Eulerian interface-sharpening methods for compressible flow

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Objective

Derive consistent interface-sharpening model & method for compressible (single & multiphase) flow problems

- PDE-based approach

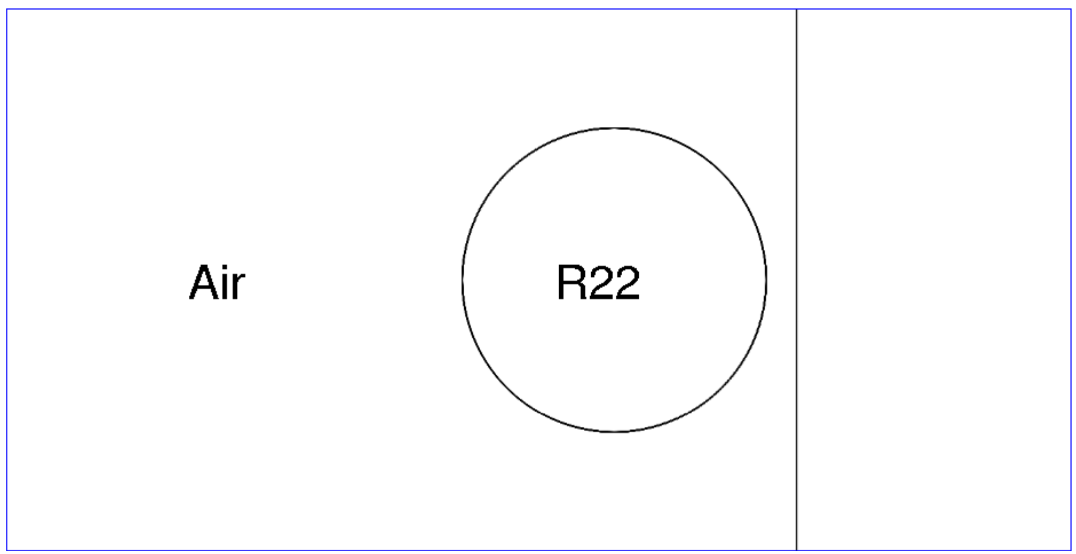
$$\partial_t q + \nabla \cdot f(q) + B \nabla q = \frac{1}{\mu} \mathcal{D}_q$$

- Algebraic-based approach (with F. Xiao, Tokyo Tech.)

1. Sharp cell edge solution reconstruction
2. Solution update: MUSCL or semi-discretize scheme

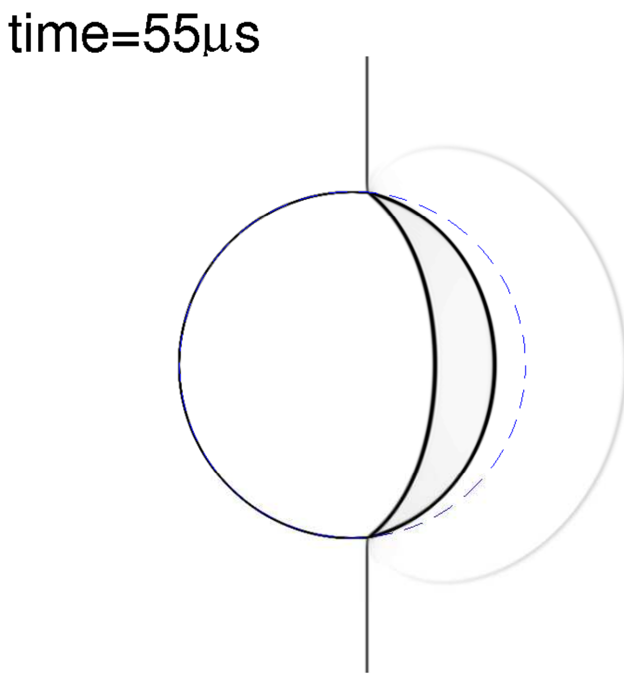
Shock in air & R22 bubble interaction

Leftward-going Mach 1.22 shock wave in air over heavier R22 bubble



Shock air-R22 bubble (cont.)

With anti-diffusion



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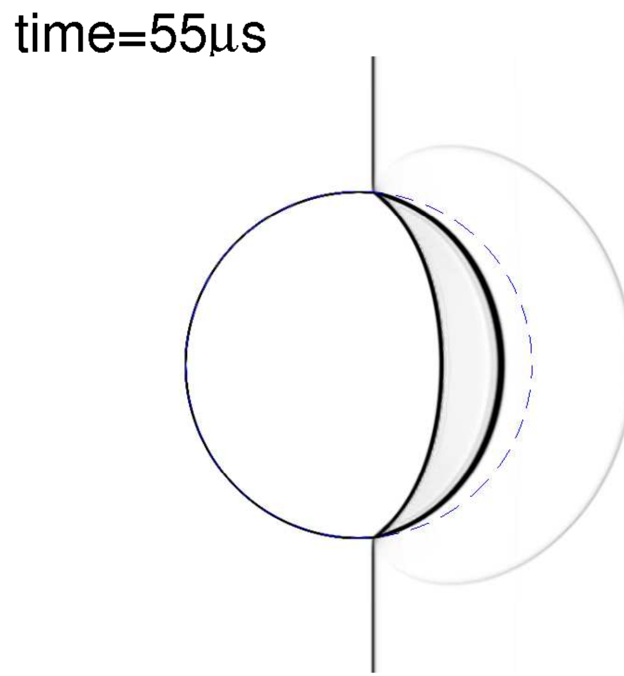
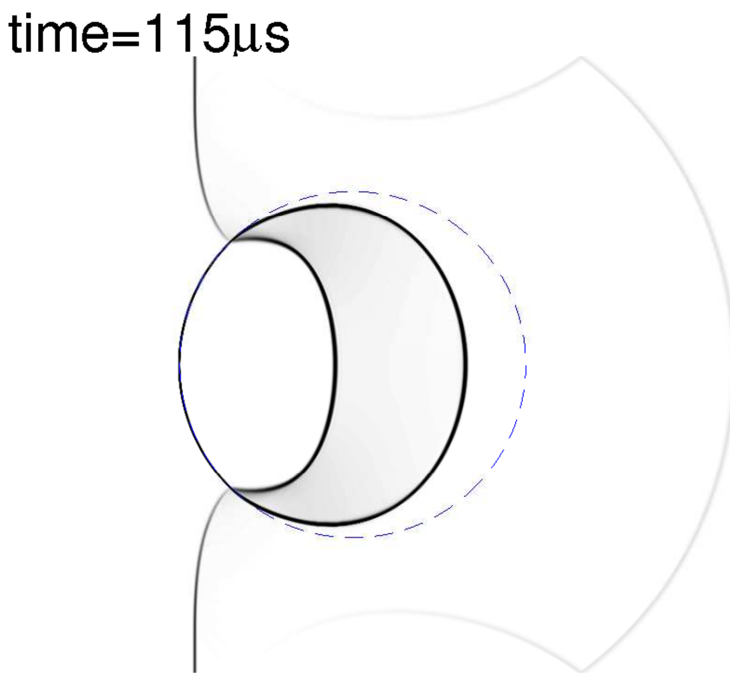


Shock in air-R22 bubble (cont.)

With anti-diffusion



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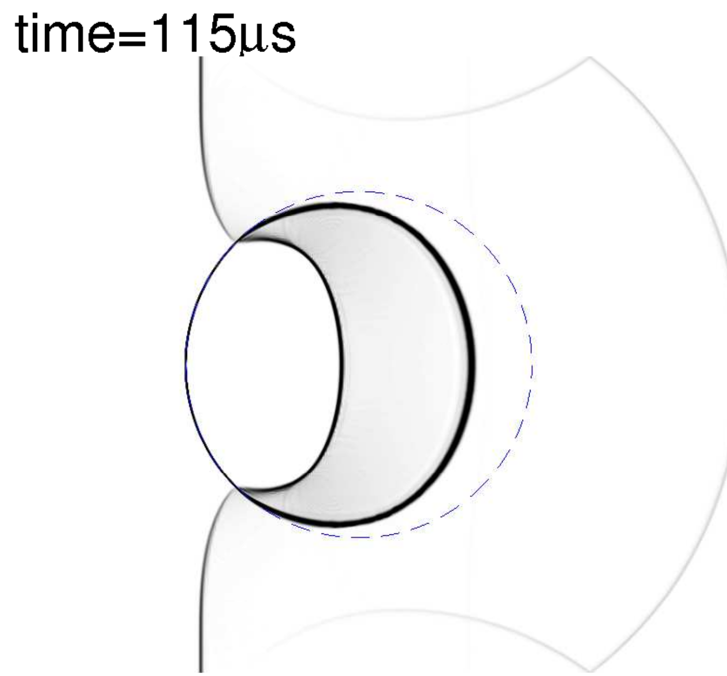
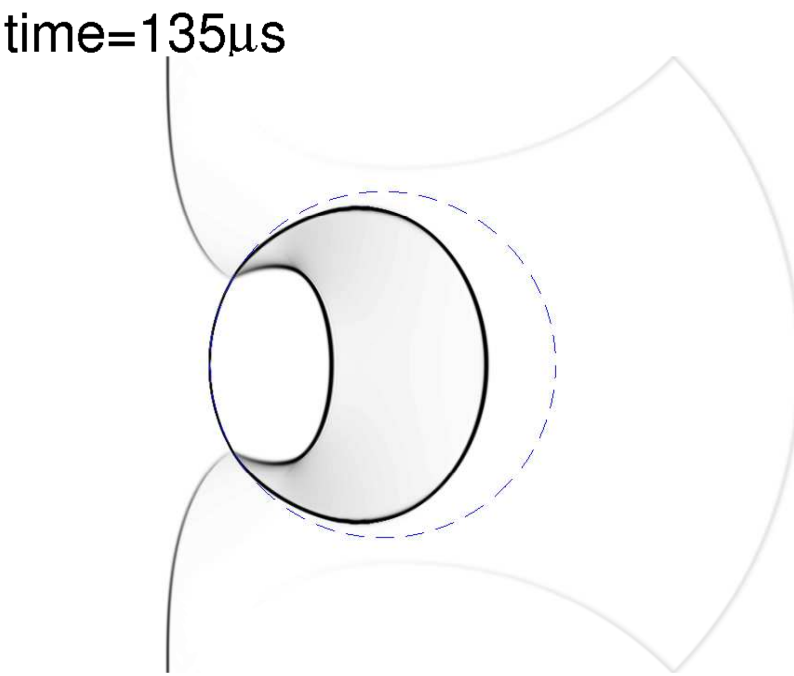


Shock air-R22 bubble (cont.)

