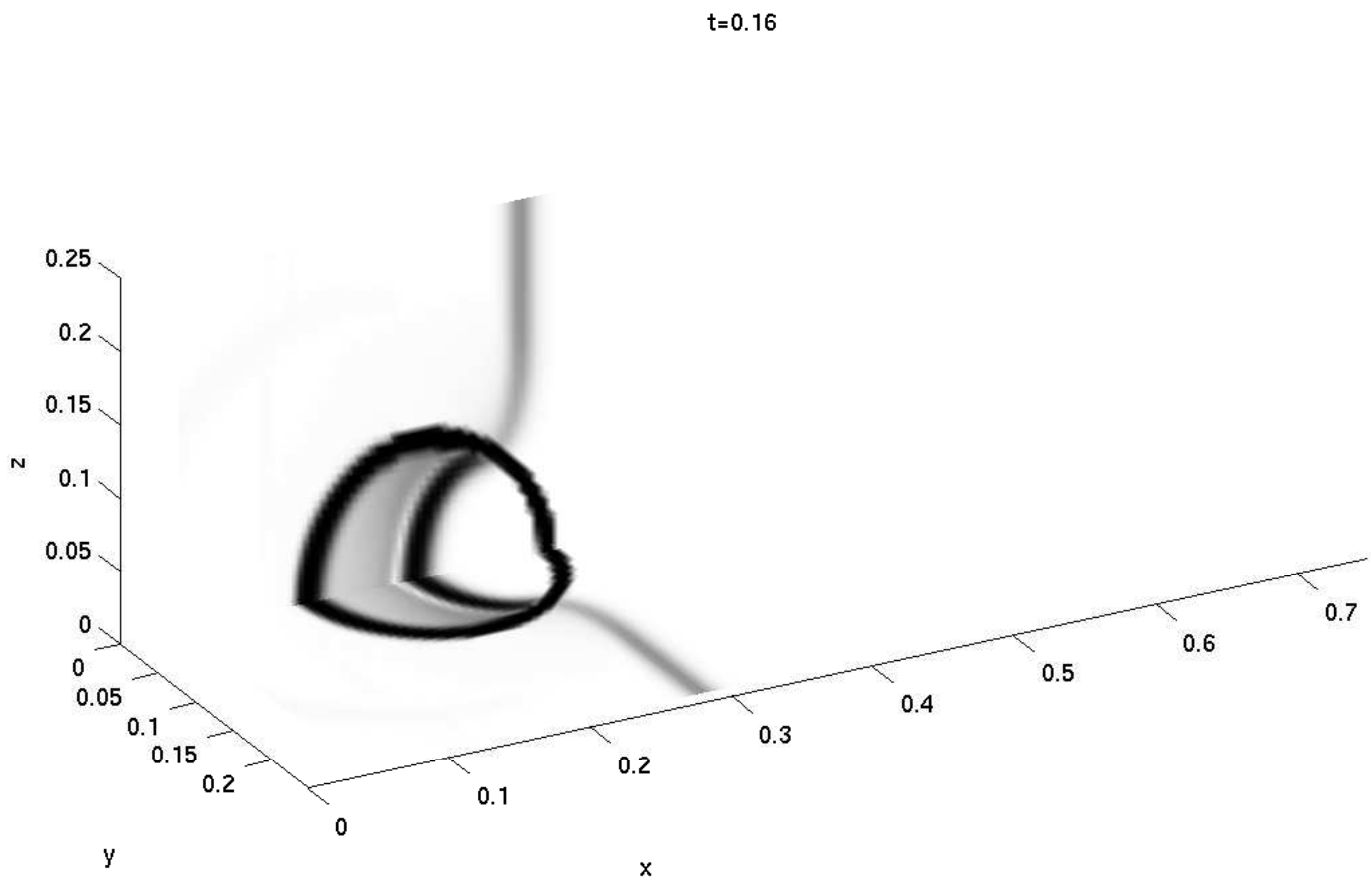
3D Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



3D Shock-Bubble (Refrigerant)

