Eulerian interface-sharpening methods for hyperbolic problems

Application to compressible multiphase flow

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Outline

- Computing monotone sharp resolution of interfaces is of fundamental importance in many practical problems of interest
- Discuss a simple Eulerian interface sharpening approach (vs. Lagrangian, interface tracking, or adaptive moving mesh) for hyperbolic problems
  - Review two PDE-based interface sharpening techniques for solving volume-fraction linear transport equation that arises, for example, from viscous incompressibe 2-phase flow
  - Extend method for computing material lines or free surfaces arising from compressible multiphase flow

Incompressible 2-phase flow: Review

Consider unsteady, incompressible, viscous, immiscible 2-phase flow with governing equations

$$\nabla \cdot \vec{u} = 0 \qquad \text{(Continuity)}$$

$$\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla p = \nabla \cdot \tau + \rho \vec{g} + \vec{f}_\sigma \qquad \text{(Momentum)}$$

$$\partial_t \alpha + \vec{u} \cdot \nabla \alpha = 0 \qquad \text{(Volume-fraction transport)}$$

Material quantities in 2-phase coexistent region are often computed by α-based weighted average as

$$\rho = \alpha \rho_1 + (1 - \alpha) \rho_2, \quad \epsilon = \alpha \epsilon_1 + (1 - \alpha) \epsilon_2, \qquad \alpha \in [0, 1],$$

where source terms are volume-fraction dependent

$$\tau = \epsilon \left(\nabla \vec{u} + \nabla \vec{u}^T \right), \quad \vec{f}_\sigma = -\sigma \kappa \nabla \alpha \quad \text{with } \kappa = \nabla \cdot \left(\frac{\nabla \alpha}{|\nabla \alpha|} \right)$$

Interface sharpening techniques

Typical interface sharpening methods for computing volume fraction in incompressible 2-phase flow include:

- Algebraic based approach
  - CICSAM (Compressive Interface Capturing Scheme for Arbitrary Meshes): Ubbink & Issa JCP 1999
  - THINC (Tangent of Hyperbola for INterface Capturing): Xiao, Honma & Kono Int. J. Numer. Meth. Fluids 2005
  - Improved THINC: Xiao *et al.*
- PDE based approah
  - Artificial compression: Harten CPAM 1977, Olsson & Kreiss JCP 2005
  - Anti-diffusion: So, Hu & Adams JCP 2011

Artificial interface compression

Our first interface-sharpening model concerns nonlinear artificial compression of form proposed by Olsson & Kreiss

$$\partial_t \alpha + \vec{u} \cdot \nabla \alpha = \frac{1}{\mu} \nabla \cdot \left[\left(D \left(\nabla \alpha \cdot \vec{n} \right) - \alpha \left(1 - \alpha \right) \right) \vec{n} \right]$$

where $\vec{n} = \nabla \alpha / |\nabla \alpha|$, $\quad D > 0$, $\quad \mu \gg 1$

Numerical method based on fractional step may apply

1. Advection step over a time step Δt to solve

$$\partial_t \alpha + \vec{u} \cdot \nabla \alpha = 0 \qquad \text{or} \qquad \partial_t \alpha + \nabla \cdot (\alpha \vec{u}) = 0$$

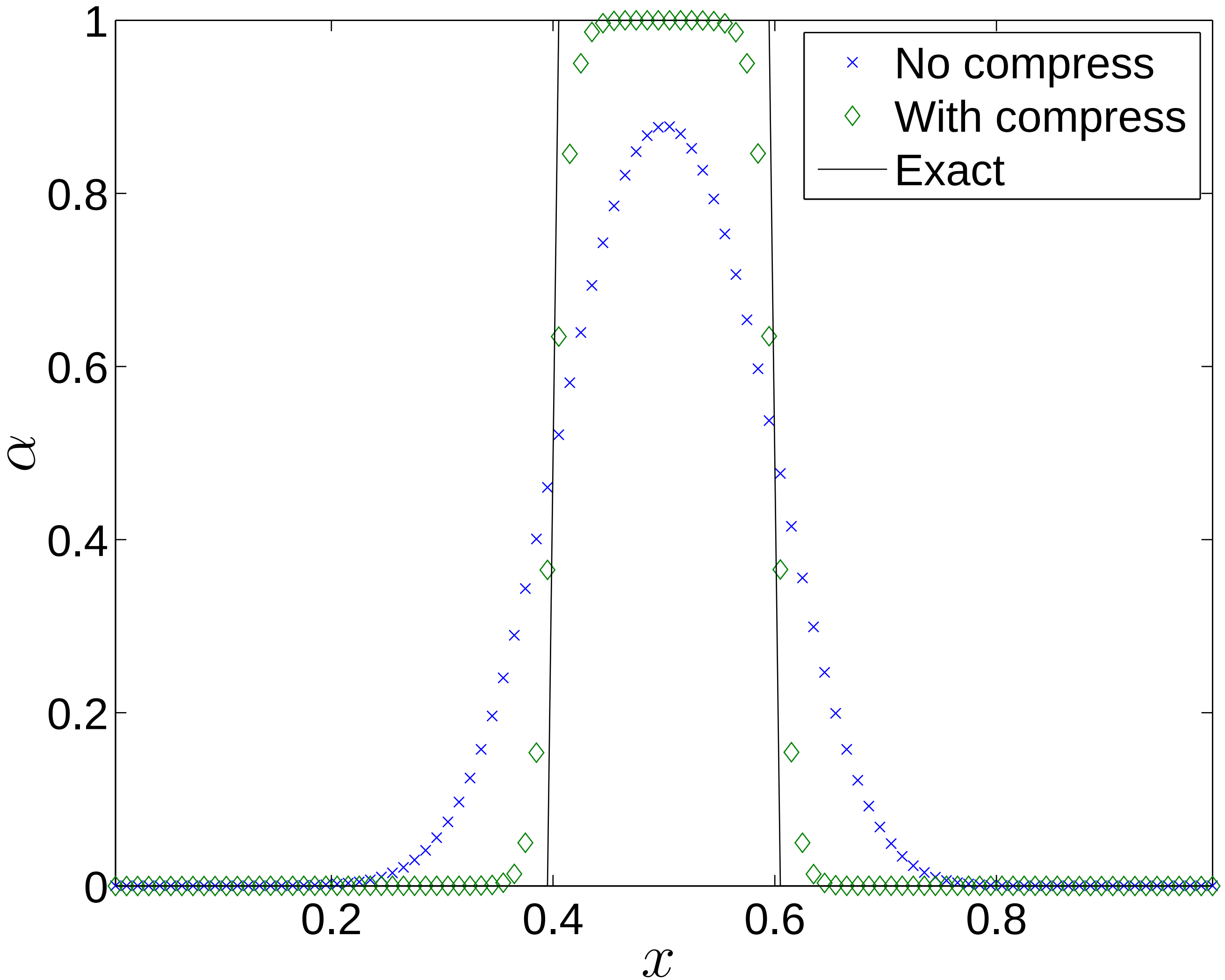
since by assumption $\nabla \cdot \vec{u} = 0$

2. Interface compression step towards τ-steady state

$$\partial_\tau \alpha = \nabla \cdot \left[\left(D \left(\nabla \alpha \cdot \vec{n} \right) - \alpha \left(1 - \alpha \right) \right) \vec{n} \right], \quad \tau = t/\mu$$

Square wave passive advection

Square-wave pluse moving with $u = 1$ after 4 periodic cycle

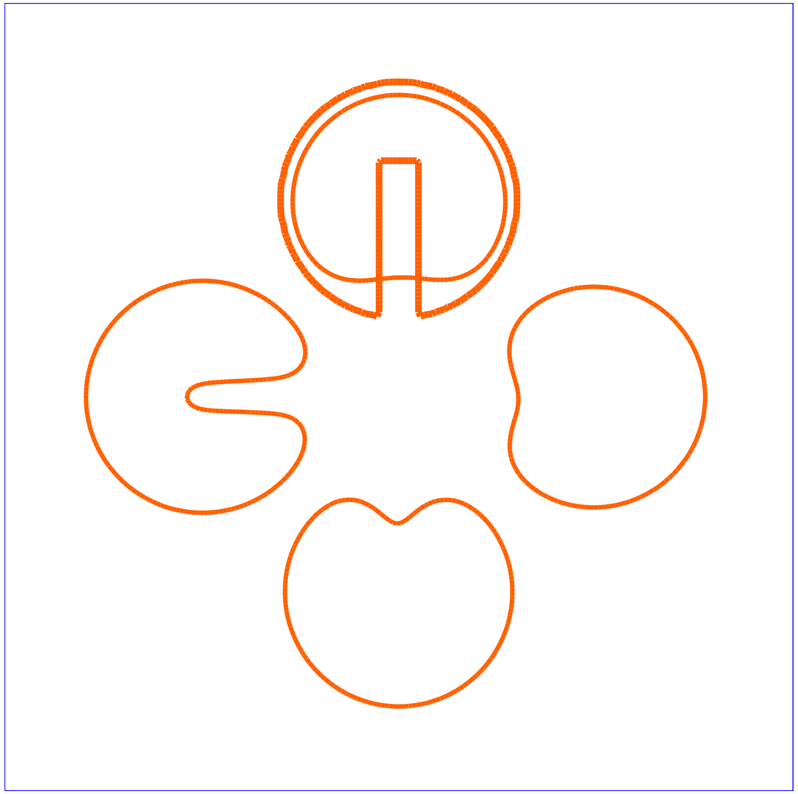


Zalesak's rotating disc

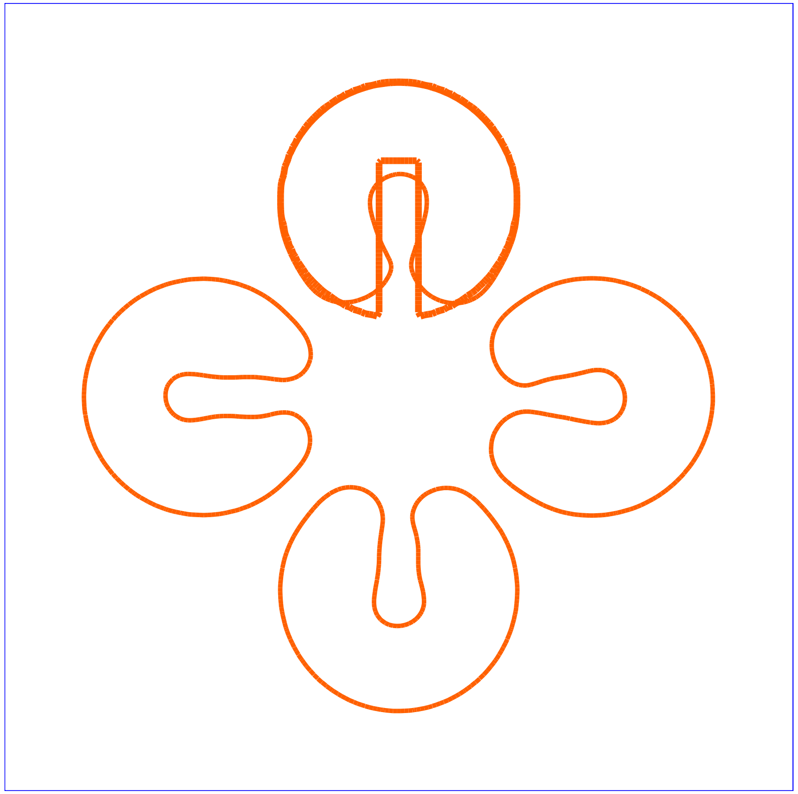
Contours $\alpha = 0.5$ at 4 different times within 1 period in that

$$\vec{u} = (1/2 - y, x - 1/2)$$

No compression



With compression

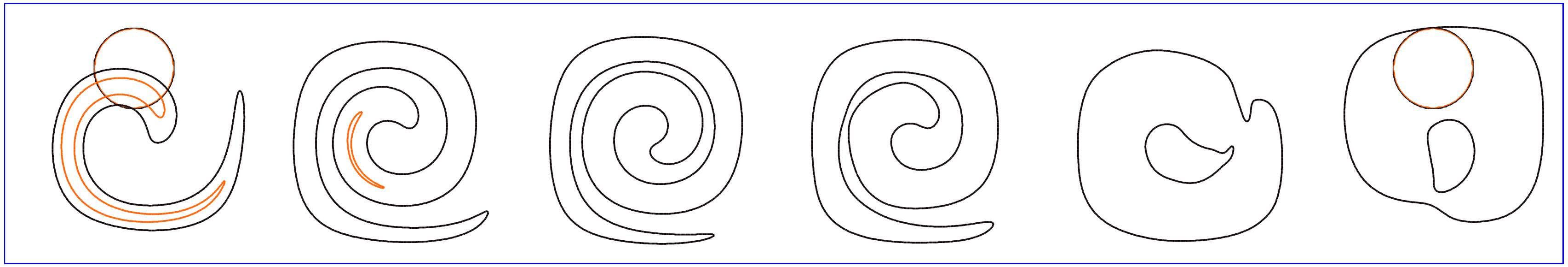


Vortex in cell

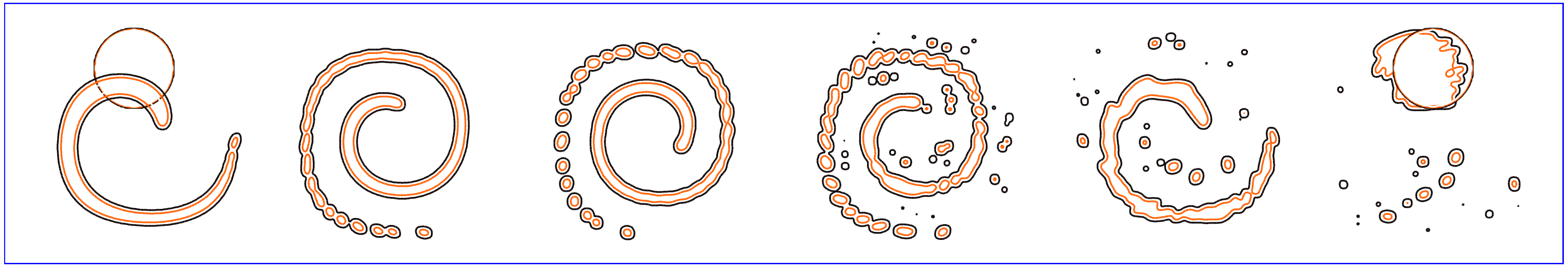
Contours $\alpha = (0.05, 0.5, 0.95)$ at 6 different times in 1 period

$$\vec{u} = \left(-\sin^2(\pi x)\sin(2\pi y), \sin(2\pi x)\sin^2(\pi y)\right)\cos(\pi t/8)$$