With anti-diffusion



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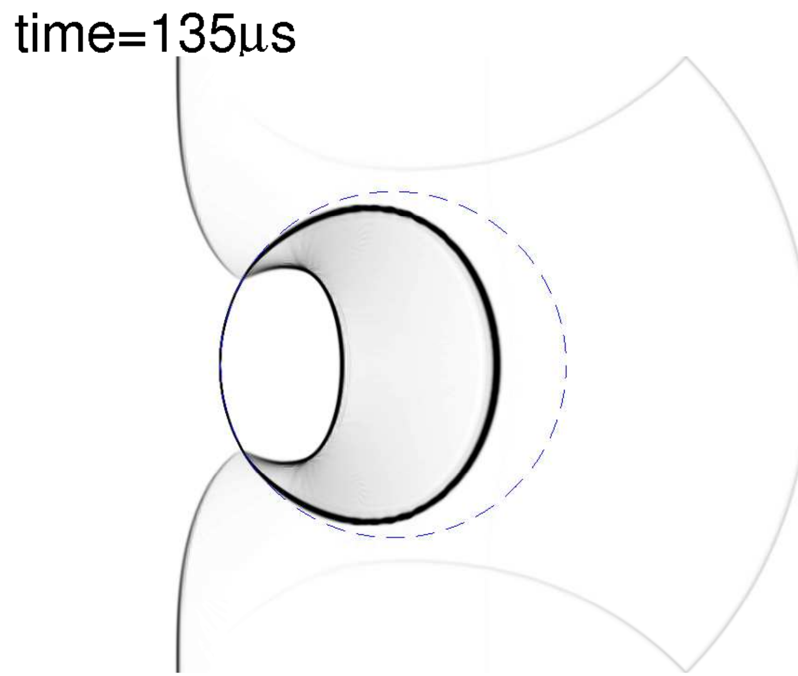
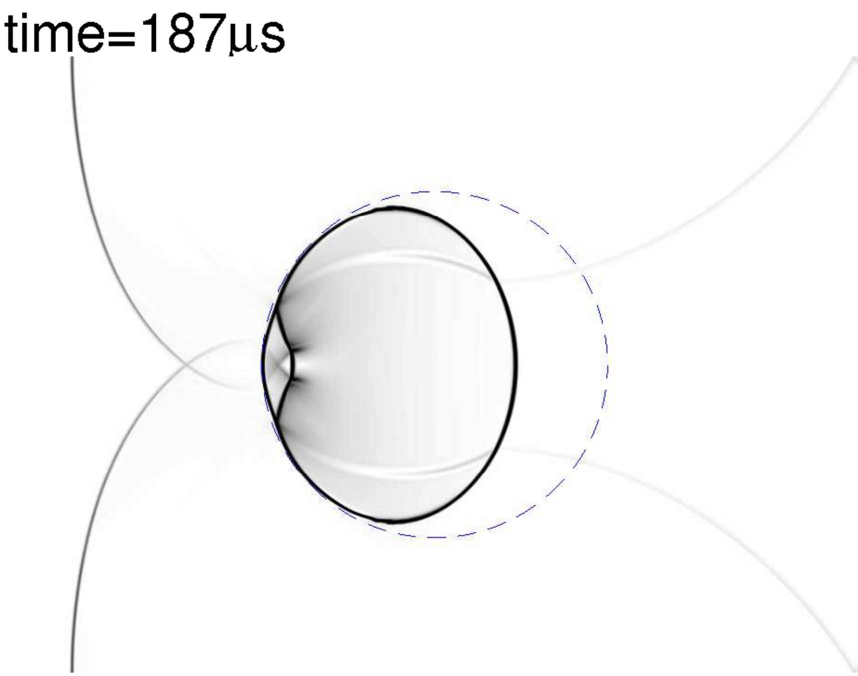


Shock air-R22 bubble (cont.)

With anti-diffusion



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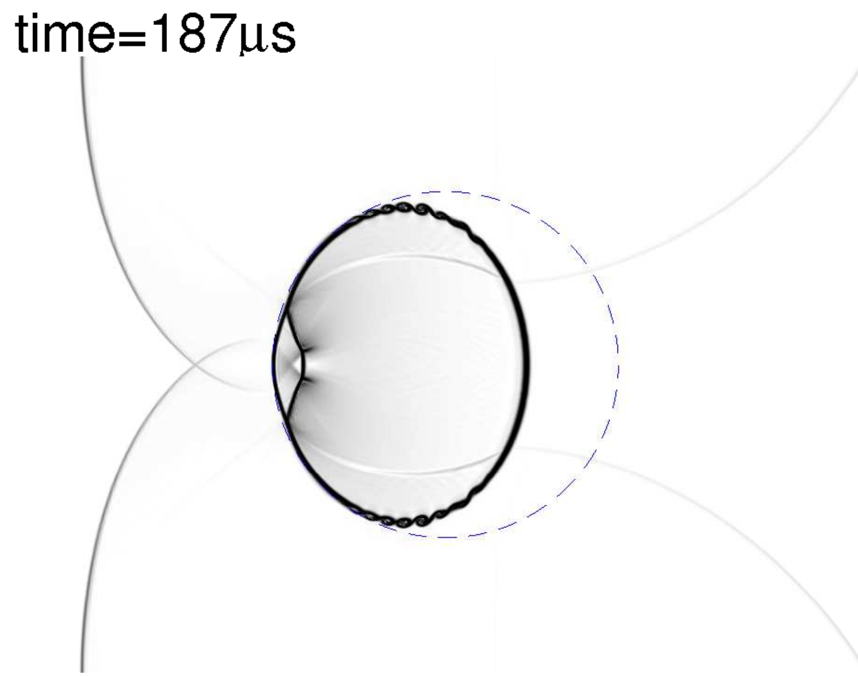
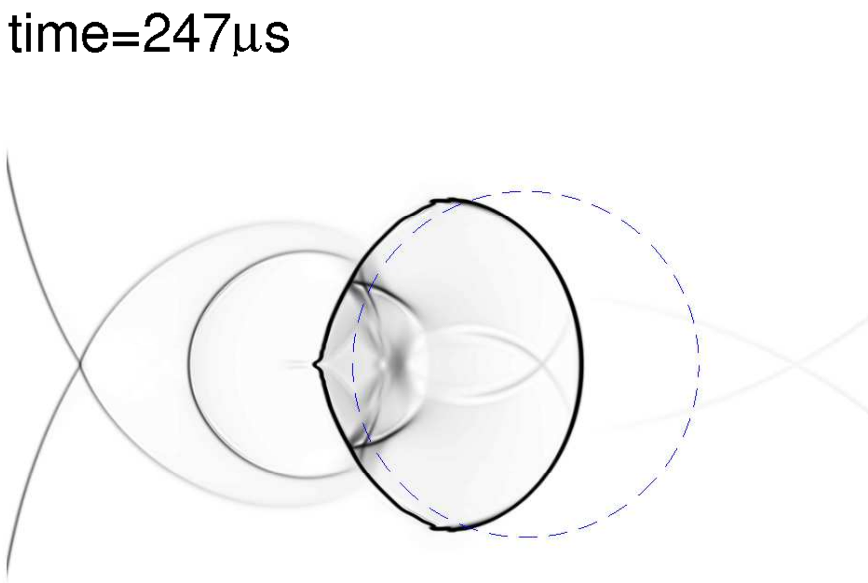


Shock air-R22 bubble (cont.)

With anti-diffusion



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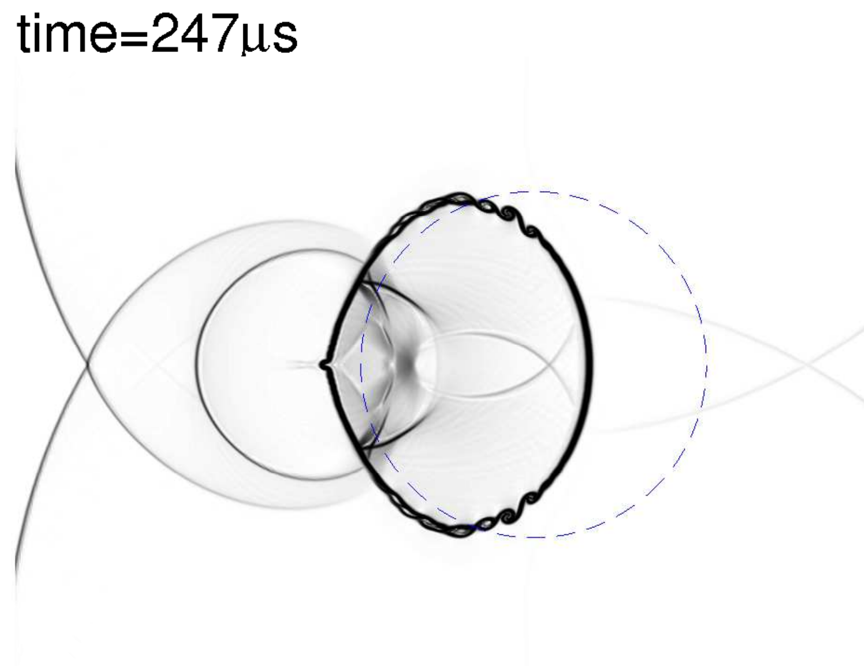
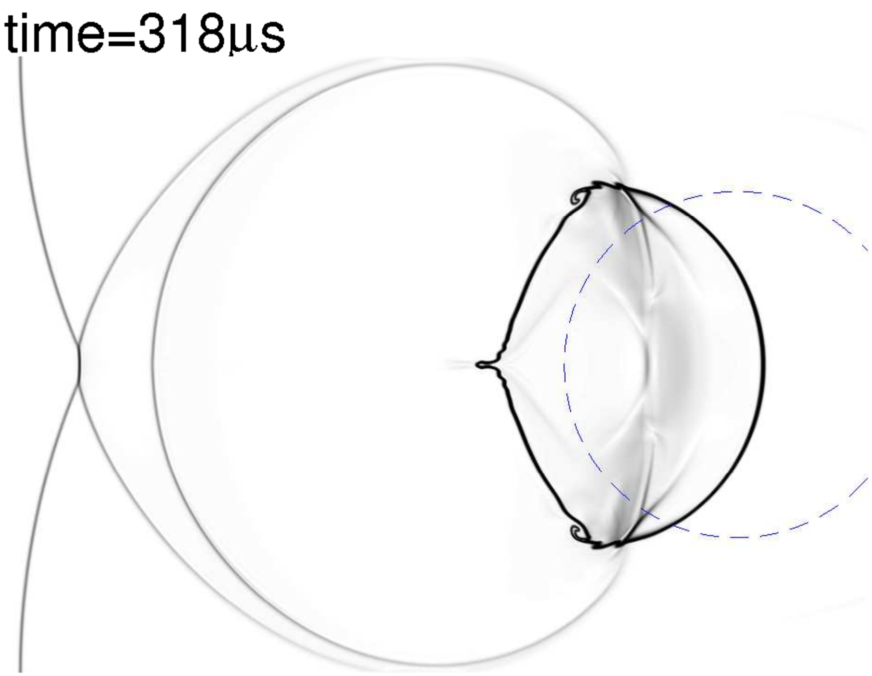


Shock air-R22 bubble (cont.)

With anti-diffusion



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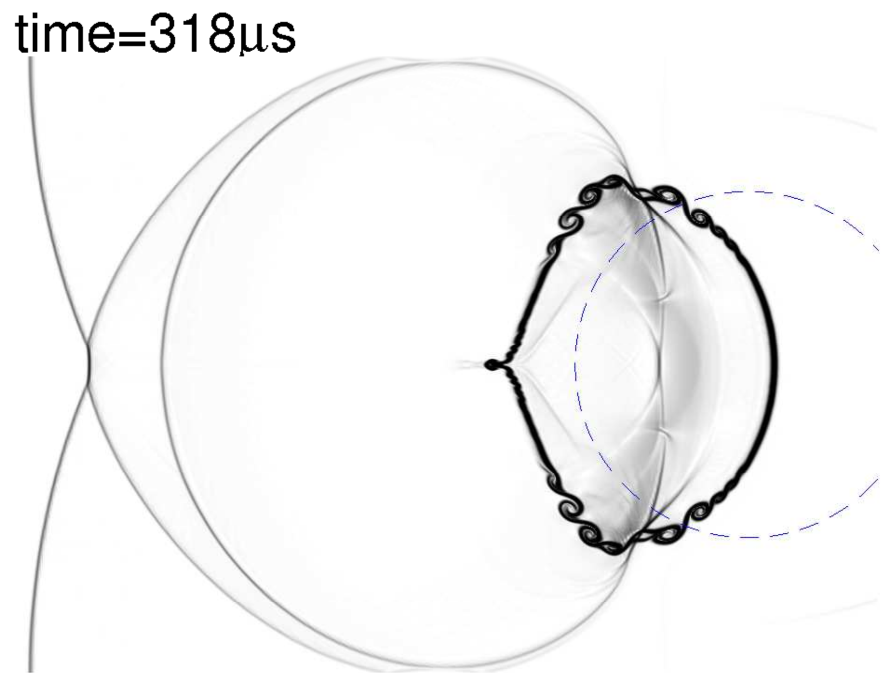
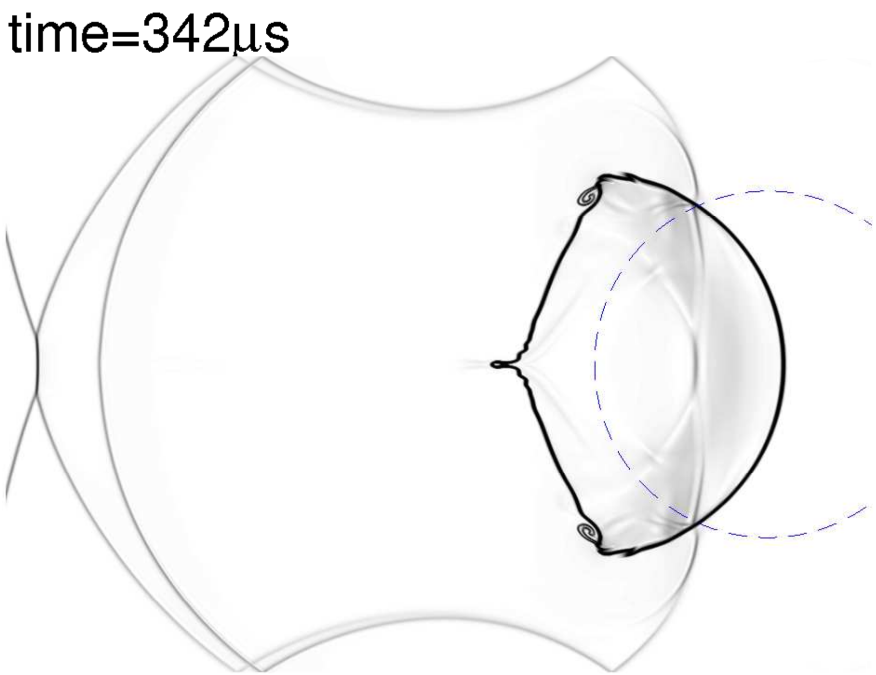


Shock air-R22 bubble (cont.)