- Numerical schlieren images: $h_0 = 0.6$, $150 \times 50 \times 50$ grid



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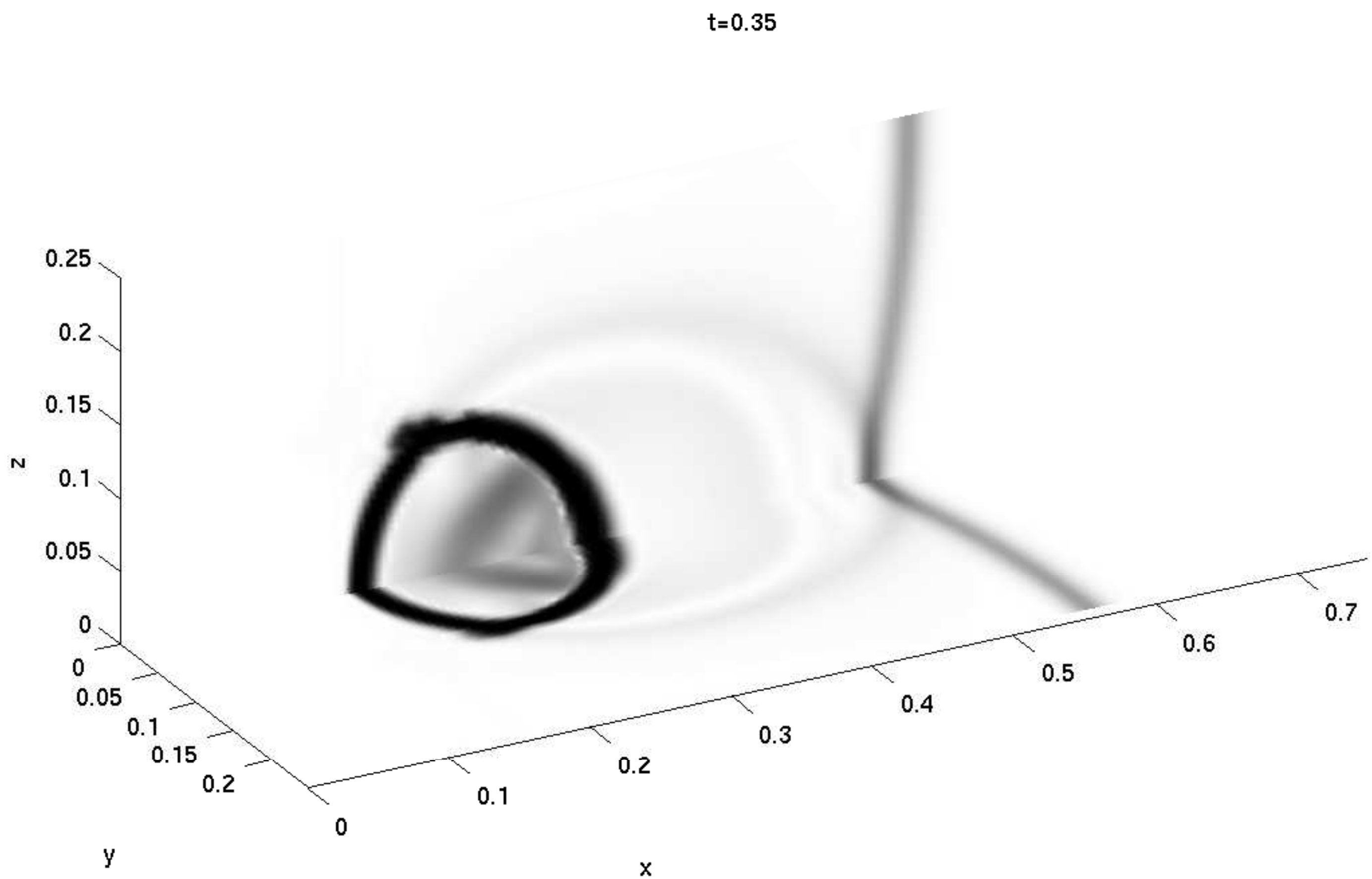