No compression



With compression



Interface compression: Remarks

Consider 1D model problem with $u > 0$ of form

$$\begin{cases} \partial_t \alpha + u \partial_x \alpha = \dfrac{1}{\mu} \partial_x \left[D \left(\partial_x \alpha \cdot \vec{n} \right) - \alpha \left(1 - \alpha \right) \right] \\ \alpha(x,0) = \alpha_0(x) = 1 / \left(1 + \exp\left(-x/D \right) \right), \quad x \in \mathbb{R}, \quad t > 0 \end{cases}$$

Exact solution for this problem is simply $\alpha(x,t) = \alpha_0(x - ut)$

When $\alpha(x,0)$ is perturbed to $\alpha_0(x) + \delta(x)$, $\delta(x) \ll 1$, we have

$$\partial_\tau \tilde{\alpha} + \partial_\xi \left(\tilde{\alpha}^2/2 \right) = \partial_\xi \left(D \partial_\xi \tilde{\alpha} \right), \qquad \tilde{\alpha}(\xi, 0) = \tilde{\alpha}_0(\xi)$$

with $\xi = x - ut$, $\tau = t/\mu$, & $\tilde{\alpha}(\xi,\tau) = 1 - \alpha(\xi,\tau)$, yielding

steady state solution $\alpha(\xi,\tau) = \tilde{\alpha}_0 \left(\xi + \xi_0 \right)$ as $\tau \to \infty$ for some suitably chosen shift ξ_0, see Sattinger (1976)

Interface compression: Remarks

If perturbation is zero mass, *i.e.*, $\int_{-\infty}^{\infty} \delta(\xi, 0) d\xi = 0$ we have true solution with $\xi_0 = 0$, see Goodman (1986)

When model is solved by a conservative method, truncation errors will be of zero mass, yielding convergence of numerical solution to exact one in time we want

In multi-D case, let $K_\alpha = D\nabla\alpha \cdot \vec{n} - \alpha(1-\alpha)$. We solve

$$\partial_\tau \alpha = \nabla \cdot [(D(\nabla\alpha \cdot \vec{n}) - \alpha(1-\alpha))\vec{n}] = K_\alpha \nabla \cdot \vec{n} + \vec{n} \cdot \nabla K_\alpha$$

yielding τ-steady state solution as $\mu \to \infty$, when $K_\alpha = 0$ & 1D profile in coordinate normal to interface

When μ finite, $K_\alpha \nabla \cdot \vec{n} + \vec{n} \cdot \nabla K_\alpha \neq 0$, strength & accuracy of curvature $\nabla \cdot \vec{n}$ plays important role in interface resolution

Interface compression runs

Methods used here are very elementary, *i.e.*,

1. Use Clawpack for advection in Step 1
2. Use simple forward Euler in time, second order in space for interface compression in Step 2
   - Diffusion coefficient $D = \varepsilon \min_{\forall i} \Delta x_i$
   - Time step $\Delta\tau$

$$\Delta\tau \le \frac{1}{2D} \sum_{i=1}^{d} \Delta x_i^2$$

   - Stopping criterion: simple 1-norm error measure

Extension to compressible flow

Shukla, Pantano & Freund (JCP 2010) proposed extension of interface-compression method for incompressible flow to compressible flow governed by reduced 2-phase model as

$$\partial_t (\alpha_1 \rho_1) + \nabla \cdot (\alpha_1 \rho_1 \vec{u}) = 0$$

$$\partial_t (\alpha_2 \rho_2) + \nabla \cdot (\alpha_2 \rho_2 \vec{u}) = 0$$

$$\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla p = 0$$

$$\partial_t (\rho E) + \nabla \cdot (\rho E \vec{u} + p \vec{u}) = 0$$

$$\partial_t \alpha_1 + \vec{u} \cdot \nabla \alpha_1 = \frac{1}{\mu} \vec{n} \cdot \nabla \left(D \nabla \alpha_1 \cdot \vec{n} - \alpha_1 (1 - \alpha_1) \right)$$

$$\partial_t \rho + \nabla \cdot (\rho \vec{u}) = \frac{1}{\mu} H(\alpha_1) \vec{n} \cdot \left(\nabla (D \nabla \rho \cdot \vec{n}) - (1 - 2\alpha_1) \nabla \rho \right)$$

Mixture pressure is computed based on isobaric closure

Compressible flow: Density correction

To see how density compression term comes from, we assume $\nabla\rho\cdot\vec{n} \sim \nabla\alpha_1\cdot\vec{n}$ & consider case when

$$K_{\alpha_1} = D\nabla\alpha_1\cdot\vec{n} - \alpha_1\left(1-\alpha_1\right) \approx 0 \quad \Longrightarrow \quad D\nabla\alpha_1\cdot\vec{n} \approx \alpha_1(1-\alpha_1)$$

yielding density diffusion normal to interface at τ-steady as

$$\nabla\left(D\nabla\rho\cdot\vec{n}\right)\cdot\vec{n} \approx \nabla\left(\alpha_1(1-\alpha_1)\right)\cdot\vec{n} = \left(1-2\alpha_1\right)\nabla\alpha_1\cdot\vec{n}$$
$$\sim \left(1-2\alpha_1\right)\nabla\rho\cdot\vec{n}$$

Define $K_\rho = \nabla(D\nabla\rho\cdot\vec{n}) - \left(1-2\alpha_1\right)\nabla\rho$ & form

$$\partial_t\rho + \nabla\cdot(\rho\vec{u}) = \frac{1}{\mu}H(\alpha_1)\vec{n}K_\rho$$

$H(\alpha_1) = \tanh\left(\alpha_1(1-\alpha_1)/D\right)^2$ is localized-interface function

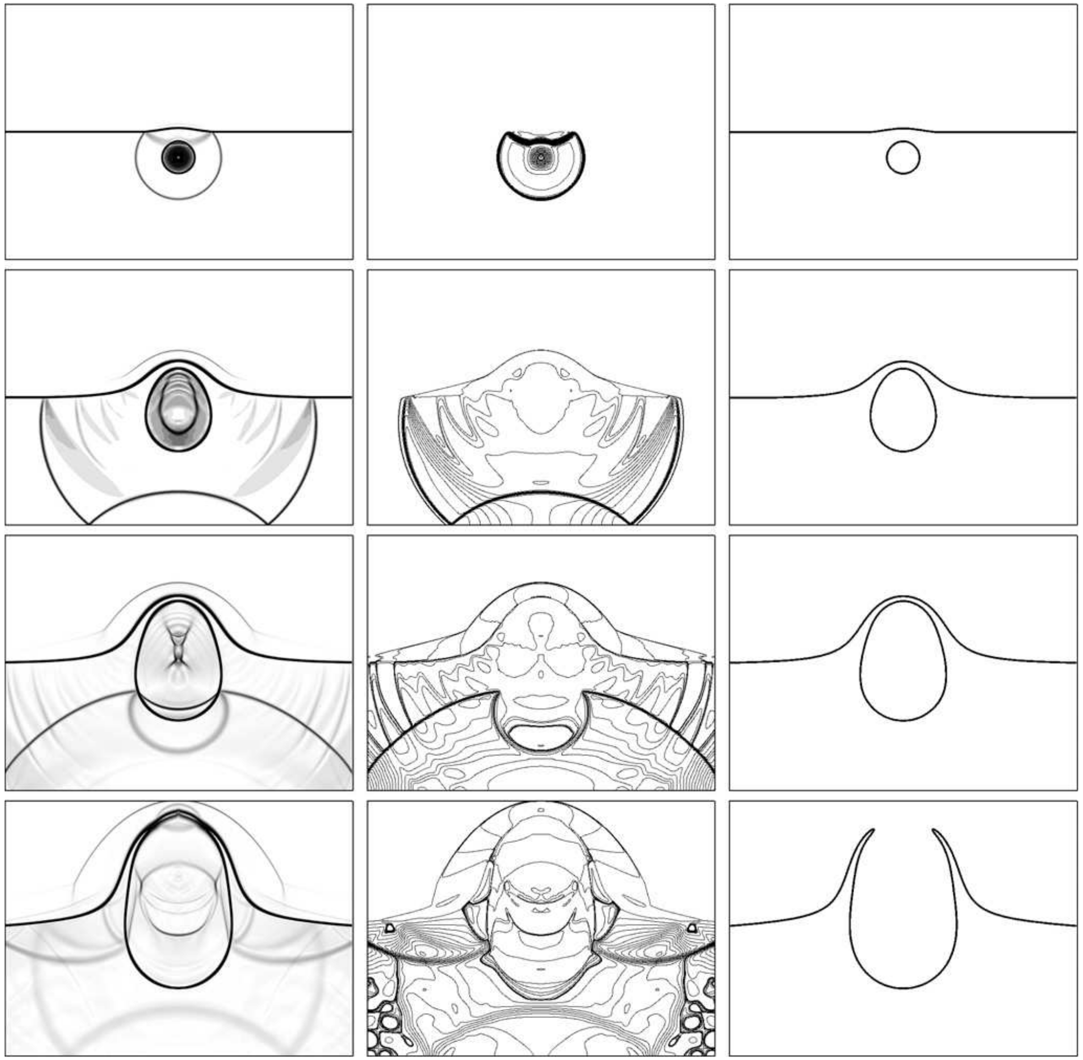
Shukla *et al.* interface compression

In each time step, Shukla's interface-compression algorithm for compressible 2-phase flow consists of following steps:

1. Solve model equation without interface-compression terms by WENO method
2. Compute primitive variable $w = (\rho_1, \rho_2, \rho, \vec{u}, p, \alpha_1)$ from conservative variables $q = (\alpha_1 \rho_1, \alpha_2 \rho_2, \rho \vec{u}, \rho E, \alpha_1)$
3. Iterate interface & density compression equations to τ-steady state until convergence
4. Update conserved variables at end of time step from primitive variables in step 2 & new values of ρ , α_1 from step 3