With anti-diffusion



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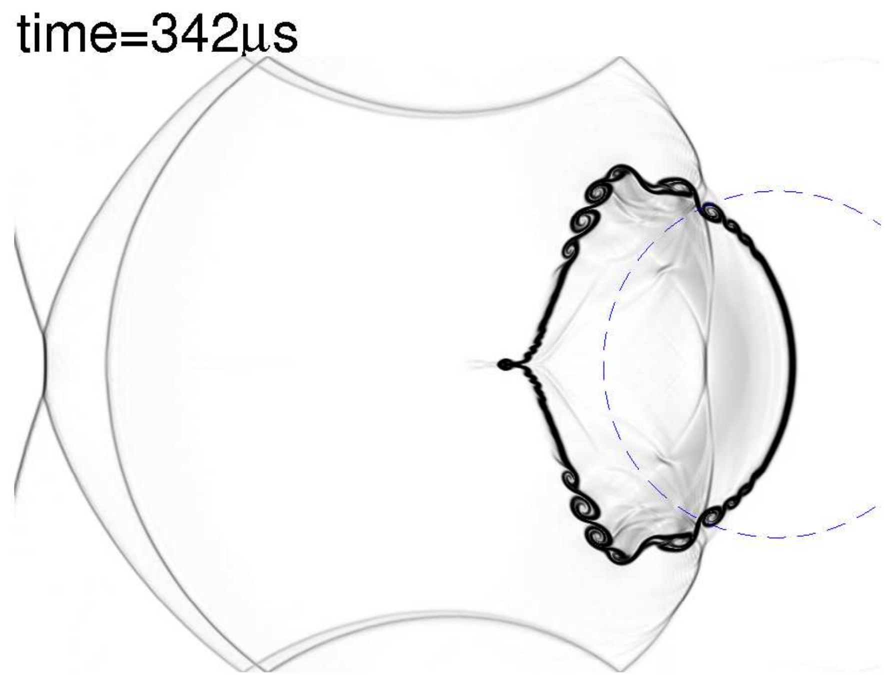
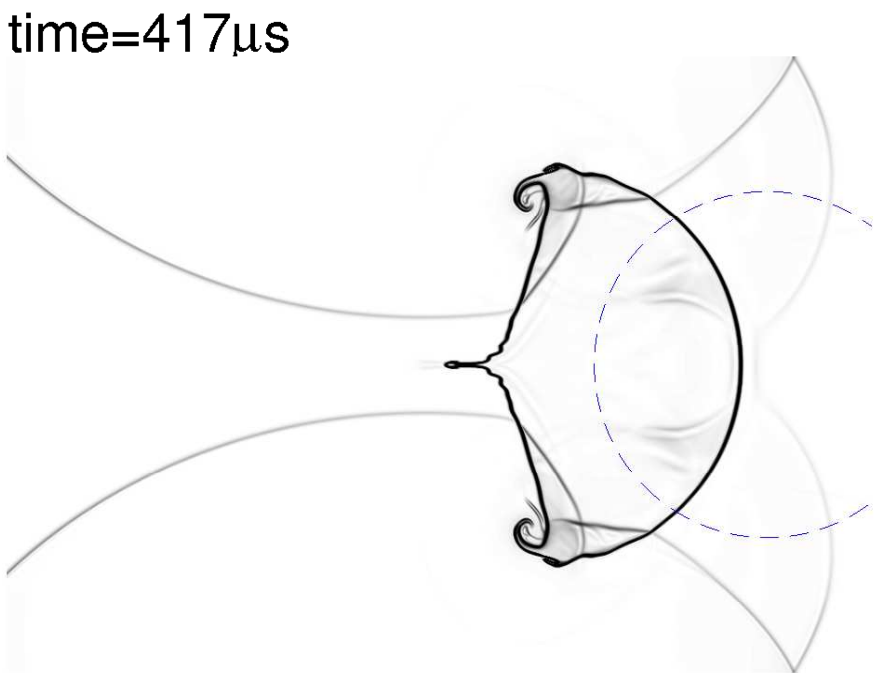


Shock air-R22 bubble (cont.)

With anti-diffusion



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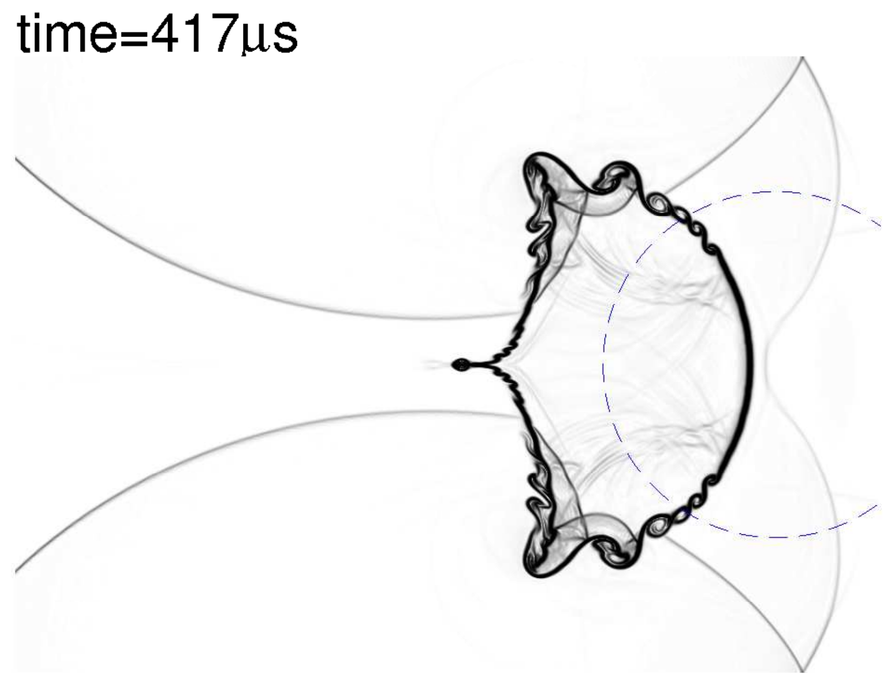


Shock air-R22 bubble (cont.)

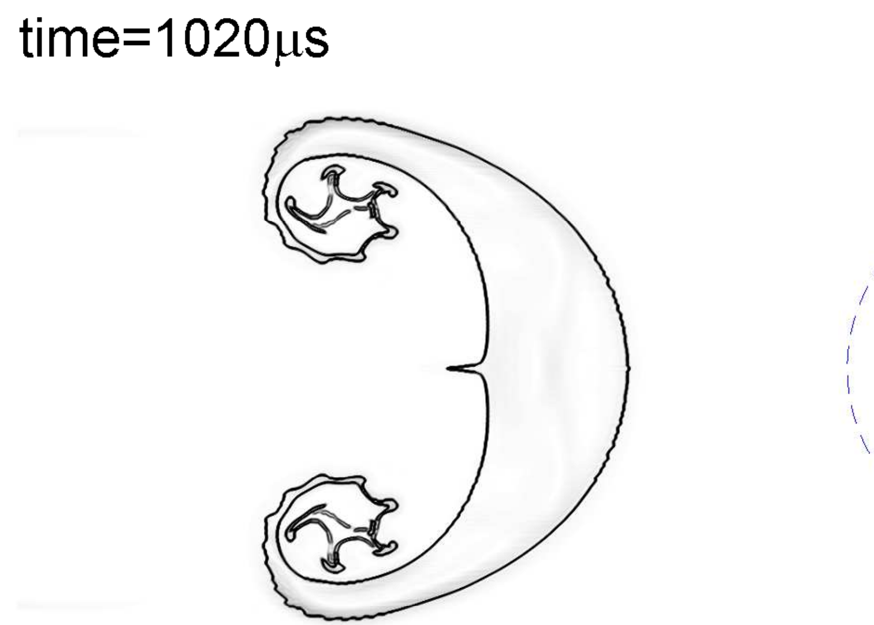
With anti-diffusion

WENO 5

Shock air-R22 bubble (cont.)

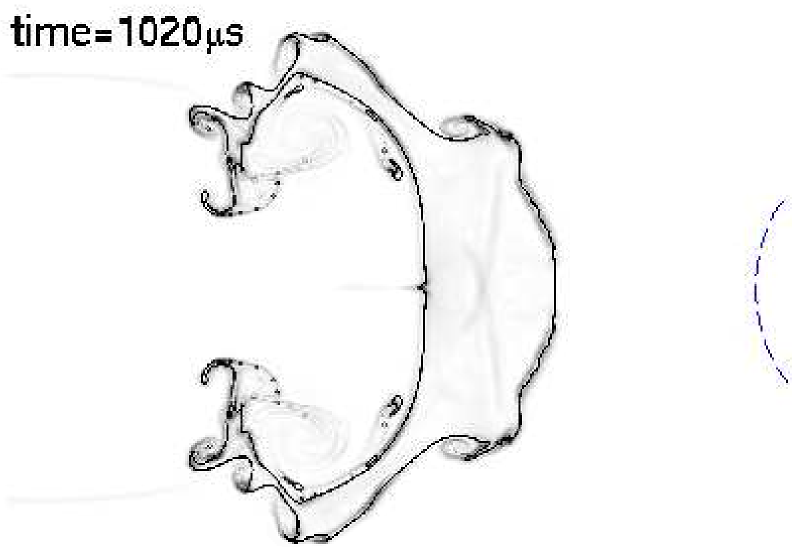
With anti-diffusion

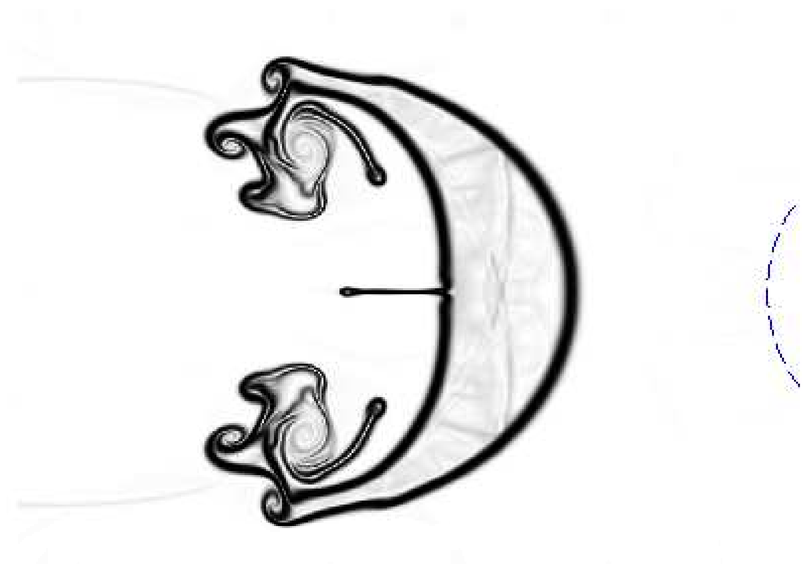
With THINC



Shock air-R22 bubble (cont.)