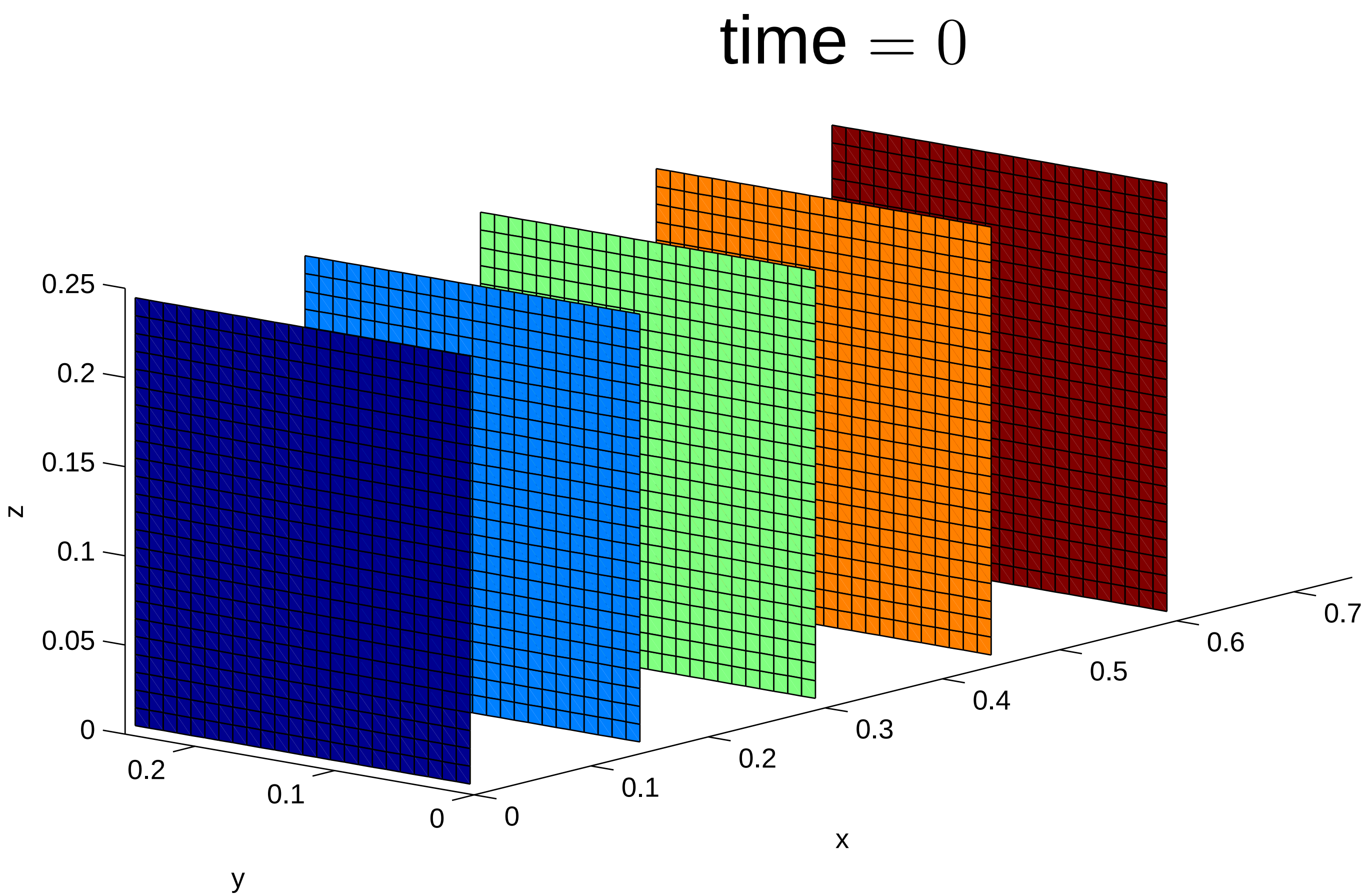


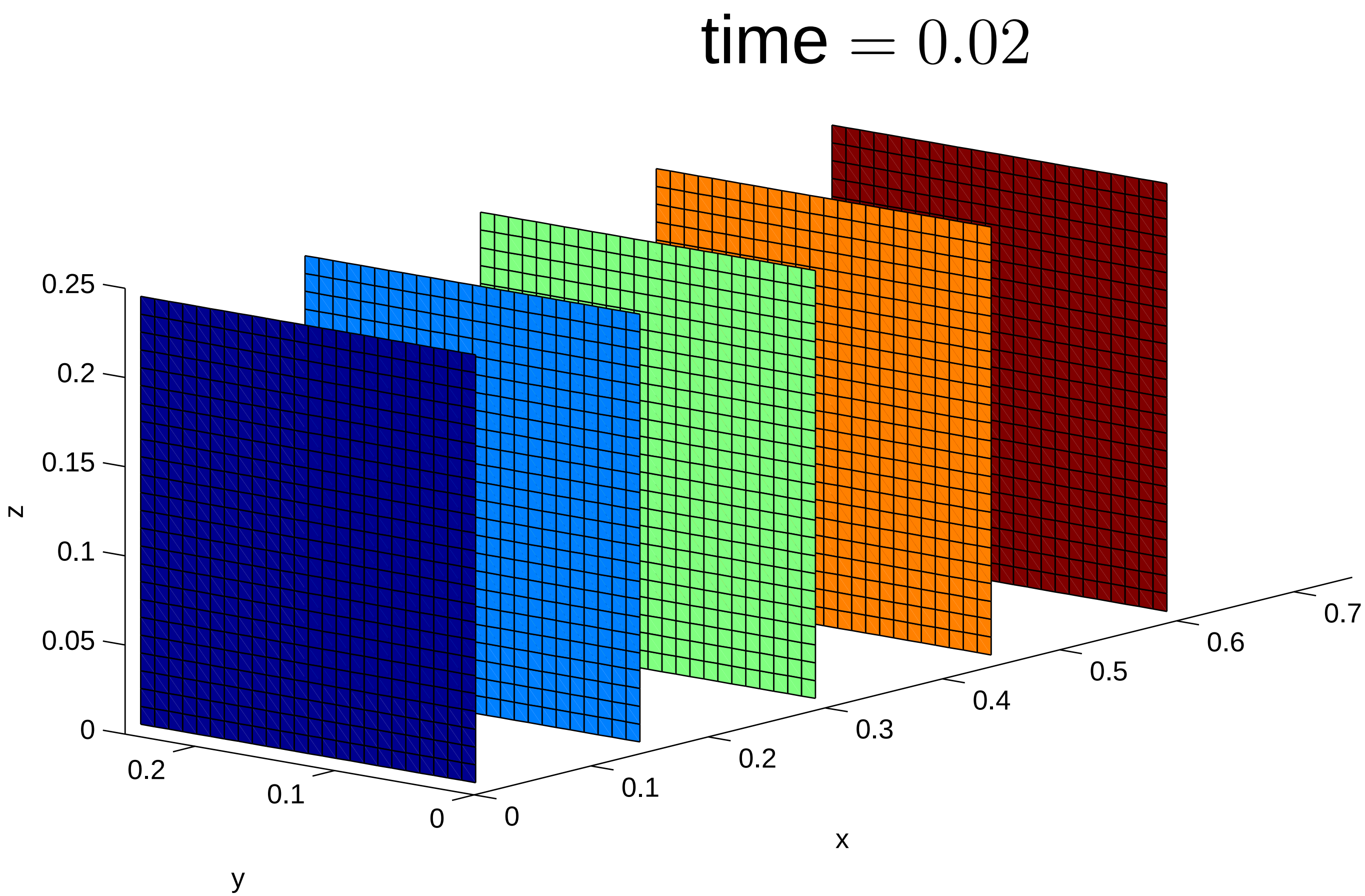
Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