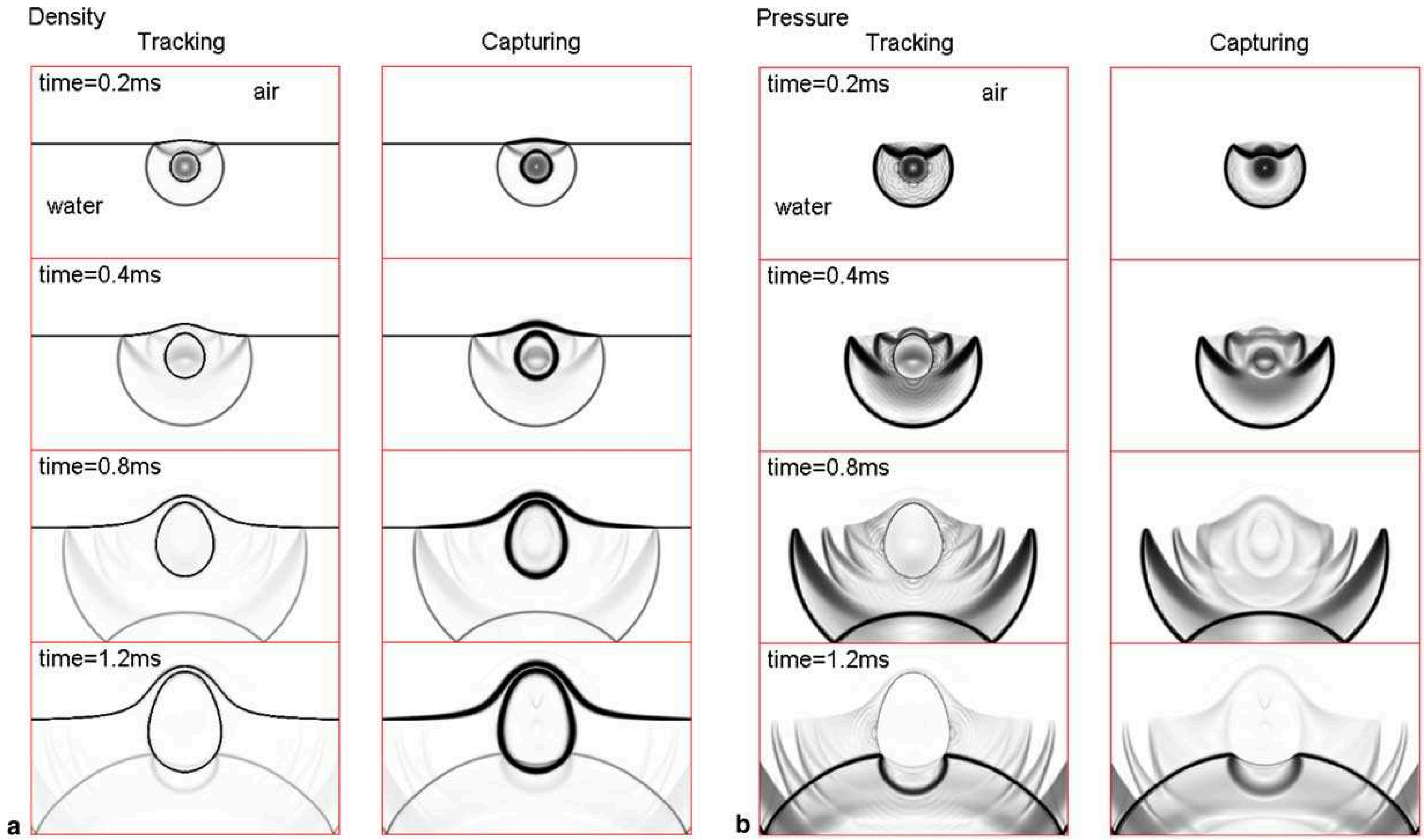
Underwater explosion (UNDEX)

Solution adpated from Shukla's paper (JCP 2010)



Underwater explosion

Solution adpated from Shyue's paper (JCP 2006)



Shukla *et al.* algorithm: Remarks

In Shukla's results there are noises in pressure contours for UNDEX means poor calculation of pressure near interface

To understand method better, consider simple interface only problem where p & $\vec{u}$ are constants in domain, while ρ & material quantities in EOS have jumps across interfaces

Assume consistent approximation in step 1 for model equation without interface-compression, yielding

$$\text{smeared} \quad (\alpha_1\rho_1, \alpha_2\rho_2, \alpha_1)^* \quad \& \quad \text{retain} \quad (\vec{u}, p)^* = (\vec{u}, p)$$

In step 3, $\rho^* = (\alpha_1\rho_1)^* + (\alpha_2\rho_2)^*$ & α_1^* are compressed to $\tilde{\rho}$ & $\tilde{\alpha_1}$, which in step 4, total mass & momentum are set

$$(\rho, \rho u)^{n+1} = (\tilde{\rho}, \tilde{\rho}\vec{u}^*) \quad \Longrightarrow \quad \vec{u}^{n+1} = \tilde{\rho}\vec{u}^*/\tilde{\rho} = \vec{u}^* \quad \text{as expected}$$

Shukla *et al.* algorithm: Remarks

In addition, for total energy, we set

$$(\rho E)^{n+1} = \left(\frac{1}{2}\rho|\vec{u}|^2 + \rho e\right)^{n+1} = \frac{1}{2}\tilde{\rho}|\vec{u}^*|^2 + \widetilde{\rho e}(?)$$

Consider stiffened gas EOS for phasic pressure $p_k = (\gamma_k - 1)(\rho e)_k - \gamma_k \mathcal{B}_k$, $k = 1, 2$. We then have

$$\widetilde{\rho e} = \sum_{k=1}^{2} \alpha_k \rho_k e_k = \sum_{k=1}^{2} \tilde{\alpha}_k \frac{p^* + \gamma_k \mathcal{B}_k}{\gamma_k - 1}$$

$$= p^* \sum_{k=1}^{2} \frac{\tilde{\alpha}_k}{\gamma_k - 1} + \sum_{k=1}^{2} \tilde{\alpha}_k \frac{\gamma_k \mathcal{B}_k}{\gamma_k - 1}$$

yielding equilibrium pressure $p^{n+1} = p^*$ if

$$\left(\frac{1}{\gamma - 1}\right)^{n+1} = \sum_{k=1}^{2} \frac{\tilde{\alpha}_k}{\gamma_k - 1} \quad \& \quad \left(\frac{\gamma \mathcal{B}}{\gamma - 1}\right)^{n+1} = \sum_{k=1}^{2} \tilde{\alpha}_k \frac{\gamma_k \mathcal{B}_k}{\gamma_k - 1}$$

Shukla *et al.* algorithm: Remarks

Next example concerns linearized Mie-Grüneisen EOS for phasic pressure $p_k = (\gamma_k - 1)\,(\rho e)_k + (\rho_k - \rho_{0k})\,\mathcal{B}_k$

$$\widetilde{\rho e} = \sum_{k=1}^{2} \alpha_k \rho_k e_k = \sum_{k=1}^{2} \frac{\tilde{\alpha}_k p^*}{\gamma_k - 1} - (\tilde{\alpha}_k \rho_k^* - \tilde{\alpha}_k \rho_{0k})\,\frac{\mathcal{B}_k}{\gamma_k - 1}$$

$$= p^* \sum_{k=1}^{2} \frac{\tilde{\alpha}_k}{\gamma_k - 1} - \sum_{k=1}^{2} (\tilde{\alpha}_k \rho_k^* - \tilde{\alpha}_k \rho_{0k})\,\frac{\mathcal{B}_k}{\gamma_k - 1}$$

yielding equilibrium pressure $p^{n+1} = p^*$ if

$$\left(\frac{1}{\gamma - 1}\right)^{n+1} = \sum_{k=1}^{2} \frac{\tilde{\alpha}_k}{\gamma_k - 1} \quad \& \quad \left(\frac{(\rho - \rho_0)\mathcal{B}}{\gamma - 1}\right)^{n+1} = \sum_{k=1}^{2} (\tilde{\alpha}_k \rho_k^* - \tilde{\alpha}_k \rho_{0k})\,\frac{\mathcal{B}_k}{\gamma_k - 1}$$

Shukla *et al.* algorithm: Remarks

In Shukla *et al.* algorithm, there is a consistent problem as

$$\sum_{k=1}^{2} (\alpha_k \rho_k)^{n+1} = \sum_{k=1}^{2} \tilde{\alpha}_k \rho_k^* \neq \tilde{\rho} = \rho^{n+1}$$

One way to remove this inconsistency is to include compression terms in partial density $\alpha_k \rho_k$ directly, $k = 1, 2$,

$$\partial_t (\alpha_k \rho_k) + \nabla \cdot (\alpha_k \rho_k \vec{u}) =$$

$$\frac{1}{\mu} H(\alpha_k) \vec{n} \cdot (\nabla (D \nabla (\alpha_k \rho_k) \cdot \vec{n}) - (1 - 2\alpha_k) \nabla (\alpha_k \rho_k))$$

We then set $\rho^{n+1} = \sum_{k=1}^{2} (\alpha_k \rho_k)^{n+1} = \sum_{k=1}^{2} \tilde{\alpha}_k \tilde{\rho}_k$

Validation of this approach is required

Positivity & accuracy

In compressible multiphase flow, positivity of volume fraction, *i.e.*, $\alpha_k \geq 0$, $\forall k$, is important due to provision of

1. information on interface location
2. information on thermodynamic states such as ρe & p in numerical "mixture" region & so ρ_k from $\alpha_k \rho_k$

It is known that devise of oscillation-free higher-order method (WENO, DG, or variant) for multiphase flow is still an open problem

In this regards, interface-sharpening of some kind should be a useful tool as opposed to Eulerian higher-order methods or other adaptive mesh methods

Anti-diffusion interface sharpening

Our second interface-sharpening model concerns anti-diffusion proposed by So, Hu & Adams (JCP 2011)

$$\partial_t \alpha + \vec{u} \cdot \nabla \alpha = -\frac{1}{\mu} \nabla \cdot (D \nabla \alpha), \qquad D > 0, \quad \mu \gg 1$$

Standard fractional step method may still apply

1. Advection step over a time step

$$\partial_t \alpha + \vec{u} \cdot \nabla \alpha = 0$$

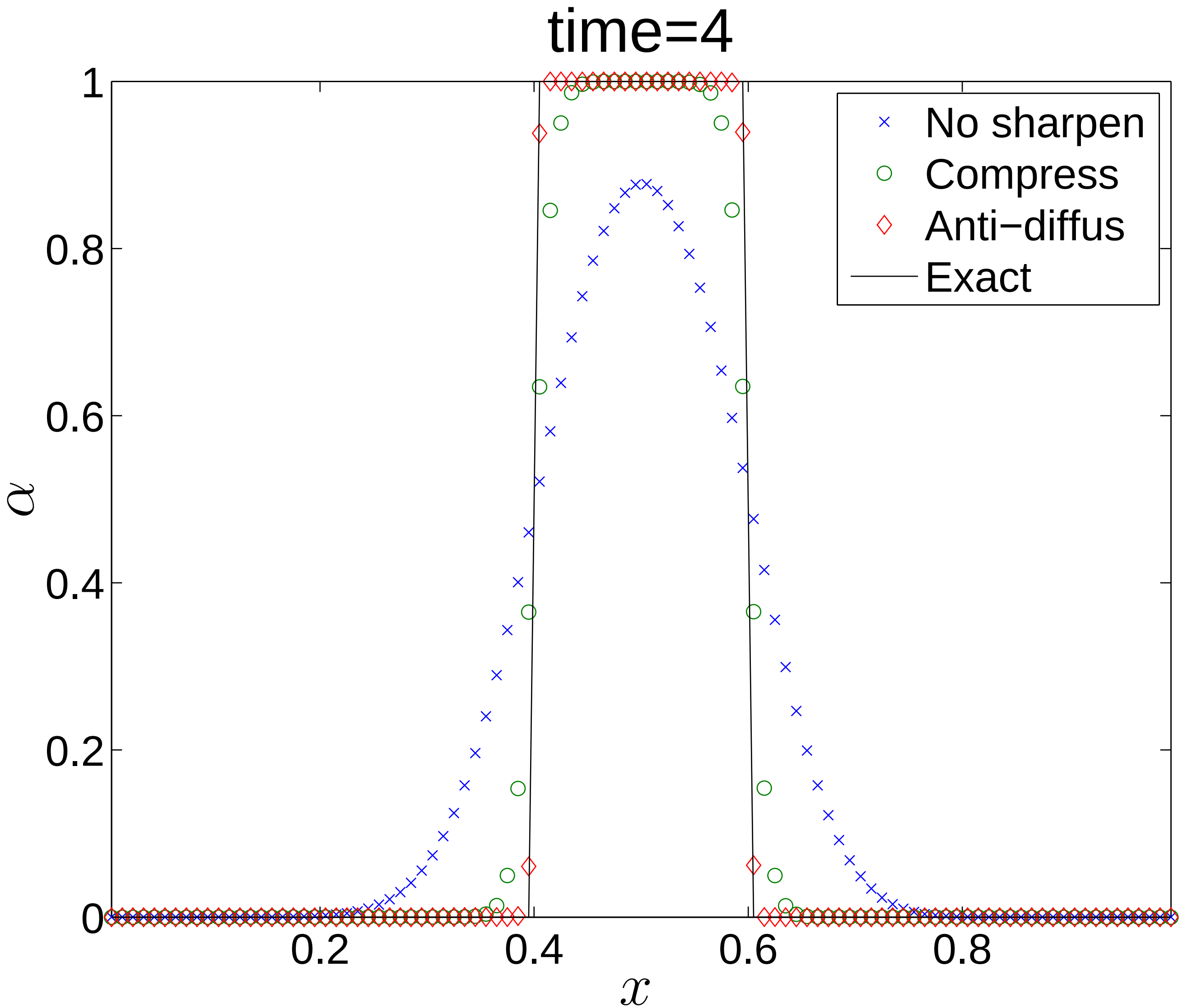
2. Anti-diffusion step towards sharp layer

$$\partial_\tau \alpha = -\nabla \cdot (D \nabla \alpha) \quad \text{or} \quad \partial_\tau \alpha = -\nabla \cdot (D \nabla \alpha \cdot \vec{n})\, \vec{n}, \quad \tau = t/\mu$$