Volume tracking (Shyue 2006) 2nd order





PDE-based interface sharpening

Incompressible 2-phase flow: PDE-based interface sharpening for volume-fraction transport

$$\partial_t \alpha + u \cdot \nabla \alpha = \frac{1}{\mu} \mathcal{D}_\alpha, \qquad \mu \in \mathbb{R} \gg 1$$

- Artificial compression: Harten CPAM 1977, Olsson & Kreiss JCP 2005

$$\mathcal{D}_\alpha := \nabla \cdot \left[\left(D(\Delta x) \nabla \alpha \cdot \vec{n} - \alpha \left(1 - \alpha \right) \right) \vec{n} \right]$$

- Anti-diffusion: So, Hu & Adams JCP 2011

$$\mathcal{D}_\alpha := -\nabla \cdot \left(D(u) \nabla \alpha \right)$$

Model interface-only problem

Interface-only problem for compressible 1-phase Euler equations with constant pressure p, velocity u, & jump in density ρ across interfaces

$$
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho u) &= 0 \quad (\text{Mass}) \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u + p I_N) &= 0 \quad (\text{Momentum}) \\
\partial_t E + \nabla \cdot (E u + p u) &= 0 \quad (\text{Energy})
\end{aligned}
$$

yielding basic transport equations for interfaces as

$$
\begin{aligned}
\partial_t \rho + u \cdot \nabla \rho &= 0 \\
u \left(\partial_t \rho + u \cdot \nabla \rho\right) &= 0 \\
\frac{u \cdot u}{2} \left(\partial_t \rho + u \cdot \nabla \rho\right) + \partial_t (\rho e) + u \cdot \nabla (\rho e) &= 0
\end{aligned}
$$

Interface-only anti-diffusion model

Shyue (2011) proposed anti-diffusion model for density

$$\partial_t \rho + u \cdot \nabla \rho = \frac{1}{\mu} \mathcal{D}_\rho, \qquad \mathcal{D}_\rho := -\nabla \cdot (D \nabla \rho)$$

To ensure velocity equilibrium for momentum

$$u \left(\partial_t \rho + u \cdot \nabla \rho\right) = \frac{1}{\mu} \mathcal{D}_{\rho u}, \qquad \mathcal{D}_{\rho u} := u \, \mathcal{D}_\rho$$

& velocity-pressure equilibrium, for total energy

$$\frac{u \cdot u}{2} \left(\partial_t \rho + u \cdot \nabla \rho\right) + \partial_t \left(\rho e\right) + u \cdot \nabla \left(\rho e\right) = \frac{1}{\mu} \mathcal{D}_E,$$

$$\mathcal{D}_E := \left[\frac{u \cdot u}{2} + \partial_\rho (\rho e)\right] \mathcal{D}_\rho$$

Mie-Grüneisen EOS $p(\rho, e) = p_\infty(\rho) + \Gamma(\rho)\rho \left[e - e_\infty(\rho)\right]$

1-phase anti-diffusion model

To deal with shock waves, anti-diffusion model for compressible 1-phase flow

$$\partial_t \rho + \nabla \cdot (\rho u) = \frac{1}{\mu}\mathcal{D}_\rho$$
$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u + p I_N) = \frac{1}{\mu}\mathcal{D}_{\rho u}$$
$$\partial_t E + \nabla \cdot (E\vec{u} + pu) = \frac{1}{\mu}\mathcal{D}_E$$

$$\mathcal{D}_\rho := -H_I \nabla \cdot (D\nabla\rho)\,,\ \mathcal{D}_{\rho u} := u\,\mathcal{D}_\rho,\ \mathcal{D}_E := \left[\frac{u\cdot u}{2} + \partial_\rho(\rho e)\right]\mathcal{D}_\rho$$

H_I : Interface indicator $= \begin{cases} 1 & \text{if near interface,} \\ 0 & \text{otherwise} \end{cases}$

Anti-diffusion method

Anti-diffusion model in compact form

$$\partial_t q + \nabla \cdot f = \frac{1}{\mu}\mathcal{D}_q$$

with q, f, & $\mathcal{D}_q$ defined (not shown)

Fractional step method:

1. Solve homogeneous equation without source terms

$$\partial_t q + \nabla \cdot f = 0$$

2. Iterate model equation with source terms

$$\partial_\tau q = \mathcal{D}_q$$

to sharp layer; $\tau = \mu t$ (pseudo time)

Numerical interface-only problem

Consistency of numerical solution for interface-only problem

Step 1: assume consistent approximation model equation without anti-diffusion,

$$\text{smeared } \rho^* \qquad \& \qquad \text{equilibrium} \quad u^* = u^n, \quad p^* = p^n$$

(may be difficult with highly nonlinear MG EOS)

Step 2: assume anti-diffusion for ρ is done consistently & stably, yielding update ρ^* to ρ^{n+1} & for momentum,

$$(\rho u)^{n+1} := (\rho u)^* + \mathcal{D}_{\rho u} = (\rho u)^* + u^* \mathcal{D}_\rho$$

$$= (\rho u)^* + u^* \left(\rho^{n+1} - \rho^*\right) = \rho^{n+1} u^* \quad \Longrightarrow \quad u^{n+1} = \frac{\rho^{n+1} u^*}{\rho^{n+1}} = u^*$$

Numerical interface-only (cont.)

For total energy, $E^{n+1} := E^* + \mathcal{D}_E$, *i.e.*,

$$(\rho K + \rho e)^{n+1} := (\rho K + \rho e)^* + (K + \partial_\rho(\rho e))^* \mathcal{D}_\rho, \qquad K = \frac{u \cdot u}{2}$$

Splitting kinetic & internal energy with MG EOS

$$(\rho K)^{n+1} := (\rho K)^* + K^* \left(\rho^{n+1} - \rho^*\right) = \rho^{n+1} K^* \quad \Longrightarrow \quad K^{n+1} = K^*$$

$$\Longrightarrow \quad u^{n+1} = u^* \quad (\text{not true if } u \text{ has transverse jump})$$

$$\left(\frac{p - p_\infty}{\Gamma} + \rho e_\infty\right)^{n+1} := \left(\frac{p - p_\infty}{\Gamma} + \rho e_\infty\right)^* + \partial_\rho \left(\frac{p - p_\infty}{\Gamma} + \rho e_\infty\right)^* \left(\rho^{n+1} - \rho^*\right)$$

$$\Longrightarrow \quad p^{n+1} = p^* \quad (\text{for linearized MG EOS only})$$

Numerical shock wave problems

Weak solution for problems with shock & rarefaction waves

Interface indicator H_I takes value zero away from interfacs, yielding standard compressible Euler equations in conservation form

Step 1, use state-of-the-art shock capturing method for entropy-satisfying weak solutions

Implementation issues

Methods used here are very elementary, *i.e.*,

Step 1: Clawpack (or Wenoclaw/Sharpclaw) for homogeneous equation over CFL-constrained Δt

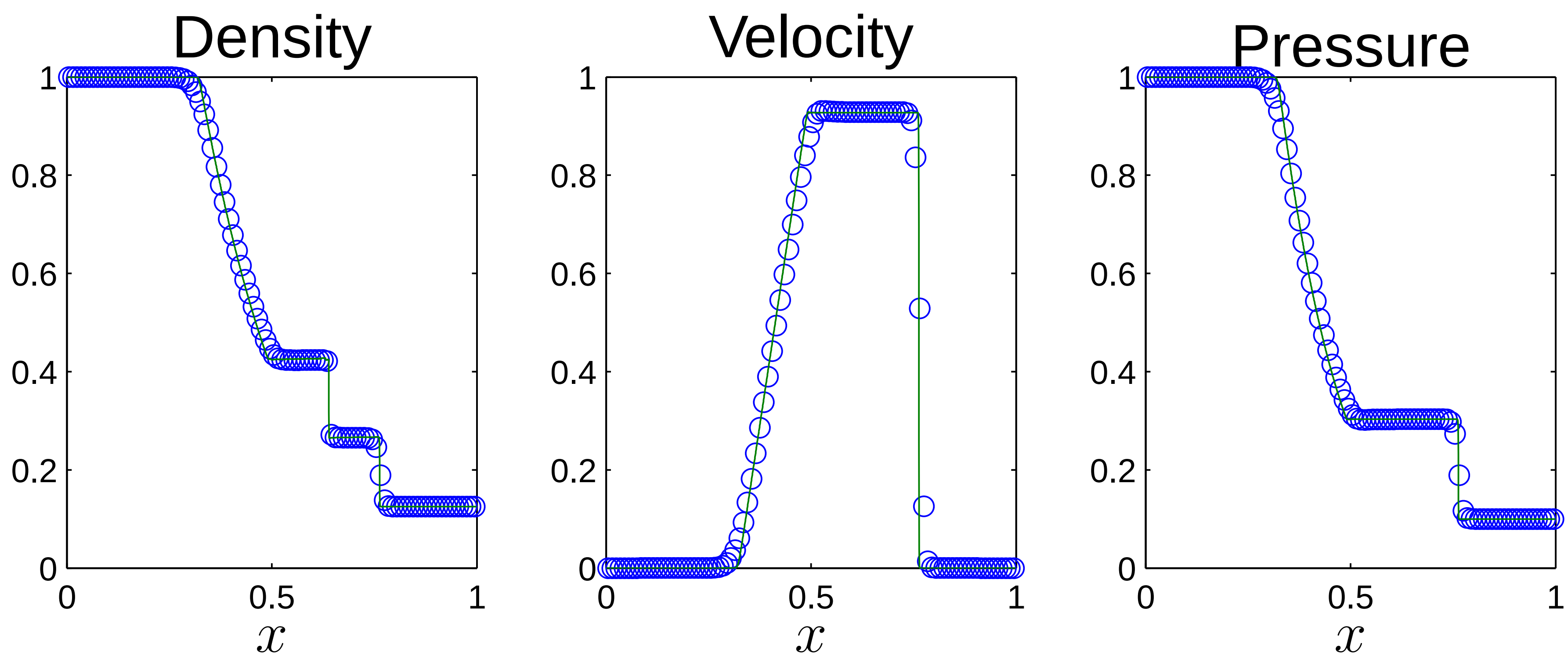
Step 2: explicit 1-step method for anti-diffusion

- Numerical regularization to $\nabla\rho$ & $\mathcal{D}_\rho$
- Diffusion coefficient $D = D(u)$ (local in space & time)
- Time step (forward Euler) $\Delta\tau \le \min\left(\Delta t, \frac{\min_{i=1}^{N} \Delta x_i^2}{2ND_{max}}\right)$
- Stopping criterion: Run $1-2$ iteration currently
- H_I is chosen based on checking jumps in ρ & p

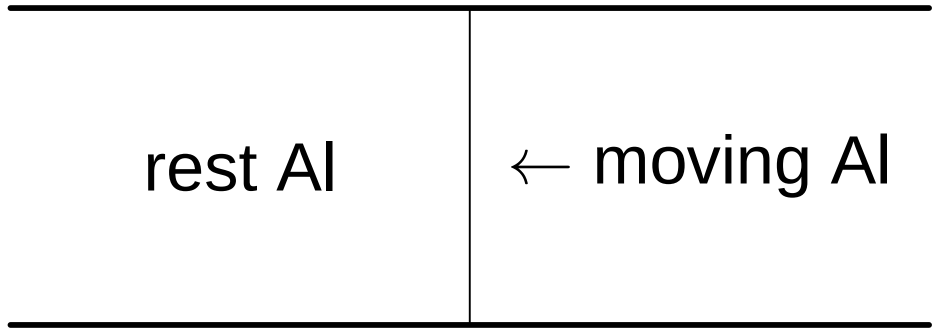
Sod Riemann problem

High-resolution result with anti-diffusion

Ideal gas: $$p(\rho, e) = (\gamma - 1)\rho e$$



Aluminum impact problem



IC:

$$(\rho,\ u,\ p)_L = \left(4\times 10^3\ \text{kg/m}^3,\ 0\ \text{m/s},\ 7.93\times 10^9\ \text{Pa}\right)$$

$$(\rho,\ u,\ p)_R = \left(2785\ \text{kg/m}^3,\ -2\times 10^3\ \text{m/s},\ 0\ \text{Pa}\right)$$