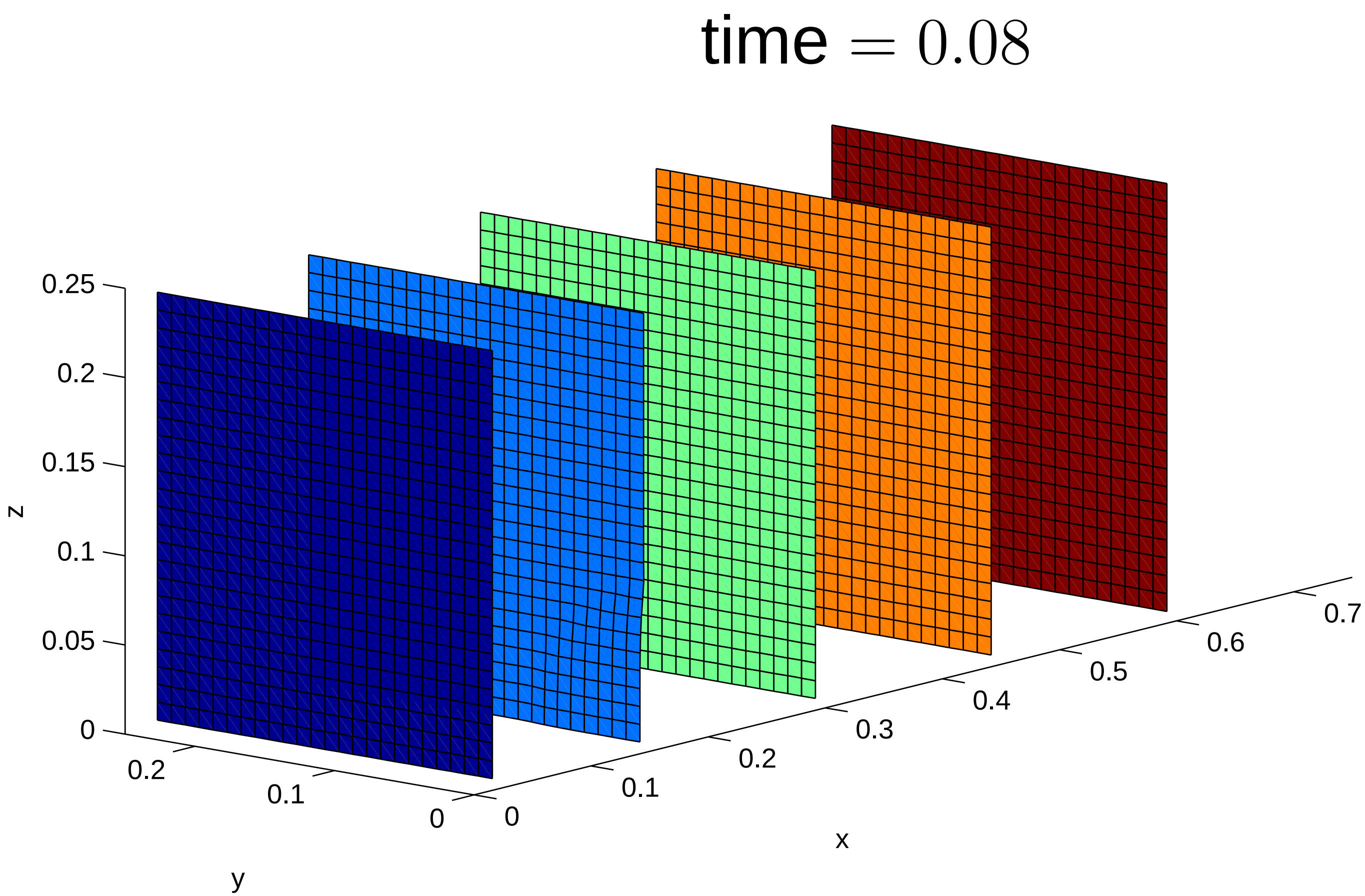
Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



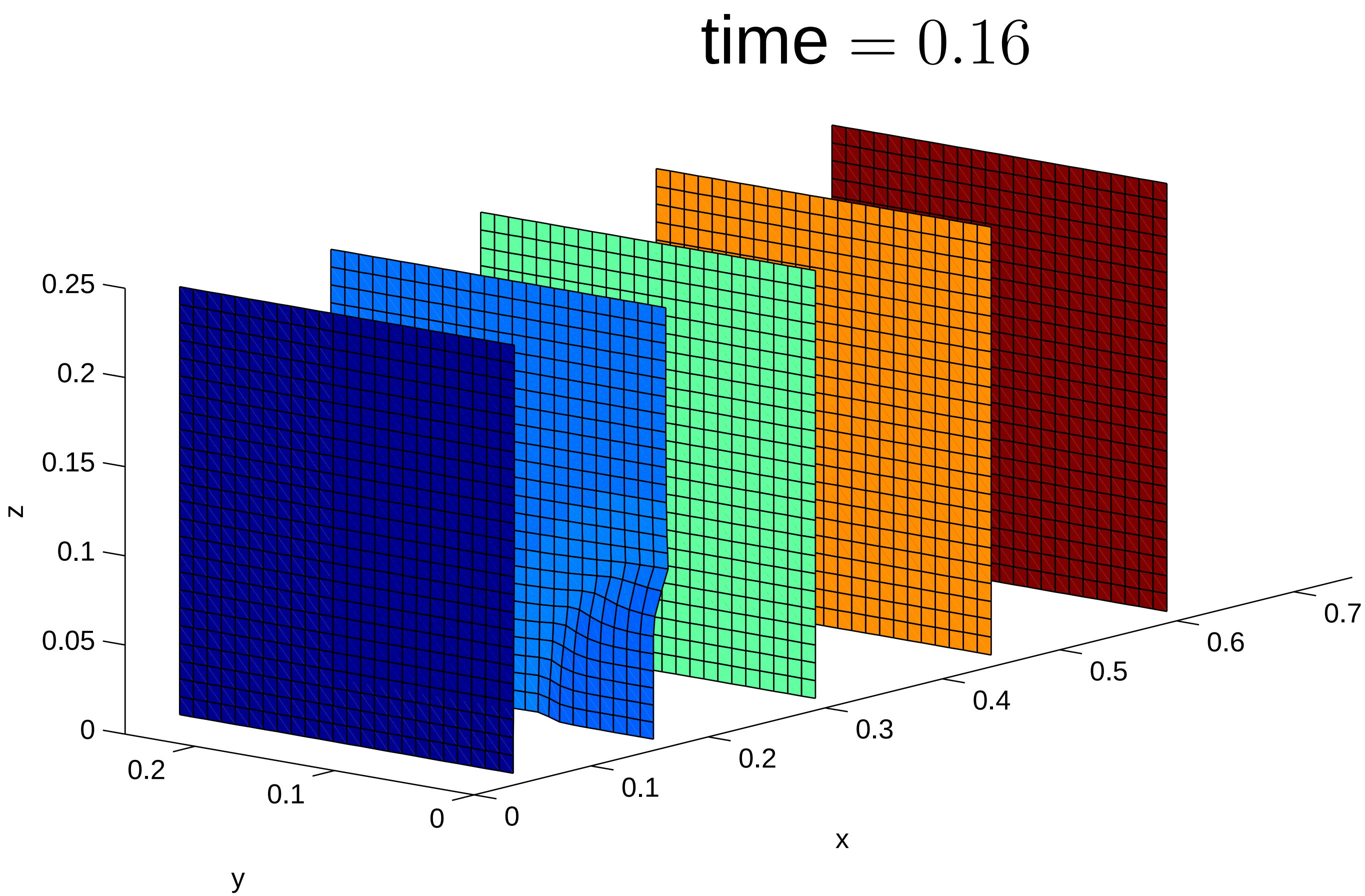
Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



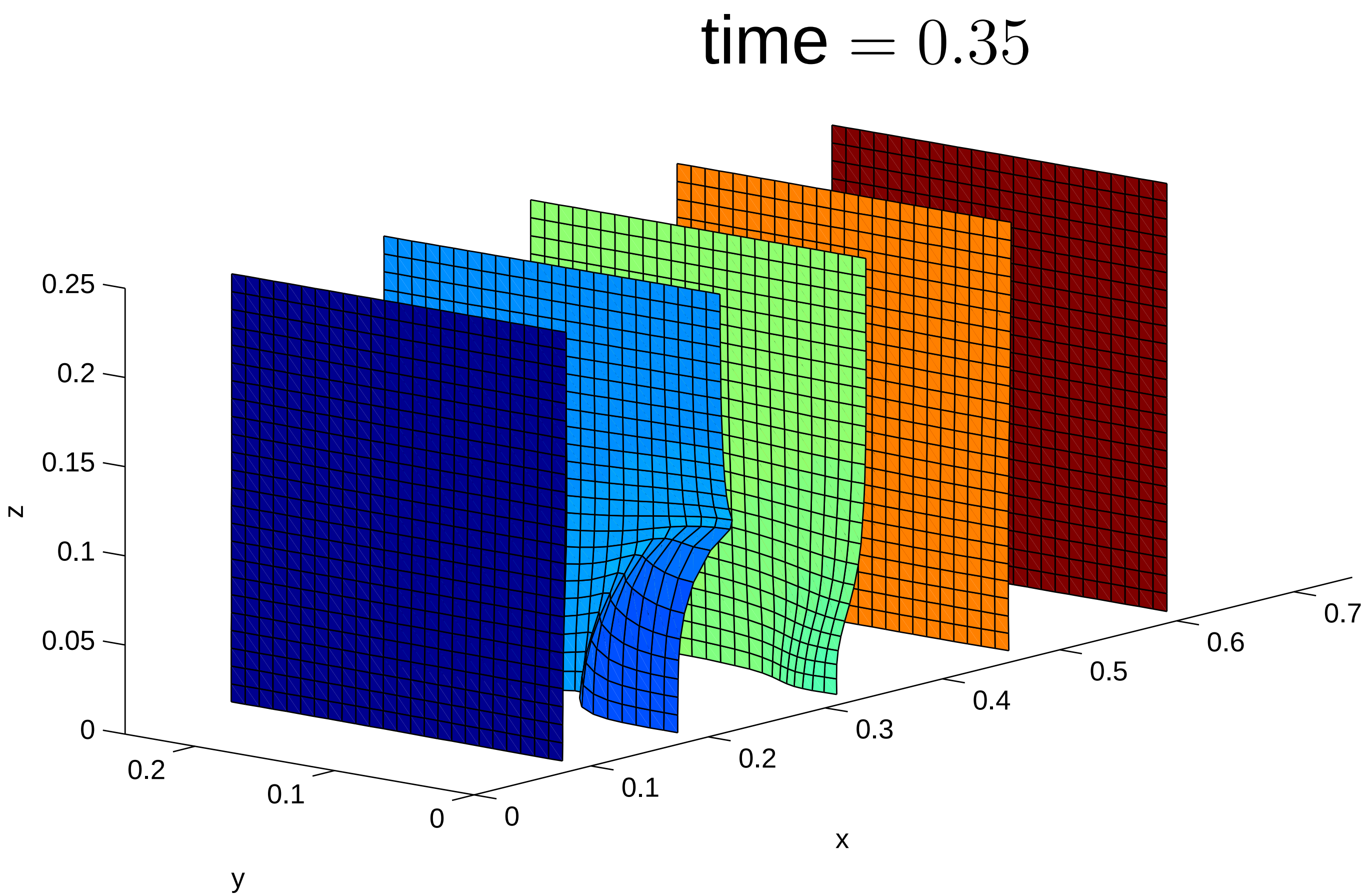
Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