Numerical regularization is required such as employ MINMOD limiter to stabilize $\nabla \alpha$ in discretization, Breuß *et al.* ('05, '07)

Square wave passive advection (revisit)

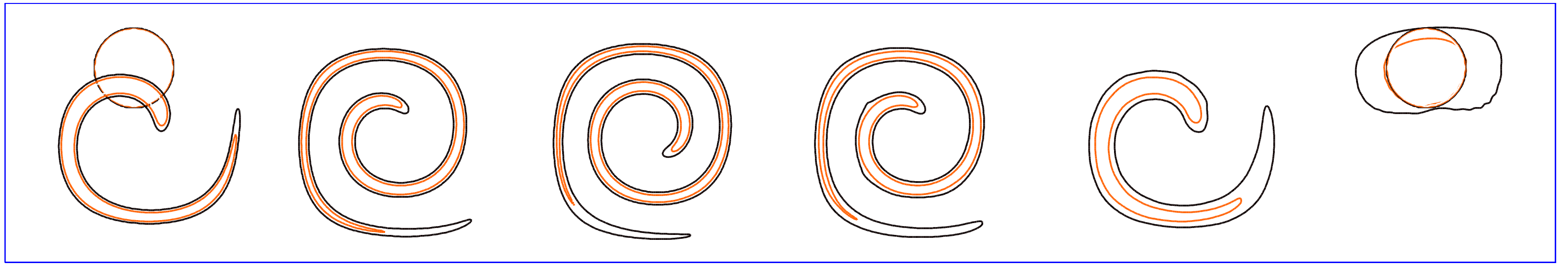
Square-wave pluse moving with $u = 1$ after 4 periodic cycle



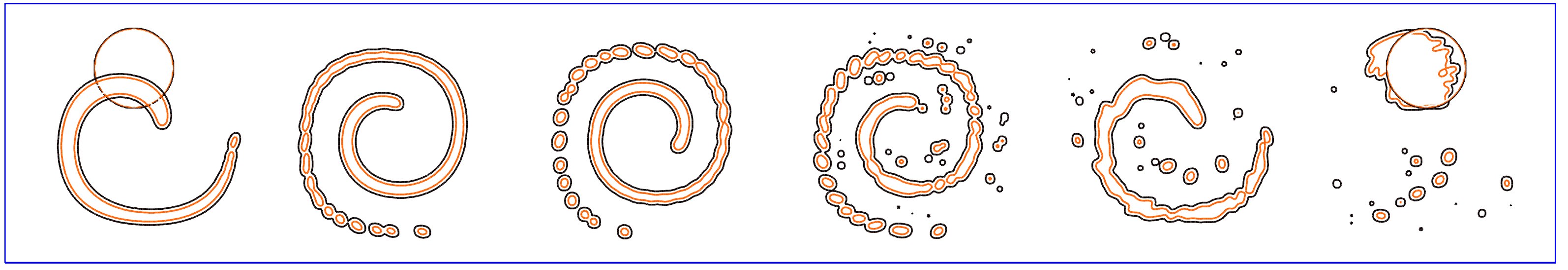
Vortex in cell (revisit)

Contours $\alpha = (0.05, 0.5, 0.95)$ at 6 different times in 1 period

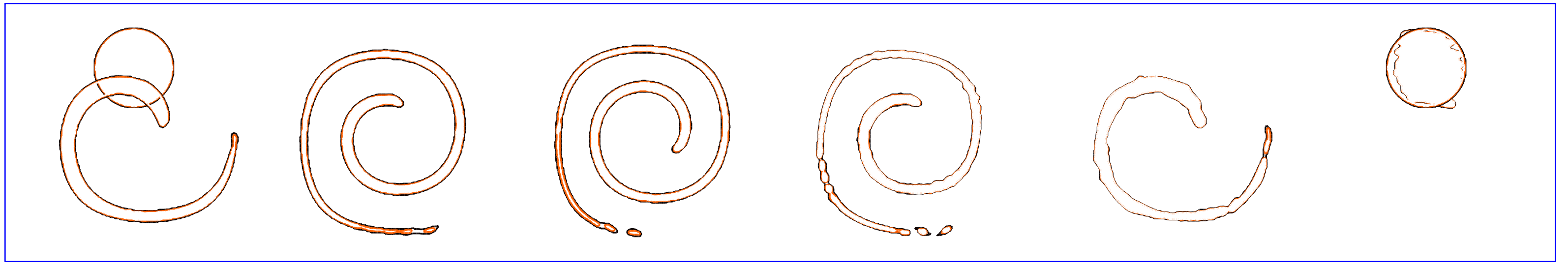
No interface sharpening (second order)



With interface compression



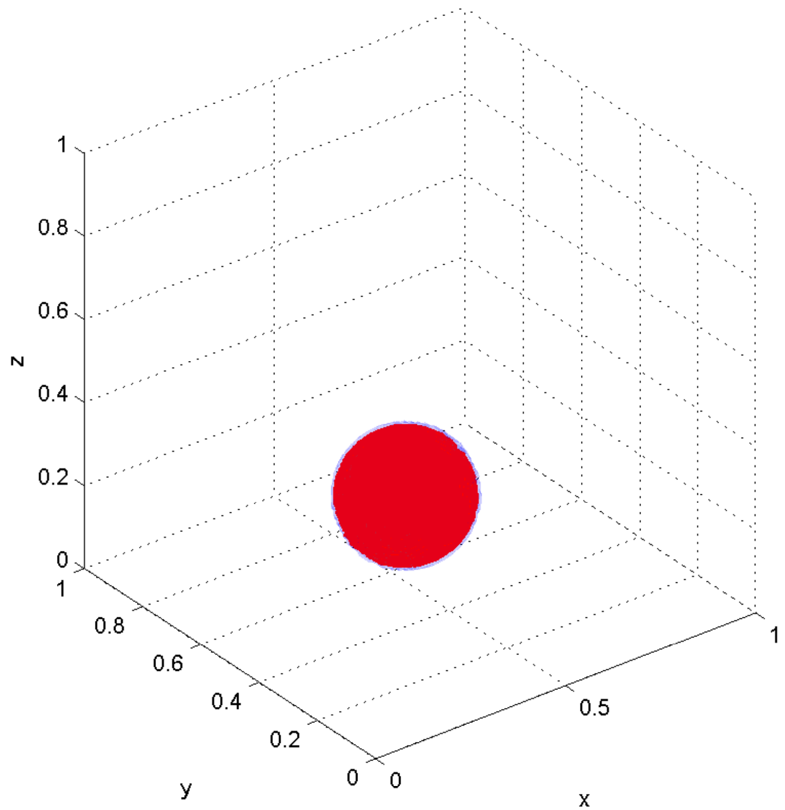
With anti-diffusion



Deformation flow in 3D

In this test, consider velocity field

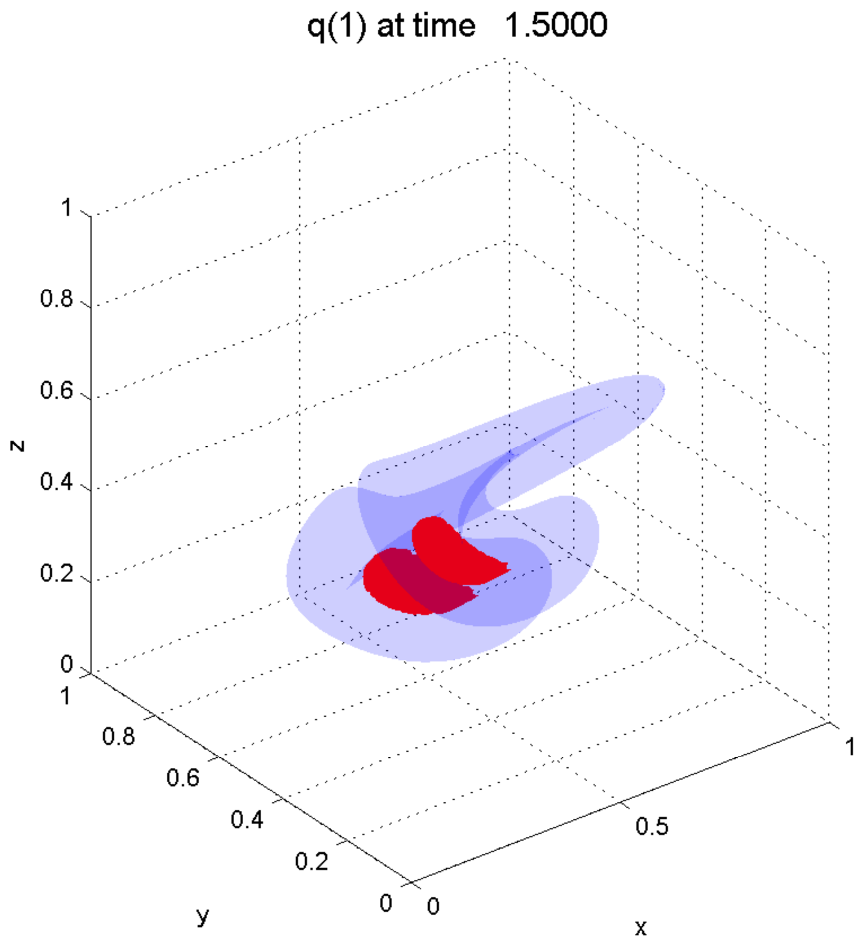
$$\vec{u} = \Big(2\sin^2(\pi x)\sin(2\pi y)\sin(2\pi z), -\sin(2\pi x)\sin^2(\pi y)\sin(2\pi z), \\ -\sin(2\pi x)\sin(2\pi y)\sin^2(\pi z)\Big)\cos(\pi t/3)$$