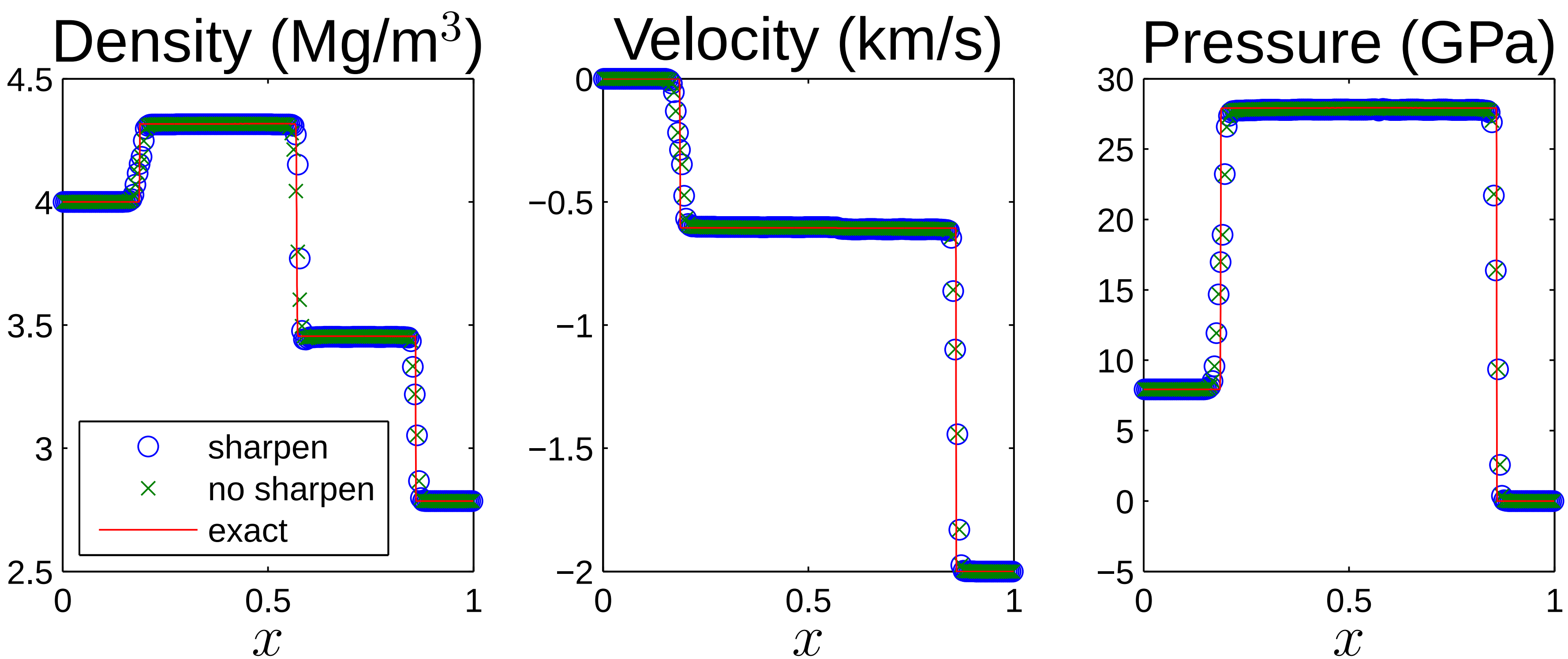
Mie-Grüneisen EOS with

$$\Gamma(\rho) = \Gamma_0(1-\eta)^\alpha, \quad p_\infty(\rho) = \frac{\rho_0 c_0^2 \eta}{(1-s\eta)^2}, \quad e_\infty(\rho) = \frac{\eta}{2\rho_0}\left(p_0 + p_\infty(\rho)\right)$$

ρ_0(kg/m^3)	c_0 (m/s)	s	Γ_0	α	p_0	e_0
2785	5328	1.338	2.0	1	0	0

Aluminum impact (cont.)

High-resolution result with anti-diffusion at time $t = 50\mu$s



1-fluid multiphase anti-diffusion

Compressible multiphase: One-fluid anti-diffusion model

$$
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho u) &= \frac{1}{\mu}\mathcal{D}_\rho \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u + p I_N) &= \frac{1}{\mu}\mathcal{D}_{\rho u} \\
\partial_t E + \nabla \cdot (E\vec{u} + p u) &= \frac{1}{\mu}\mathcal{D}_E \\
\partial_t \phi_j + u \cdot \nabla \phi_j &= \rho d_\rho \phi_j \nabla \cdot u + \frac{1}{\mu}\mathcal{D}_{\phi_j}
\end{aligned}
$$

Mixture pressure modeled by Mie-Grüneisen EOS:

$$
\phi_1 = \frac{1}{\Gamma(\rho)}, \quad \phi_2 = \frac{p_\infty(\rho)}{\Gamma(\rho)}, \qquad \phi_3 = \rho e_\infty(\rho)
$$

$\mu \to \infty$ reduces to fluid-mixture model (Shyue JCP 2001)

5-equation 2-phase flow model

Unsteady, inviscid, compressible homogeneous 2-phase flow governed by 5-equation model

$$
\begin{aligned}
\partial_t\left(\alpha_1\rho_1\right) + \nabla\cdot\left(\alpha_1\rho_1 u\right) &= 0 \quad && \text{(Phasic } 1 \text{ continuity)}\\
\partial_t\left(\alpha_2\rho_2\right) + \nabla\cdot\left(\alpha_2\rho_2 u\right) &= 0 \quad && \text{(Phasic } 2 \text{ continuity)}\\
\partial_t\left(\rho u\right) + \nabla\cdot\left(\rho u\otimes u + pI_N\right) &= 0 \quad && \text{(Momentum)}\\
\partial_t E + \nabla\cdot\left(E u + p u\right) &= 0 \quad && \text{(Energy)}\\
\partial_t \alpha_1 + u\cdot\nabla\alpha_1 &= 0 \quad && \text{(Volume fraction)}
\end{aligned}
$$

Phasic pressure p_k follows Mie-Grüneisen EOS

$$
p_k(\rho_k, e_k) = p_{\infty,k}(\rho_k) + \Gamma_k(\rho_k)\rho_k\left[e_k - e_{\infty,k}(\rho_k)\right] \qquad k = 1,2
$$

5-equation model (cont.)

- Isobaric closure leads to mixture pressure , *i.e.*,

Substitute $p = p_1 = p_2$ in $\rho e = \sum_{k=1}^{2} \rho_k e_k,$ yielding

$$p = \left(\rho e - \sum_{k=1}^{2} \alpha_k \rho_k e_{\infty,k}(\rho_k) + \sum_{k=1}^{2} \alpha_k \frac{p_{\infty,k}(\rho_k)}{\Gamma_k(\rho_k)} \right) \Big/ \sum_{k=1}^{2} \frac{\alpha_k}{\Gamma_k(\rho_k)}$$

- Model is hyperbolic with mixture acoustic impedance Allaire *et al.* (JCP 2002)

$$\rho c^2 = \sum_{k=1}^{2} \alpha_k \rho_k c_k^2, \qquad c_k : \text{phasic sound speed}$$

5-equation model (cont.)

In cavitated regions, $p < p_{\text{sat}}$, cutoff model

- Non-conservative energy correction

$$E := E_{\text{sat}} = \sum_{k=1}^{2} \alpha_k (\rho_k e_k)_{\text{sat}} + \rho K$$

cutoff phasic internal energy is

$$(\rho_k e_k)_{\text{sat}} = \frac{p_{\text{sat}} - p_{\infty,k}(\rho_{\text{sat}})}{\Gamma_k(\rho_{\text{sat}})} + \rho_{\text{sat}} e_{\infty,k}(\rho_{\text{sat}})$$

- α-based energy-preserving correction (Shyue 2012)

$$\alpha_1 := (\alpha_1)_{\text{sat}} = \frac{\rho e - (\rho_2 e_2)_{\text{sat}}}{(\rho_1 e_1)_{\text{sat}} - (\rho_2 e_2)_{\text{sat}}}$$

5-equation anti-diffusion model

Proposed 5-equation anti-diffusion model

$$\partial_t(\alpha_1\rho_1) + \nabla\cdot(\alpha_1\rho_1 u) = \frac{1}{\mu}\mathcal{D}_{\alpha_1\rho_1}$$

$$\partial_t(\alpha_2\rho_2) + \nabla\cdot(\alpha_2\rho_2 u) = \frac{1}{\mu}\mathcal{D}_{\alpha_2\rho_2}$$

$$\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u + pI_N) = \frac{1}{\mu}\mathcal{D}_{\rho u}$$

$$\partial_t E + \nabla\cdot(Eu + pu) = \frac{1}{\mu}\mathcal{D}_{E}$$

$$\partial_t\alpha_1 + u\cdot\nabla\alpha_1 = \frac{1}{\mu}\mathcal{D}_{\alpha_1}$$

Exist two ways to set $\mathcal{D}_z$, $z = \alpha_1\rho_1, \ldots, \alpha_1$, in literature

5-equation anti-diffusion (cont.)

- α-ρ based (Shyue 2011)

$$\mathcal{D}_{\alpha_1} := -\nabla\cdot(D\nabla\alpha_1), \quad \mathcal{D}_{\alpha_k\rho_k} := -H_I\nabla\cdot(D\nabla\alpha_k\rho_k), \quad k=1,2$$

$$\mathcal{D}_\rho := \sum_{k=1}^{2}\mathcal{D}_{\alpha_k\rho_k}, \quad \mathcal{D}_{\rho u} := u\mathcal{D}_\rho,$$

$$\mathcal{D}_E := K\mathcal{D}_\rho + \sum_{k=1}^{2}\partial_{\alpha_k\rho_k}(\rho_k e_k)\mathcal{D}_{\alpha_k\rho_k} + \sum_{k=1}^{2}\rho_k e_k\mathcal{D}_{\alpha_k}$$

- α-based only (So, Hu, & Adams JCP 2012)

$$\mathcal{D}_{\alpha_1} := -\nabla\cdot(D\nabla\alpha_1), \quad \mathcal{D}_{\alpha_k\rho_k} := \rho_k\mathcal{D}_{\alpha_k}, \quad k=1,2, \quad \mathcal{D}_{\alpha_2} := -\mathcal{D}_{\alpha_1}$$

$$\mathcal{D}_\rho := \sum_{k=1}^{2}\mathcal{D}_{\alpha_k\rho_k}, \quad \mathcal{D}_{\rho u} := u\mathcal{D}_\rho, \quad \mathcal{D}_E := K\mathcal{D}_\rho + \sum_{k=1}^{2}\rho_k e_k\mathcal{D}_{\alpha_k}$$

Use $\mathcal{D}_{\alpha_1} := \nabla\cdot\left[\left(D\nabla\alpha_1\cdot\vec{n} - \alpha_1\left(1-\alpha_1\right)\right)\vec{n}\right]$ (Shyue 2012)

5-equation interface-compression

Shukla, Pantano & Freund (JCP 2010)

$$\partial_t (\alpha_1 \rho_1) + \nabla \cdot (\alpha_1 \rho_1 u) = 0$$

$$\partial_t (\alpha_2 \rho_2) + \nabla \cdot (\alpha_2 \rho_2 u) = 0$$

$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u + p I_N) = 0$$

$$\partial_t E + \nabla \cdot (E u + p u) = 0$$

$$\partial_t \alpha_1 + u \cdot \nabla \alpha_1 = \frac{1}{\mu} n \cdot \nabla \left(D \nabla \alpha_1 \cdot n - \alpha_1 (1 - \alpha_1) \right)$$

$$\partial_t \rho + \nabla \cdot (\rho u) = \frac{1}{\mu} H_I(\alpha_1) n \cdot \left(\nabla (D \nabla \rho \cdot n) - (1 - 2\alpha_1) \nabla \rho \right)$$