$



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  - Viscous flow extension
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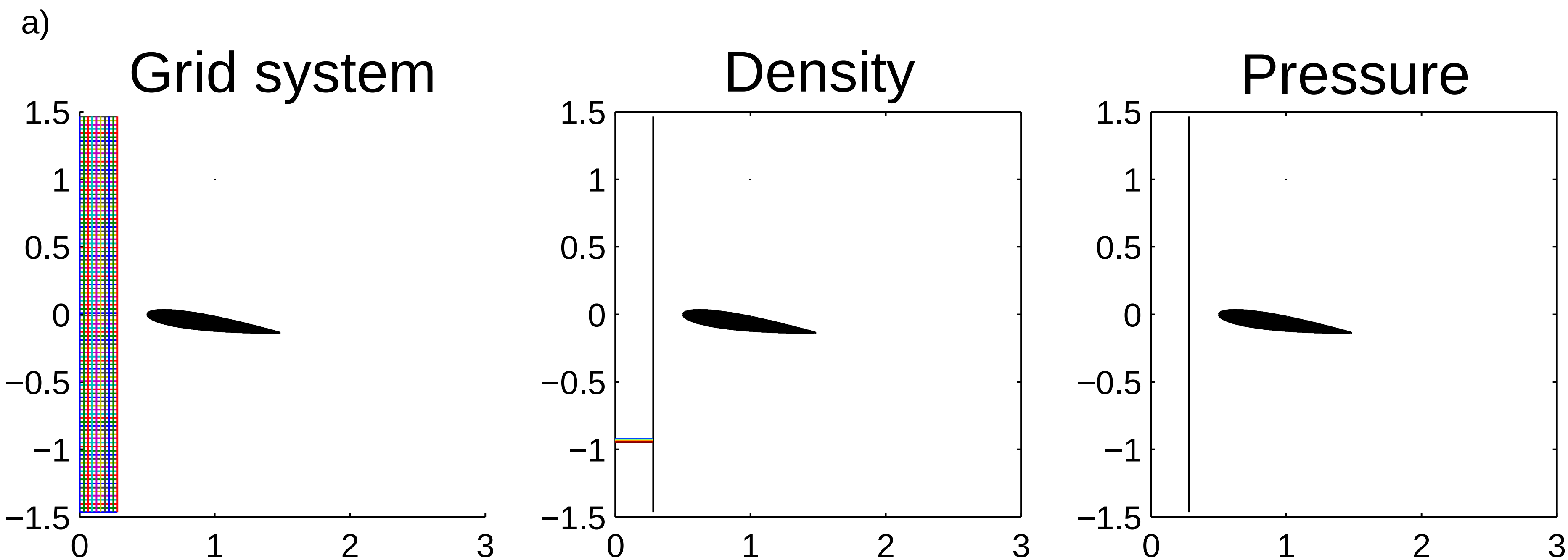
Thank You

Automatic Time-Marching Grid

- Supersonic NACA0012 over heavier gas