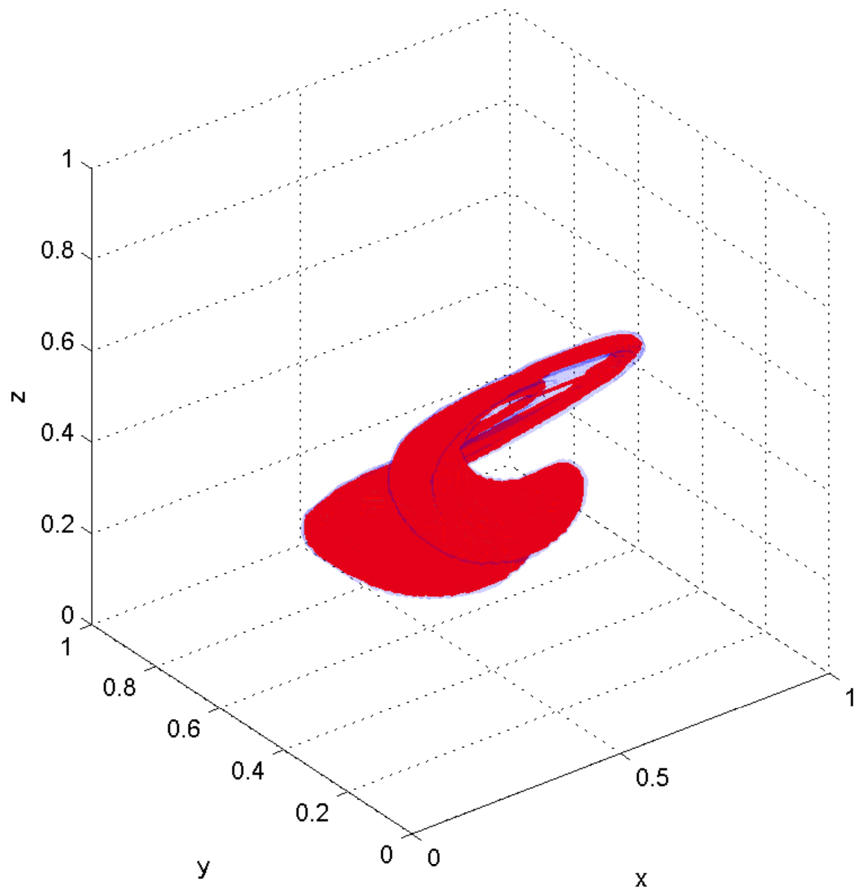


Deformation flow in $3D$

No anti-diffusion

With anti-diffusion

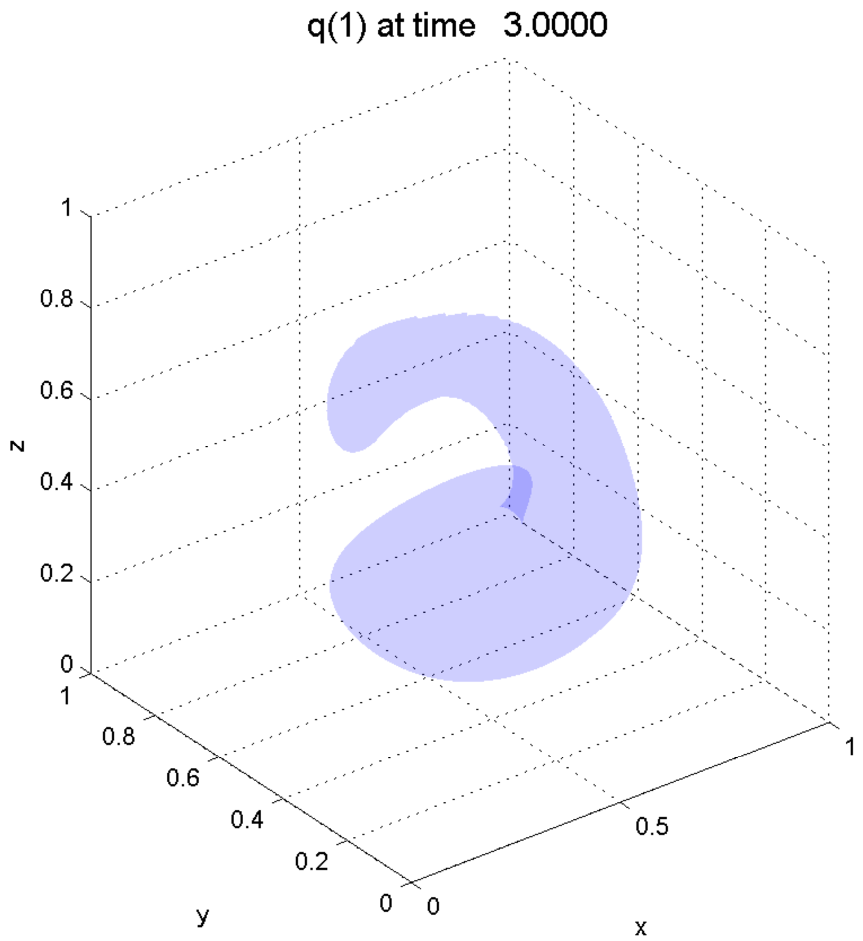


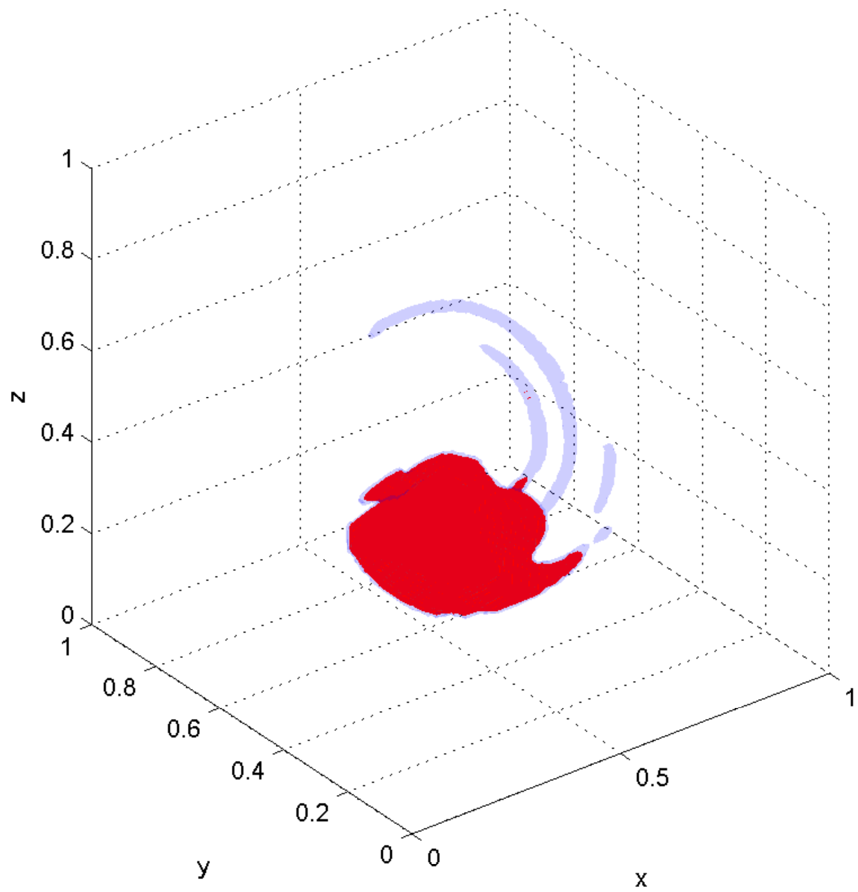


Deformation flow in $3D$

No anti-diffusion

With anti-diffusion





Anti-diffusion runs

Methods used here are essentially the same as artificial interface compression runs, *i.e.*,

1. Use Clawpack for advection in Step 1
2. Use first order explicit method for anti-diffusion in Step 2
   - Diffusion coefficient $D = \max |\vec{u}|$
   - Time step $\Delta\tau$

$$\Delta\tau \le \frac{1}{2D} \sum_{i=1}^{d} \Delta x_i^2$$

   - Stopping criterion: some measure of interface sharpness

Anti-diffusion to compressible flow

Reduced 2-phase model with anti-diffusion (Shyue 2011)

$$\partial_t \alpha_1 + \vec{u} \cdot \nabla \alpha_1 = -\frac{1}{\mu} \nabla \cdot \left(D \nabla \alpha_1 \right) = -\frac{1}{\mu} \mathcal{K}_{\alpha_1}$$

$$\partial_t \left(\alpha_1 \rho_1 \right) + \nabla \cdot \left(\alpha_1 \rho_1 \vec{u} \right) = -\frac{1}{\mu} H_I \nabla \cdot \left(D \nabla \alpha_1 \rho_1 \right) = -\frac{1}{\mu} H_I \mathcal{K}_{\alpha_1 \rho_1}$$

$$\partial_t \left(\alpha_2 \rho_2 \right) + \nabla \cdot \left(\alpha_2 \rho_2 \vec{u} \right) = -\frac{1}{\mu} H_I \nabla \cdot \left(D \nabla \alpha_2 \rho_2 \right) = -\frac{1}{\mu} H_I \mathcal{K}_{\alpha_2 \rho_2}$$

$$\partial_t \left(\rho \vec{u} \right) + \nabla \cdot \left(\rho \vec{u} \otimes \vec{u} \right) + \nabla p = -\frac{1}{\mu} H_I \vec{u} \, \nabla \cdot \left(D \nabla \rho \right) = -\frac{1}{\mu} H_I \mathcal{K}_{\rho \vec{u}}$$

$$\partial_t \left(\rho E \right) + \nabla \cdot \left(\rho E \vec{u} + p \vec{u} \right) = -\frac{1}{\mu} H_I \left(\mathcal{K}_{\rho |\vec{u}|^2/2} + \mathcal{K}_{\rho e} \right)$$

Isobaric closure for mixture pressure is used as usual

H_I denotes interface indicator, $\mathcal{K}$ denotes “diffusion” term

Anti-diffusion to compressible flow

To find $\mathcal{K}_{\rho|\vec{u}|^2/2}$ assuming $|\vec{u}|^2$ is constant, we observe

$$\nabla\left(\frac{1}{2}\rho|\vec{u}|^2\right) = \frac{1}{2}|\vec{u}|^2\nabla\rho \quad \text{yielding} \quad \mathcal{K}_{\rho|\vec{u}|^2/2} = \frac{1}{2}|\vec{u}^2|\nabla\cdot(D\nabla\rho)$$

To find $\mathcal{K}_{\rho e}$, we need to know equation of state. Now in stiffened gas case with $p_k = (\gamma_k - 1)(\rho e)_k - \gamma_k \mathcal{B}_k$,

$$\nabla(\rho e) = \nabla\left(\sum_{k=1}^{2}\alpha_k\rho_k e_k\right) = \nabla\left(\sum_{k=1}^{2}\alpha_k\frac{p+\gamma_k\mathcal{B}_k}{\gamma_k - 1}\right)$$

$$= \sum_{k=1}^{2}\left(\frac{p+\gamma_k\mathcal{B}_k}{\gamma_k - 1}\right)\nabla\alpha_k = \left(\frac{p+\gamma_1\mathcal{B}_1}{\gamma_1 - 1} - \frac{p+\gamma_2\mathcal{B}_2}{\gamma_2 - 1}\right)\nabla\alpha_1$$

$$= \beta\,\nabla\alpha_1 \quad \text{yielding} \quad \mathcal{K}_{\rho e} = \beta\,\nabla\cdot(D\nabla\alpha_1)$$

Anti-diffusion to compressible flow

We next consider case with linearized Mie-Grüneisen EOS $p_k = (\gamma_k - 1)\,(\rho e)_k + (\rho_k - \rho_{0k})\,\mathcal{B}_k \; k = 1, 2$, & proceed same procedure as before

$$\nabla(\rho e) = \nabla \left(\sum_{k=1}^{2} \alpha_k \rho_k e_k \right) = \nabla \left(\sum_{k=1}^{2} \alpha_k \frac{p - (\rho_k - \rho_{0k})\mathcal{B}_k}{\gamma_k - 1} \right)$$

$$= \sum_{k=1}^{2} \frac{p + \rho_{0k}\mathcal{B}_k}{\gamma_k - 1} \nabla \alpha_k + \sum_{k=1}^{2} \frac{\mathcal{B}_k}{\gamma_k - 1} \nabla\left(\alpha_k \rho_k\right)$$

$$= \beta_0 \nabla \alpha_1 + \sum_{k=1}^{2} \beta_k \nabla(\alpha_k \rho k)$$

We choose $\mathcal{K}_{\rho e} = \beta_0 \, \nabla \cdot (D \nabla \alpha_1) + \sum_1^2 \beta_k \, \nabla \cdot (D \nabla \alpha_k \rho_k)$

Anti-diffusion to compressible flow

Write anti-diffusion model in compact form

$$\partial_t q + \nabla \cdot \vec{f} + B \nabla q = -\frac{1}{\mu} \psi(q)$$

with q, $\vec{f}$, B, & ψ defined (not shown)

In each time step, proposed anti-diffusion algorithm for compressible 2-phase flow consists of following steps:

1. Solve model equation without anti-diffusion terms

$$\partial_t q + \nabla \cdot \vec{f} + B \nabla q = 0$$

2. Iterate model equation with anti-diffusion terms

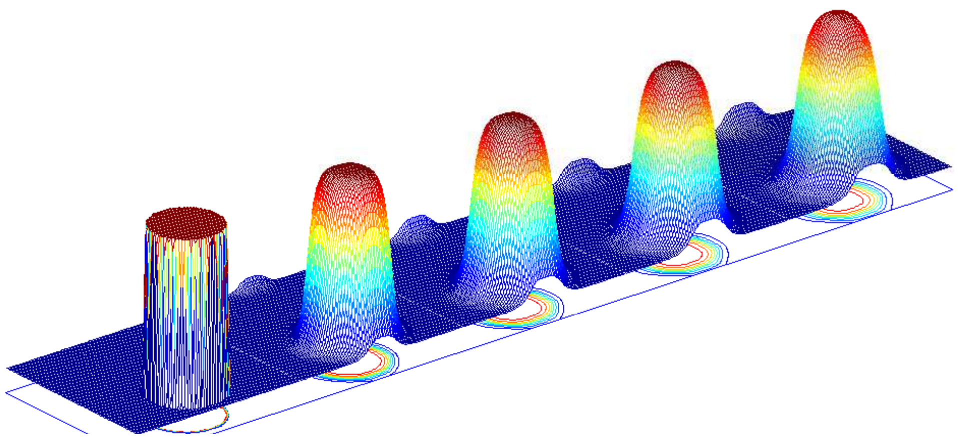
$$\partial_\tau q = -\psi(q)$$

to sharp layer

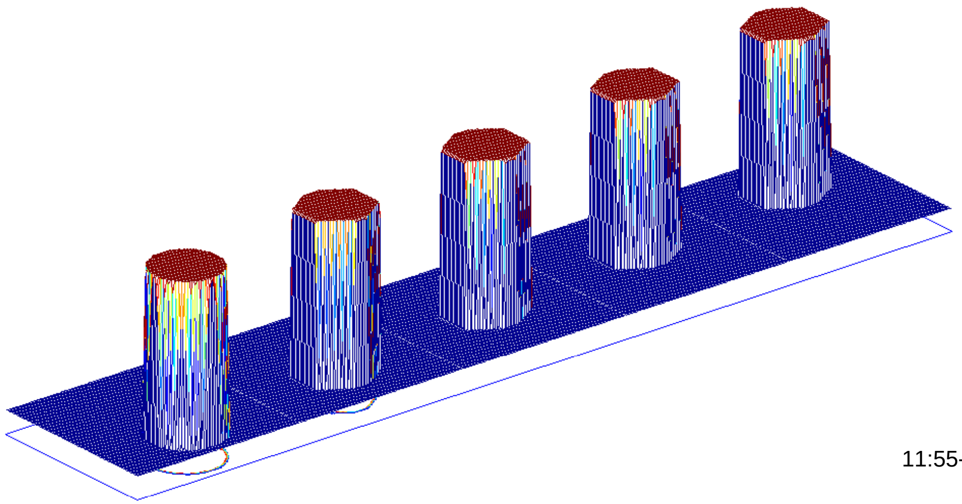
Circular water column in uniform flow

Density surface plot (moving speed $\vec{u} = (1, 1/10)$)