Nonliner compression & linear diffusion for interface sharpening & stability

Method proposed there is unstable numerically

PDE-based interface sharpening

PDE-based interface-sharpening model in compact form

$$\partial_t q + \nabla \cdot f + B \nabla q = \frac{1}{\mu} \mathcal{D}_q$$

with q, f, B, & $\mathcal{D}_q$ defined (not shown)

Fractional step method is used

1. Solve homogenous equation without source terms

$$\partial_t q + \nabla \cdot f + B \nabla q = 0$$

2. Iterate model equation with source terms

$$\partial_\tau q = \mathcal{D}_q$$

to sharp layer; $\tau = \mu t$ (pseudo time)

Numerical interface-only problem

Consistency of numerical solution for interface-only problem

Step 1: assume consistent approximation model equation without anti-diffusion,

smeared $(\alpha_1\rho_1)^*, (\alpha_2\rho 2)^*, \alpha_1^* \quad \& \quad$ equilibrium $u^* = u^n, p^* = p^n$

(can be done even with highly nonlinear MG EOS)

Step 2: update $(\alpha_1\rho_1)^*$, $(\alpha_2\rho_2)^*$, α_1^* to $(\alpha_1\rho_1)^{n+1}$, $(\alpha_2\rho_2)^{n+1}$, α_1^{n+1} *via* anti-diffusion, & for momentum,

$$(\rho u)^{n+1} := (\rho u)^* + \mathcal{D}_{\rho u} = (\rho u)^* + u^* \mathcal{D}_\rho \quad \Longrightarrow \quad u^{n+1} = u^*$$

Numerical interface-only (cont.)

For total energy, $E^{n+1} := E^* + \mathcal{D}_E$, *i.e.*, α-based

$$(\rho K + \rho e)^{n+1} := (\rho K + \rho e)^* + K^* \mathcal{D}_\rho + \sum_{k=1}^{2} (\rho_k e_k)^* \mathcal{D}_{\alpha_k \rho_k}$$

Splitting kinetic & internal energy with MG EOS

$$(\rho K)^{n+1} := (\rho K)^* + K^* \left(\rho^{n+1} - \rho^*\right) = \rho^{n+1} K^* \quad \Longrightarrow \quad K^{n+1} = K^*$$

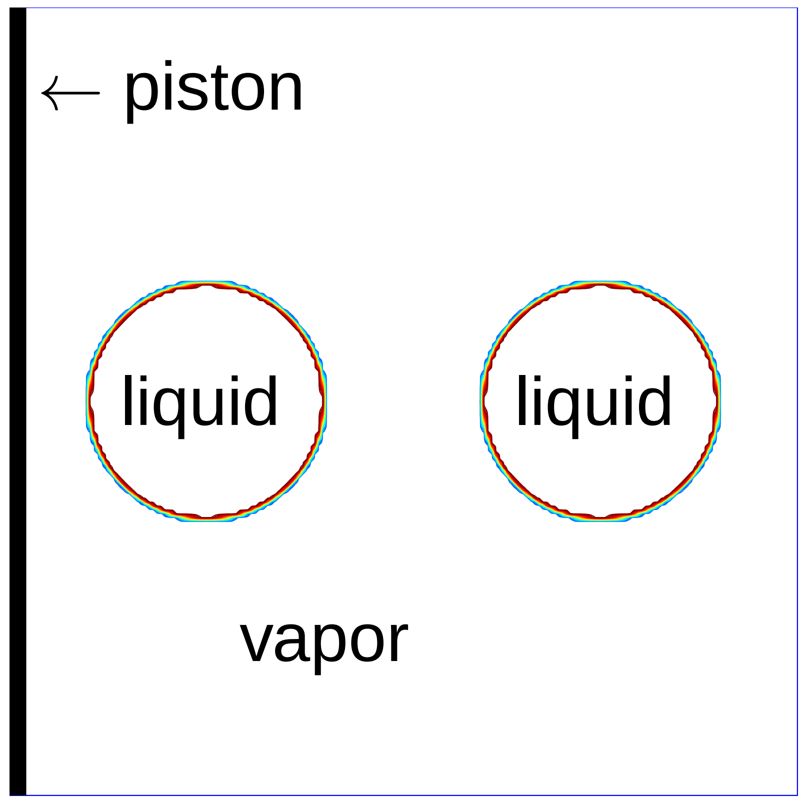
$$\sum_{k=1}^{2} (\alpha_k \rho_k e_k)^{n+1} := \sum_{k=1}^{2} (\alpha_k \rho_k e_k)^* + \sum_{k=1}^{2} (\rho_k e_k)^* \left(\alpha_k^{n+1} - \alpha_k^*\right)$$

$$\sum_{k=1}^{2} \left[\alpha_k \left(\frac{p - p_{\infty,k}}{\Gamma_k} + \rho_k e_{\infty,k}\right)\right]^{n+1} := \sum_{k=1}^{2} \left(\frac{p - p_{\infty,k}}{\Gamma_k} + \rho_k e_{\infty,k}\right)^* \alpha_k^{n+1}$$

$$\Longrightarrow \quad p^{n+1} = p^* \quad \text{(for general MG EOS)}$$

Piston-induced liquid drops depression

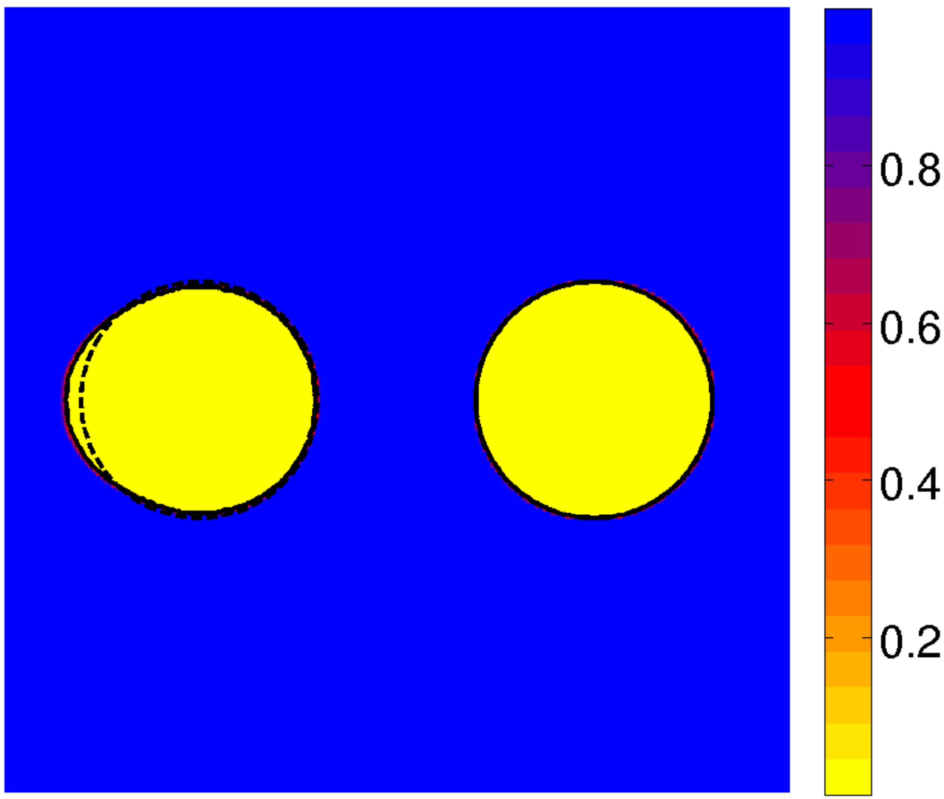
Liquid & vapor governed by stiffened gas EOS
Piston velocity $u = -100$m/s



Liquid drops depression

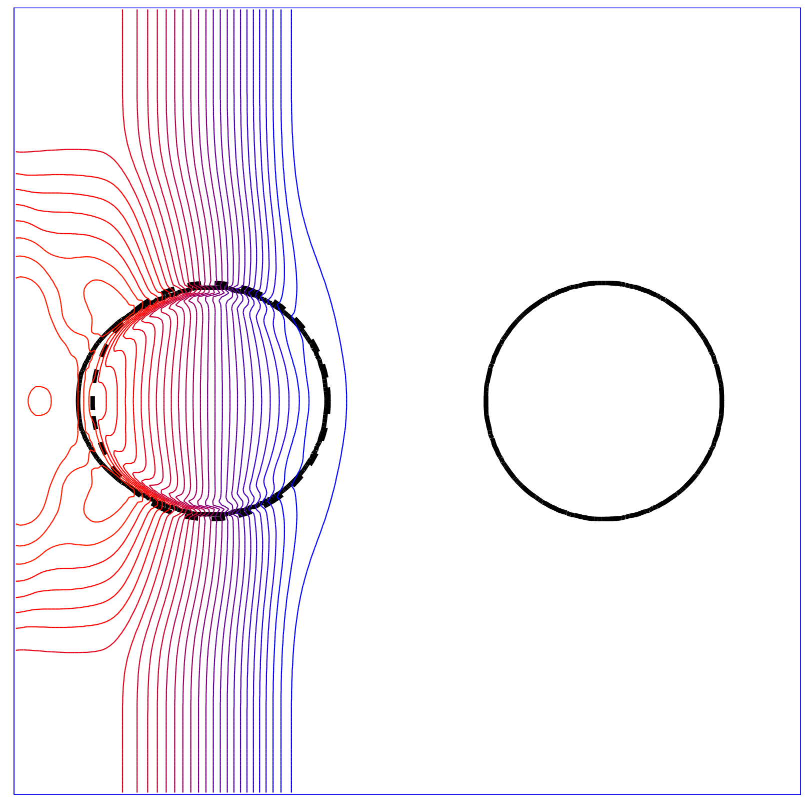
Vapor mass fraction

t=2ms



Mixture pressure

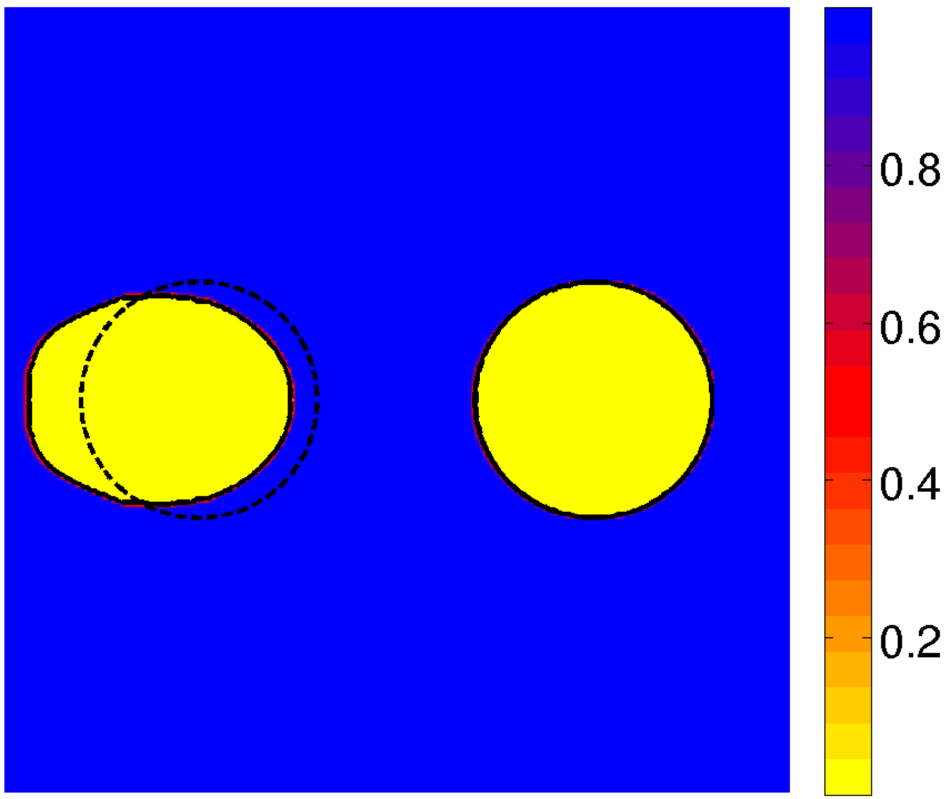
t=2ms



Liquid drops depression (cont.)