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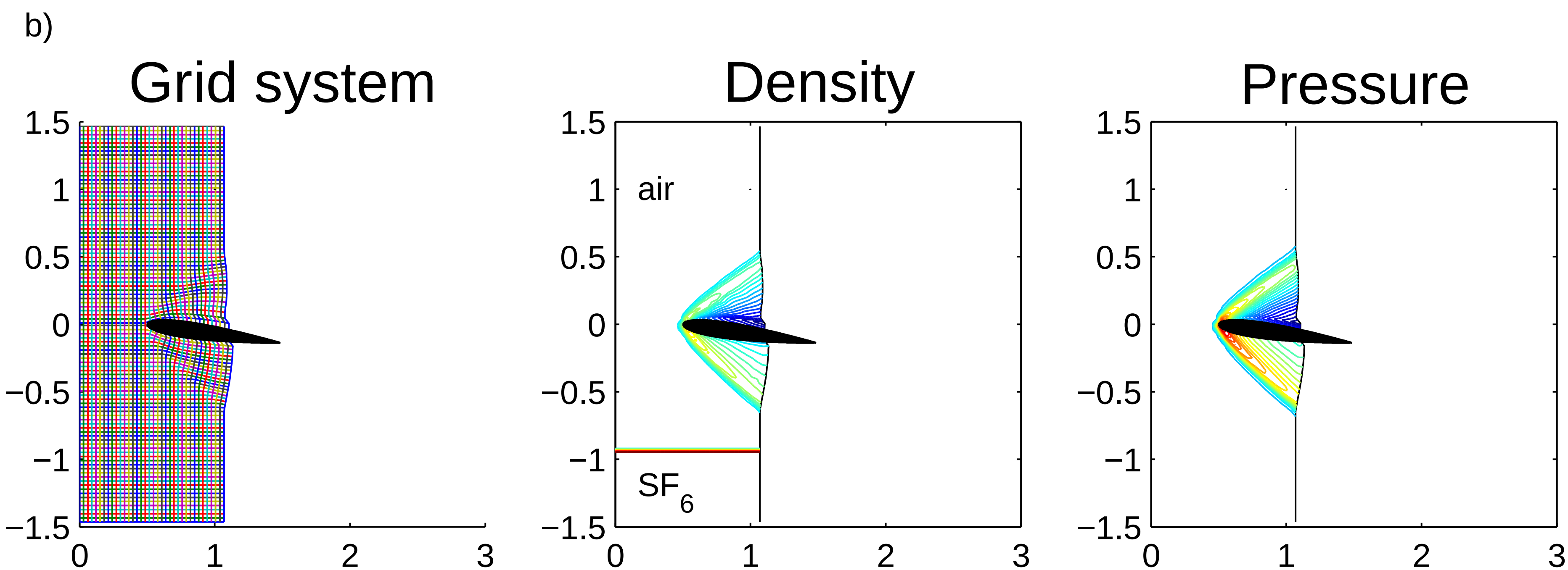

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