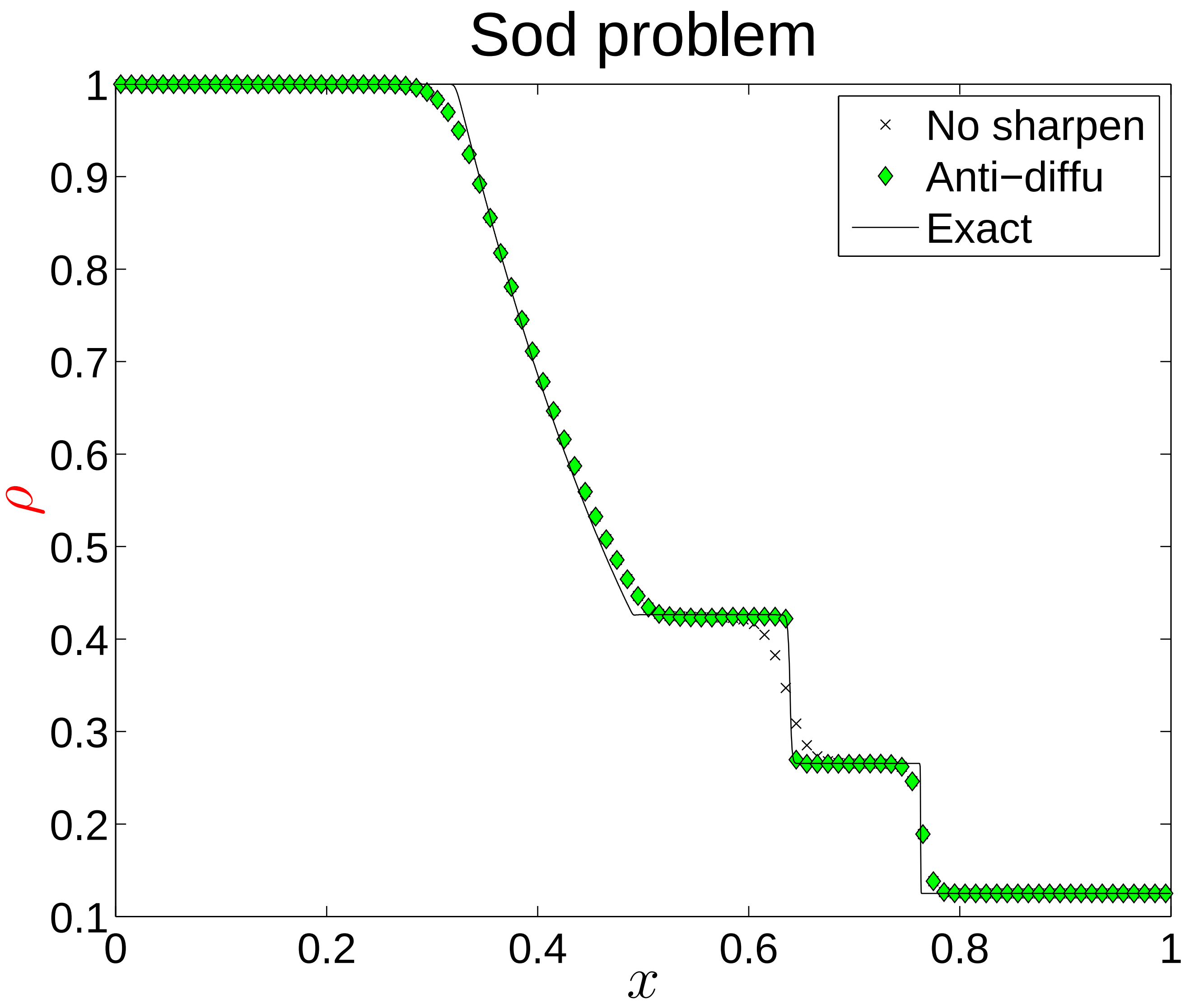
No anti-diffusion



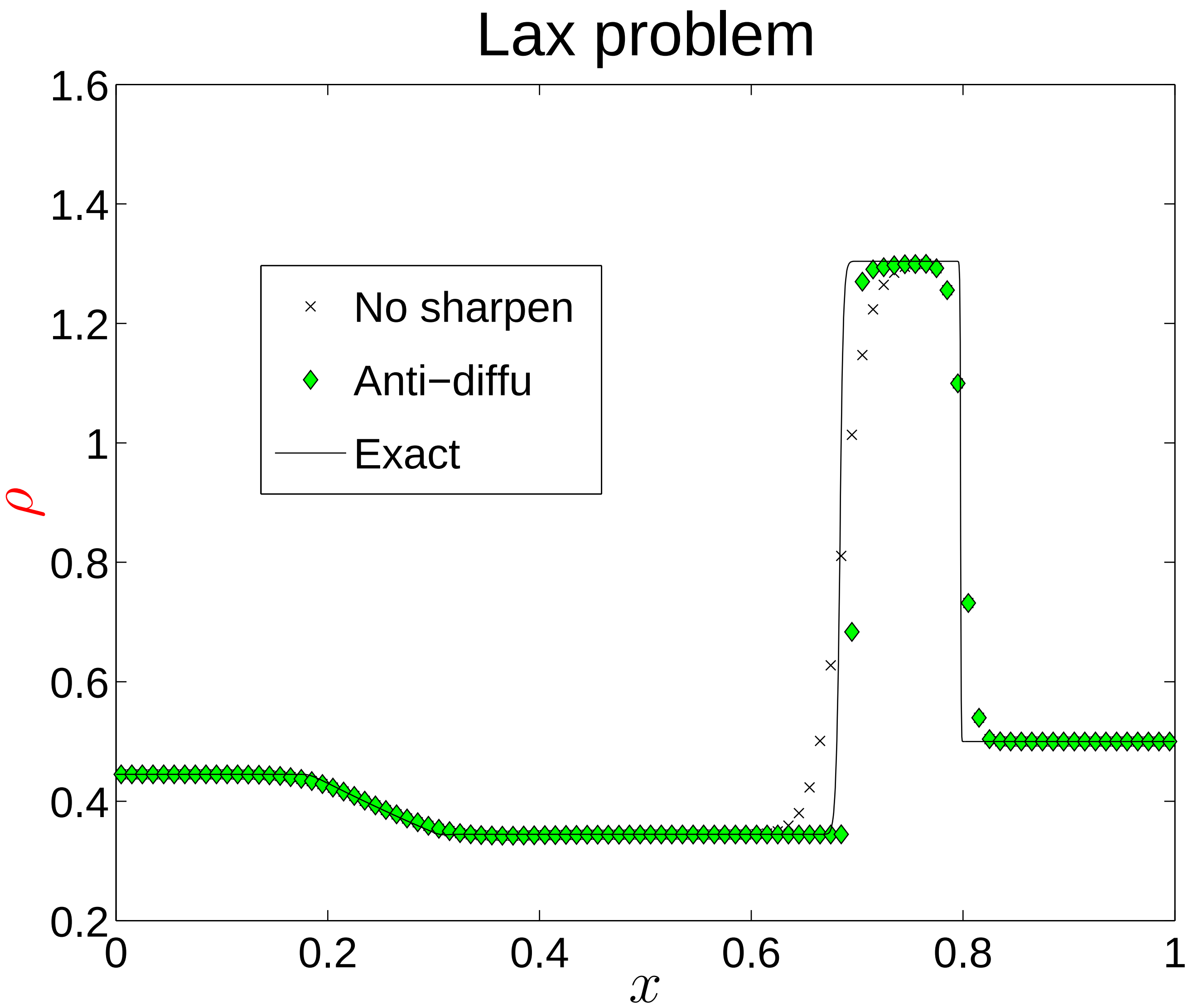
With anti-diffusion



Single-phase Riemann problem

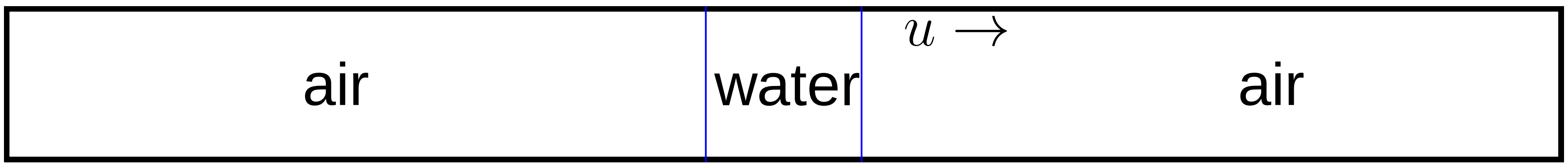


Single-phase Riemann problem

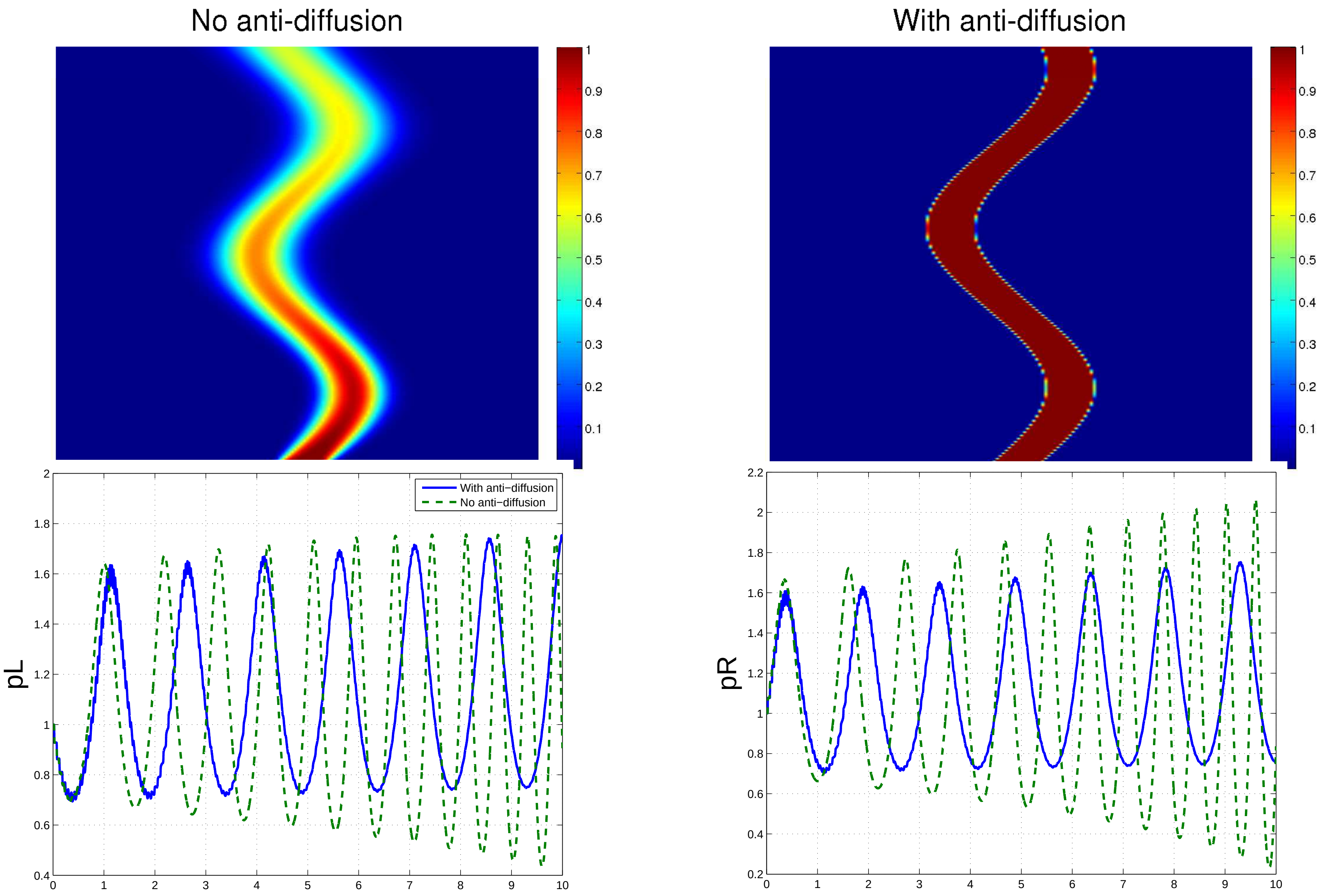


Oscillating water column

- Initially, in closed shock tube, water column moves at $u = 1$ from left to right, yielding air compression at right & air expansion at left
- Subsequently, pressure difference built up across water column resulting deceleration of column of water to right, makes a stop, & then acceleration to left; a reverse pressure difference built up across water column redirecting flow from left to right again
- Eventually, water column starts to oscillate