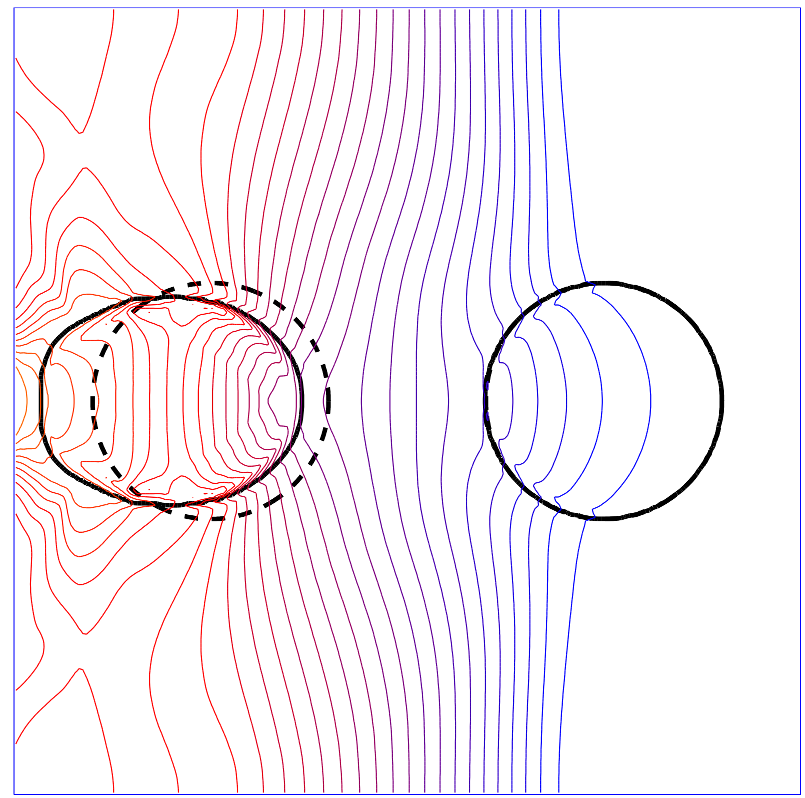
Vapor mass fraction

t=4ms



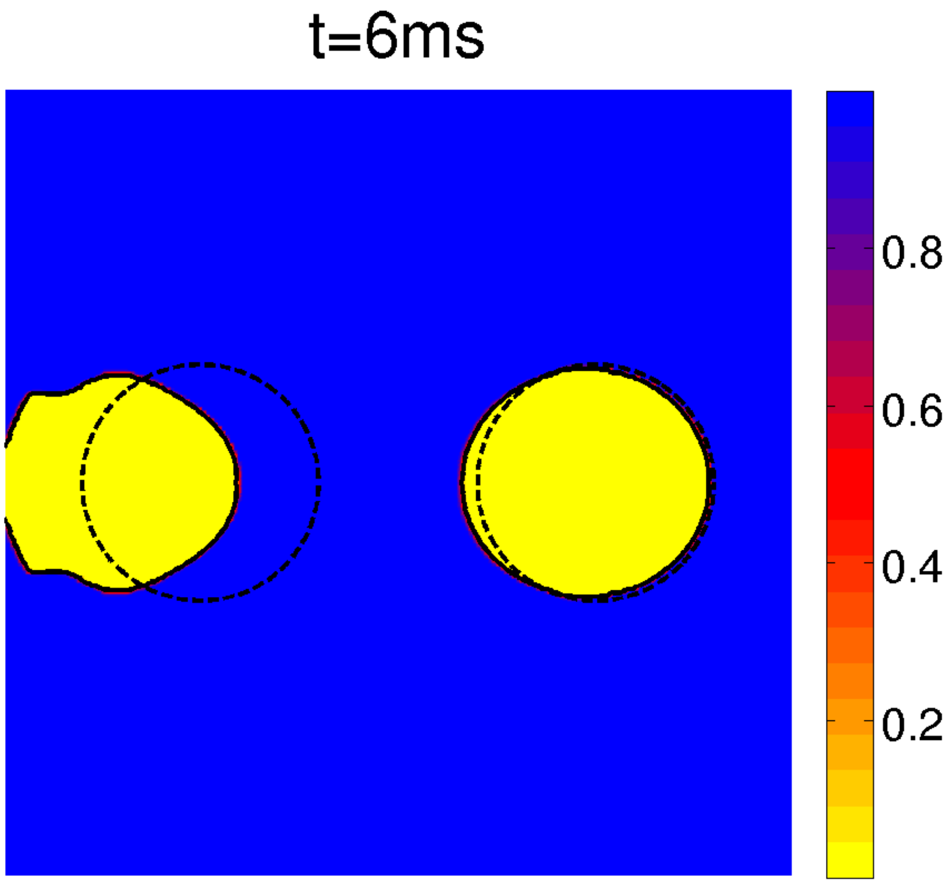
Mixture pressure

t=4ms

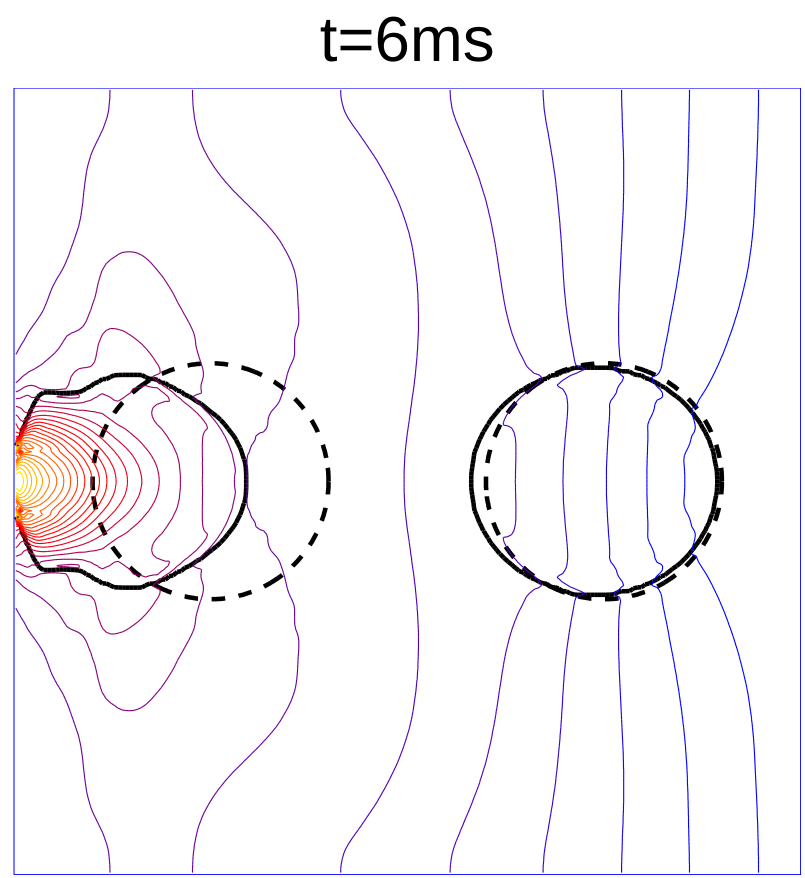


Liquid drops depression (cont.)

Vapor mass fraction



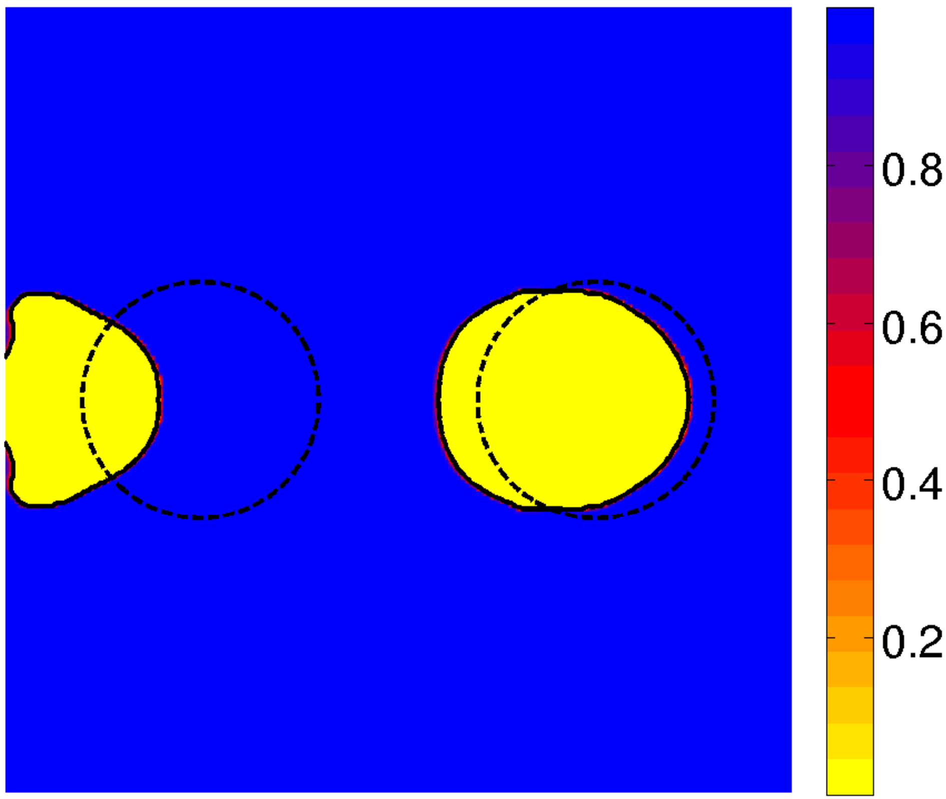
Mixture pressure



Liquid drops depression (cont.)