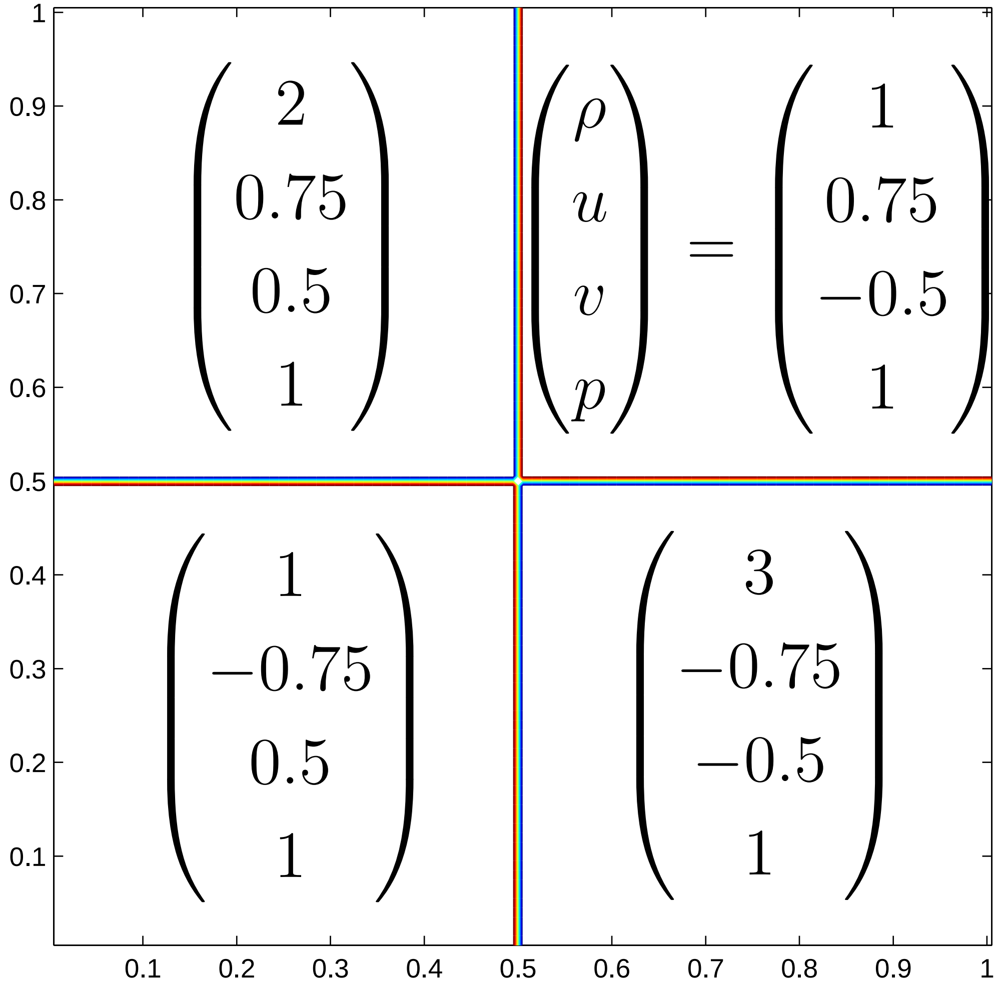


Oscillating water column



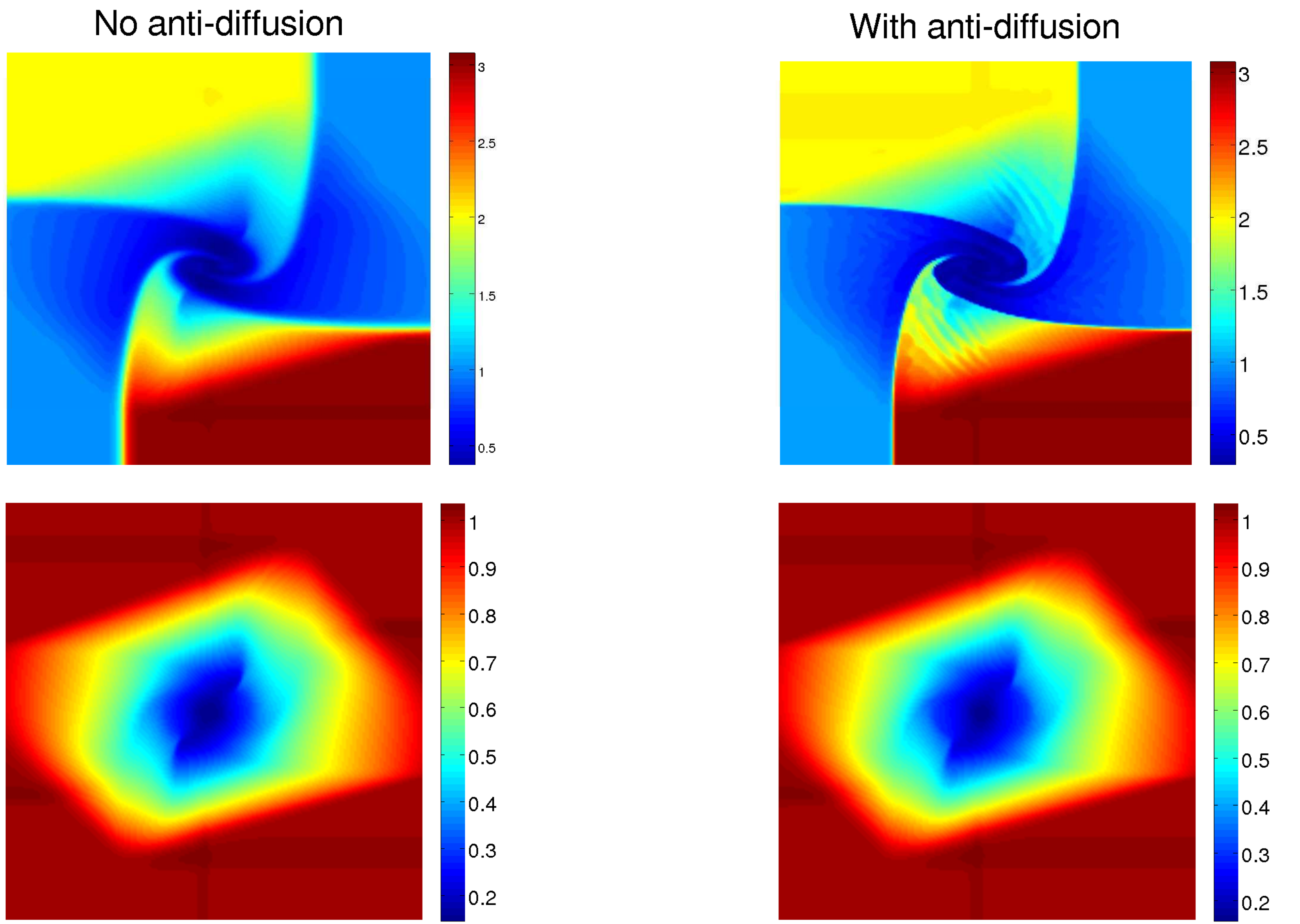
2D Riemann Problem

With initial 4-slip lines wave pattern



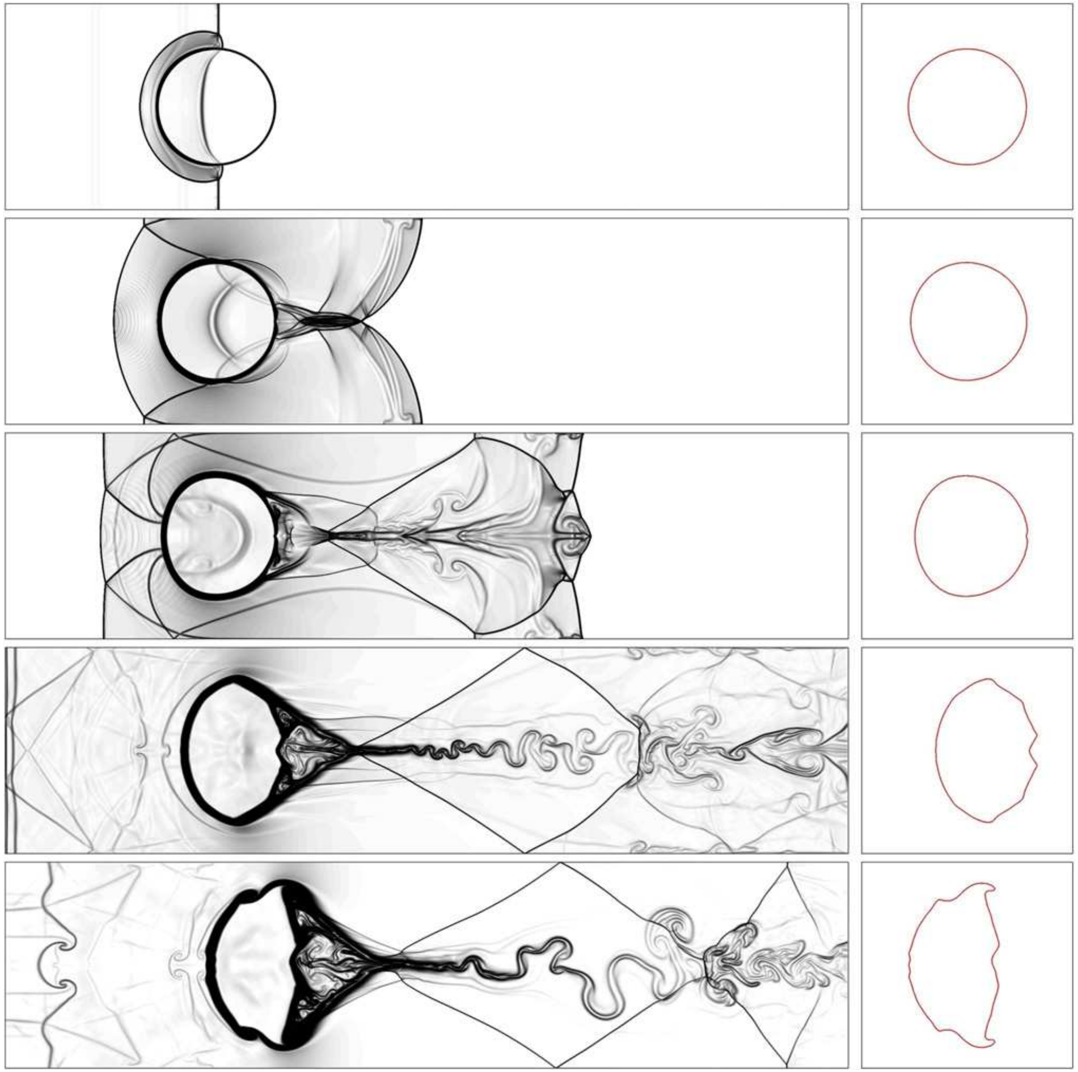
2D Riemann Problem

Density on top & pressure on bottom



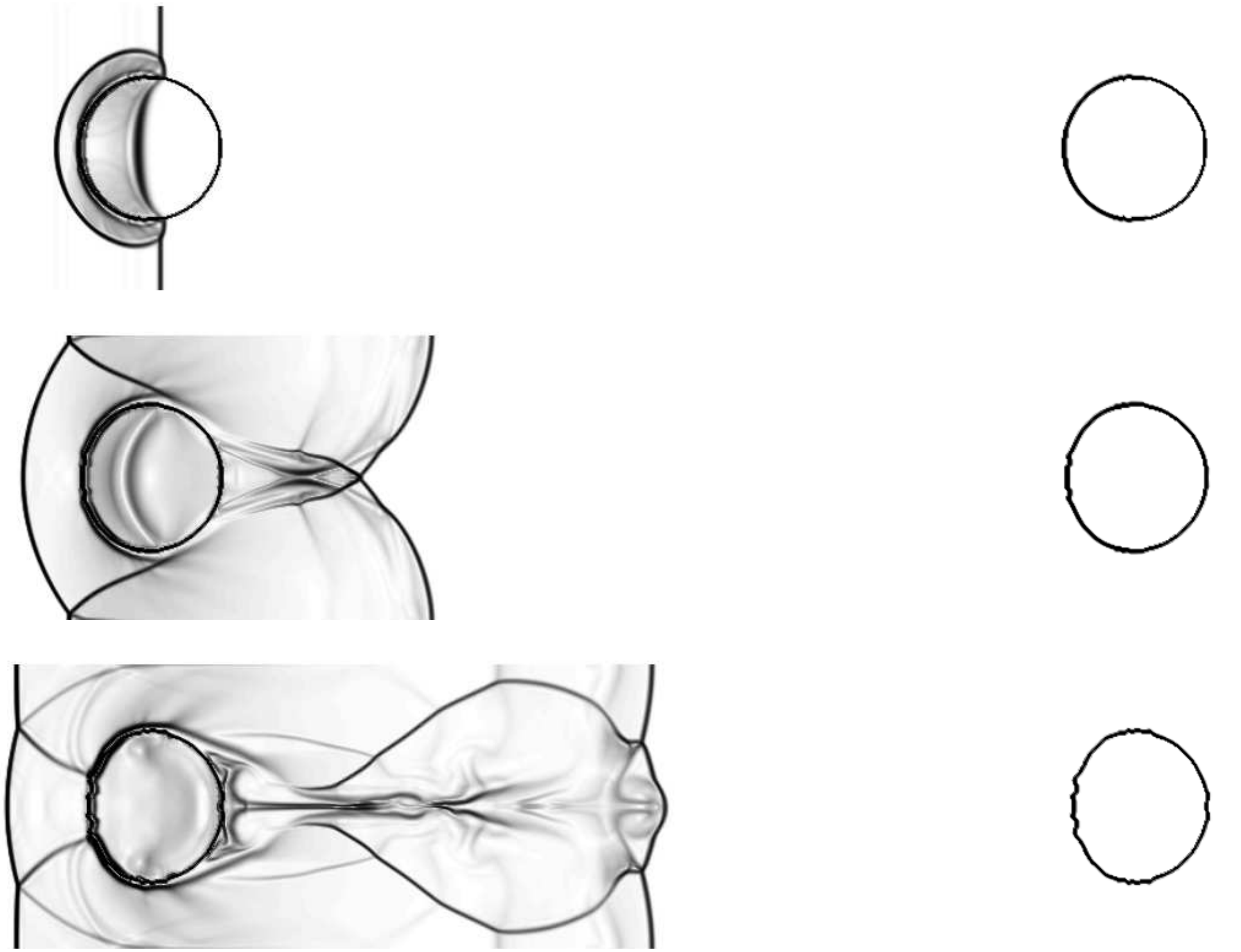
Shock in air & water cylinder

Solution adpated from Shukla's paper (JCP 2010)



Shock in air & water cylinder

Schlieren images for anti-diffusion results at times $t = 0.15$, 0.4, 0.65 (volume fraction on right)



Anti-diffusion sharpening: Remarks

1. Local interface identification
   - Algebraic

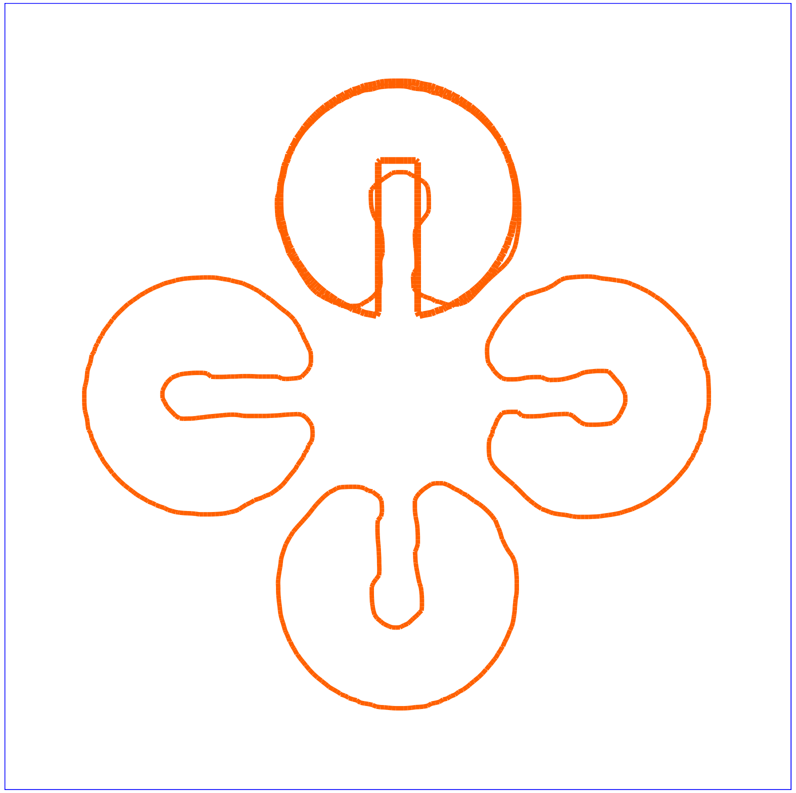
$$H(\alpha) = \tanh\left(\alpha(1-\alpha)/D\right)^2 \quad \text{(Shukla \textit{et al.})}$$

   - Physical-jump (density, volume fraction, ...)
2. Diffusion coefficient definition
   - Global $D = \max|\vec{u}|$ or $D_j = \max|u_j|$
   - Local $D = \max_{M(C)}|\vec{u}|$ or $D_j^C = \max_{M(C)}|u_j|$
3. Stopping criterion
   - Run $1-2$ anti-diffusion iteration currently
   - Interface-shapreness measure (?)

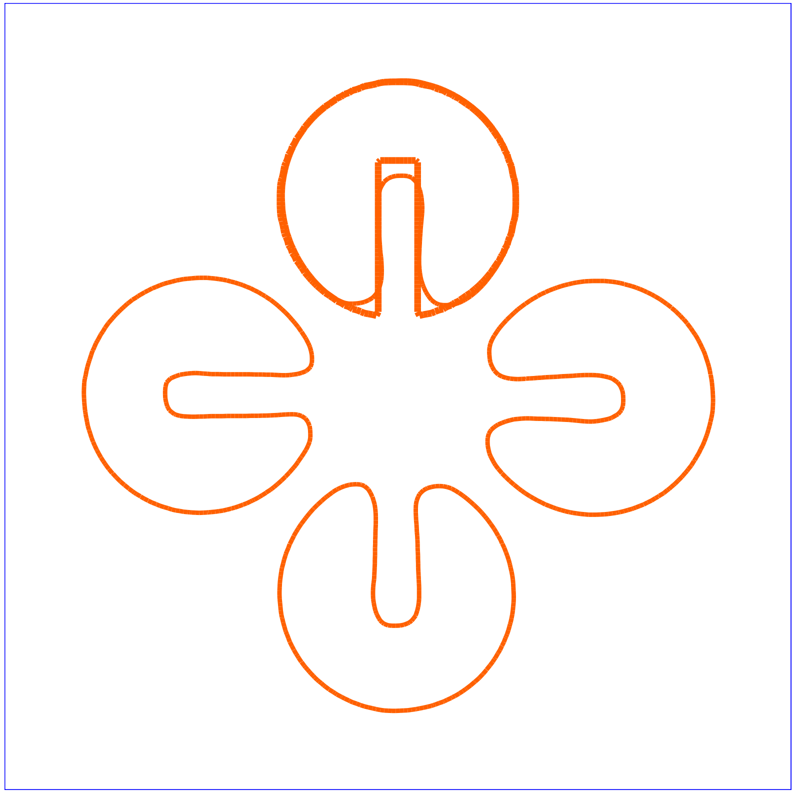
Effect of diffusion coefficient

Anti-diffusion results for Zalesak's rotating disc

Global D_j

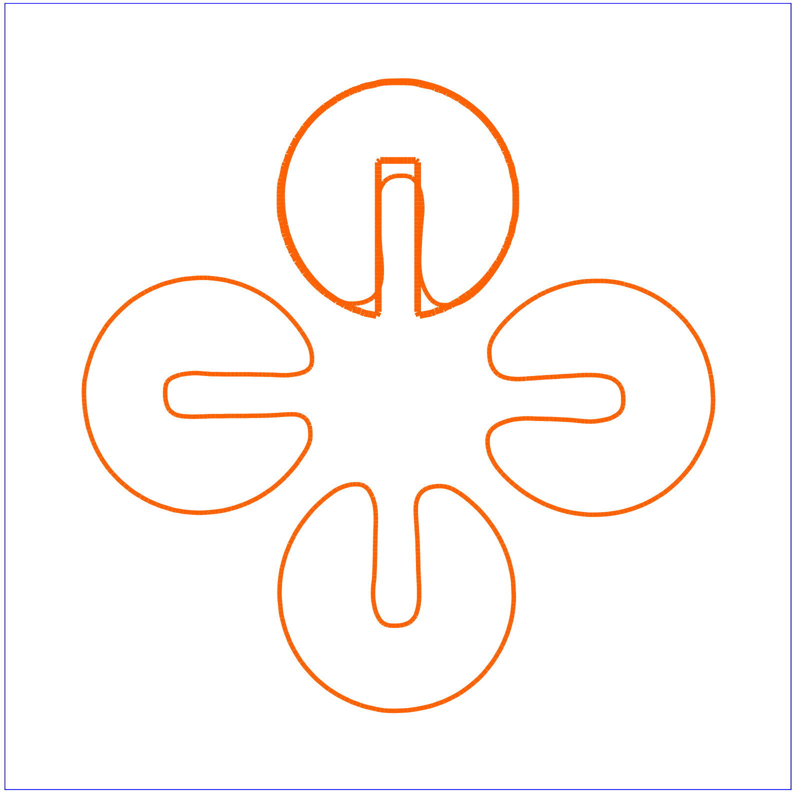
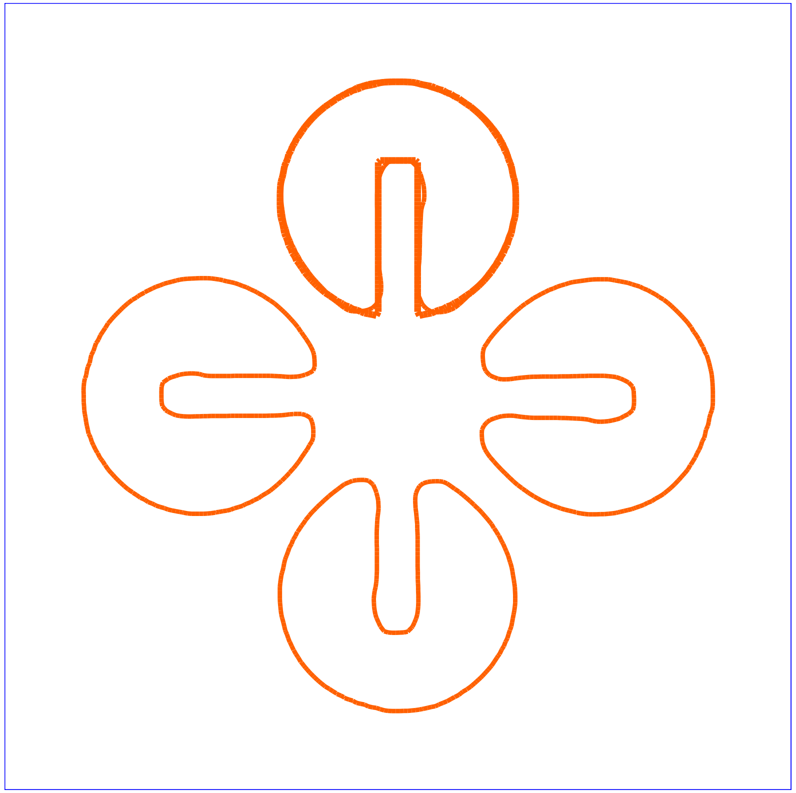


Local D_j



Effect of stopping criterion

Anti-diffusion results for Zalesak's rotating disc

Local D_j: 1 step	Local D_j: 2 step
	

Future perspectives

Extend method to mapped grid & model with phase transition proposed by Saurel, Petipas, Abgrall (JFM 2008)

$$\partial_t (\alpha_1 \rho_1) + \nabla \cdot (\alpha_1 \rho_1 \vec{u}) = 0$$

$$\partial_t (\alpha_2 \rho_2) + \nabla \cdot (\alpha_2 \rho_2 \vec{u}) = 0$$

$$\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla p = 0$$

$$\partial_t (\rho E) + \nabla \cdot (\rho E \vec{u} + p \vec{u}) = 0$$

$$\partial_t \alpha_1 + \vec{u} \cdot \nabla (\alpha_1 \vec{u}) = \alpha_1 \frac{\bar{K}_S}{K_S^1} \nabla \cdot \vec{u} + \frac{\tilde{K}_\Gamma}{\tilde{K}_S} Q_1 + \frac{\tilde{K}_c}{\tilde{K}_S} \rho \dot{Y}$$

$$\bar{K}_S = \left(\frac{\alpha_1}{K_S^1} + \frac{\alpha_2}{K_S^2} \right)^{-1}, \quad \tilde{K}_S = \left(\frac{K_S^1}{\alpha_1} + \frac{K_S^2}{\alpha_2} \right), \quad \tilde{K}_\Gamma = \left(\frac{\Gamma_1}{\alpha_1} + \frac{\Gamma_2}{\alpha_2} \right)$$

$\tilde{K}_c = \left(\frac{c_1^2}{\alpha_1} + \frac{c_2^2}{\alpha_2} \right), \; K_S^\iota = \rho_\iota c_\iota^2, \; Q_1 = H(T_2 - T_1), \; \dot{Y}$ (mass trans.)

Thank you