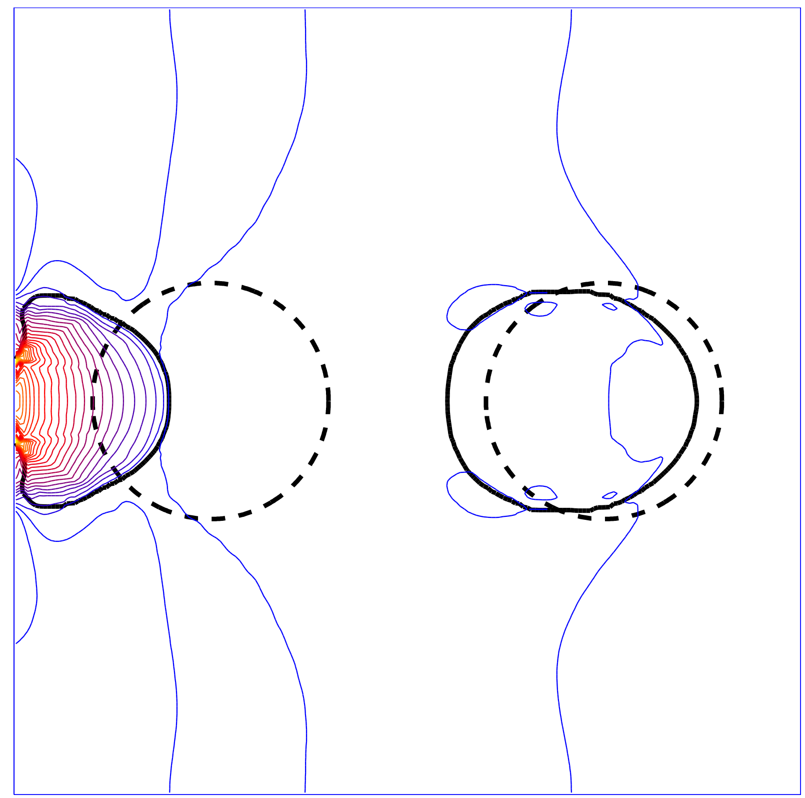
Vapor mass fraction

t=8ms



Mixture pressure

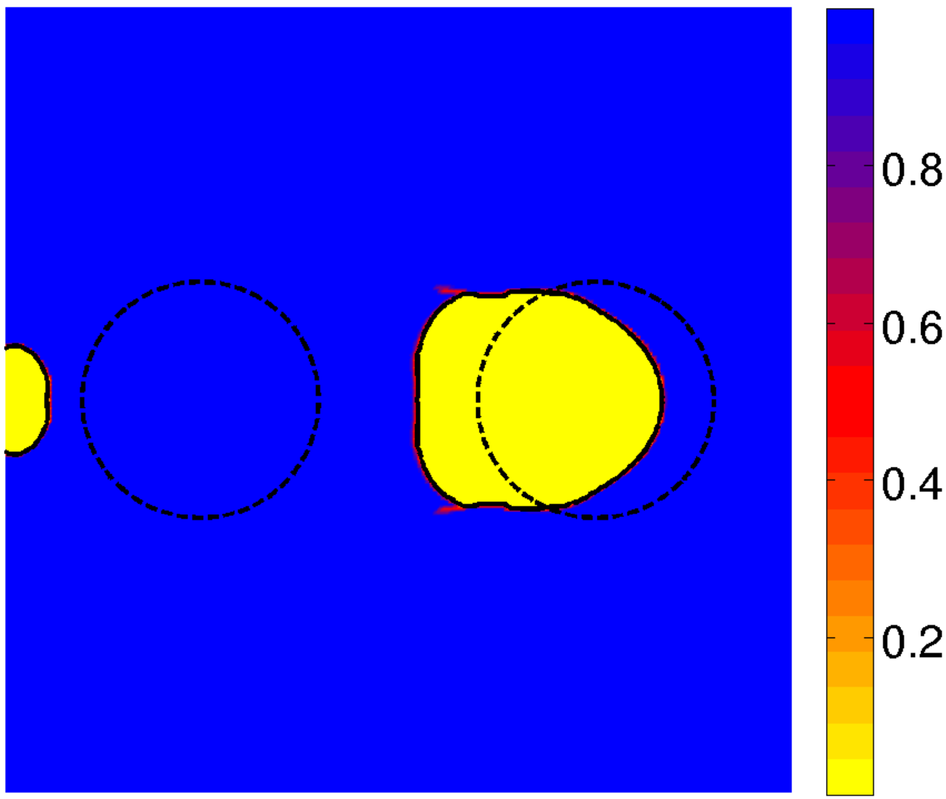
t=8ms



Liquid drops depression (cont.)

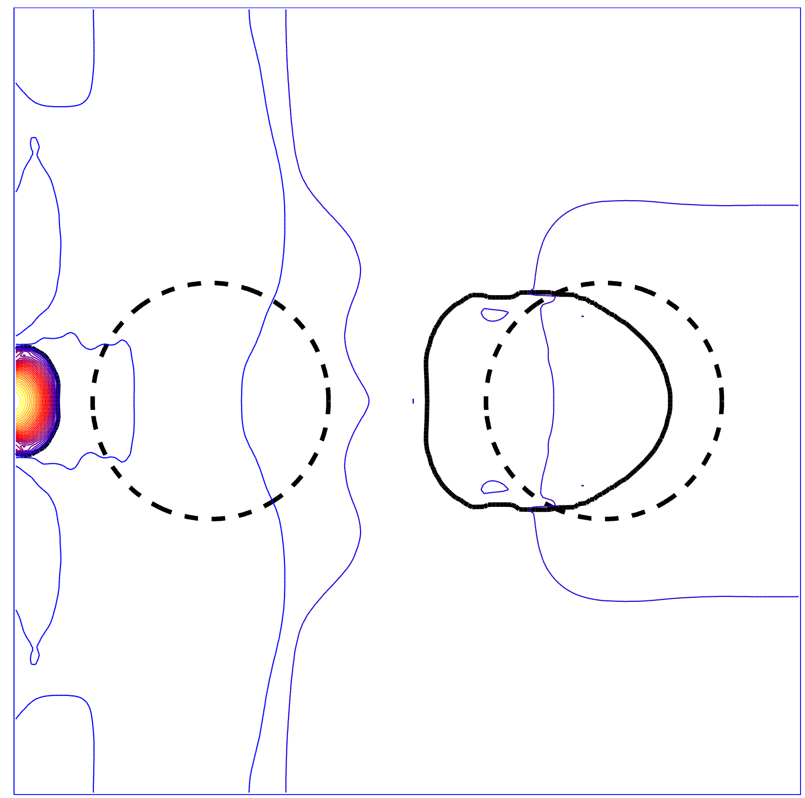
Vapor mass fraction

t=10ms



Mixture pressure

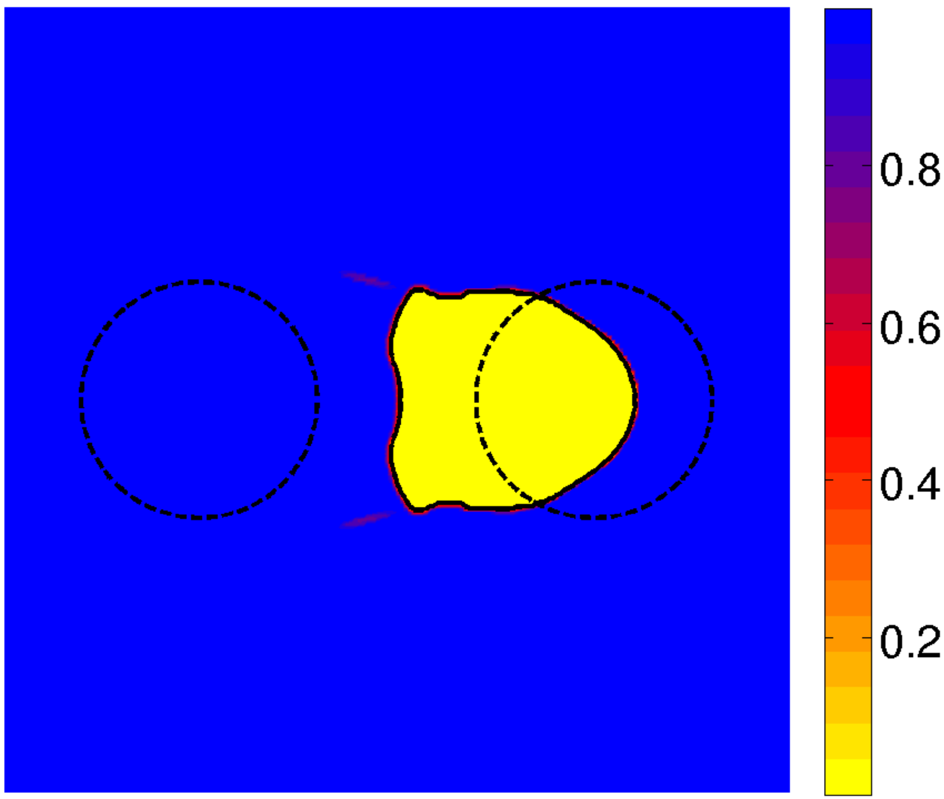
t=10ms



Liquid drops depression (cont.)

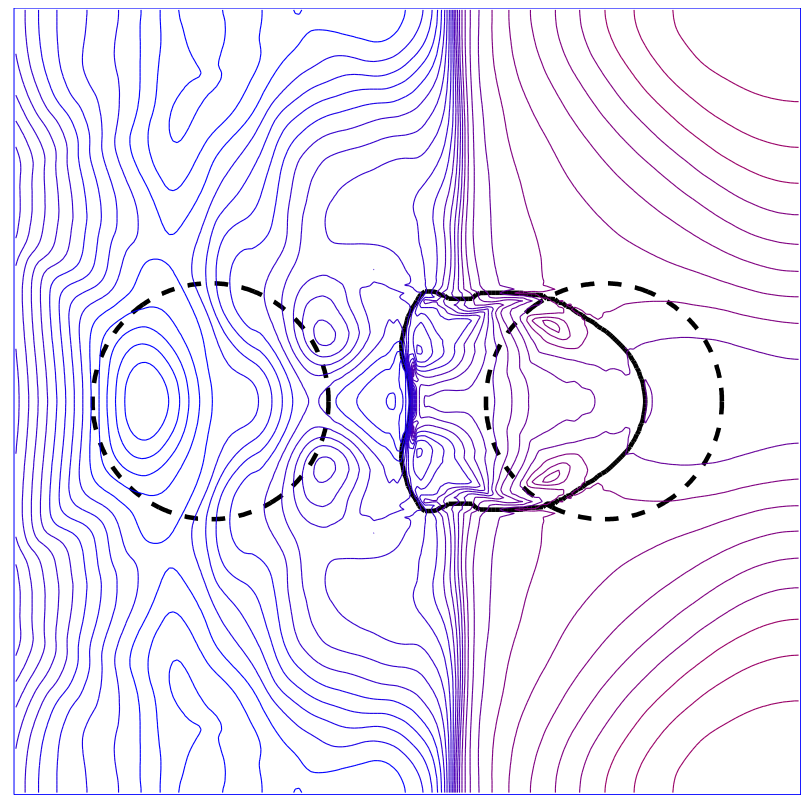
Vapor mass fraction

t=12ms



Mixture pressure

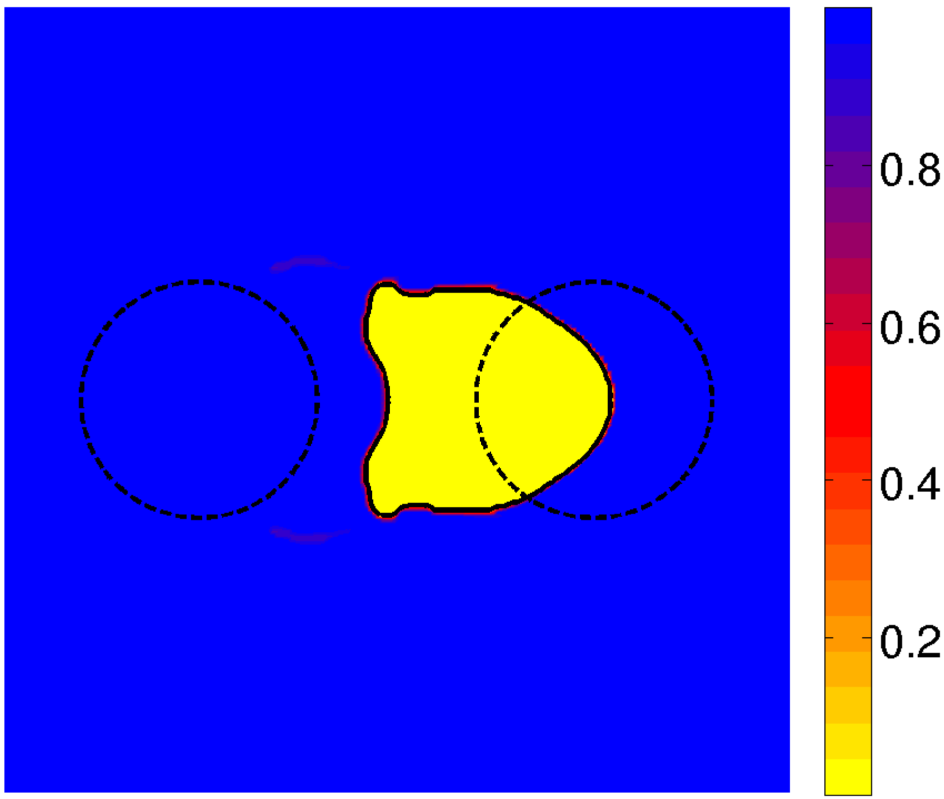
t=12ms



Liquid drops depression (cont.)

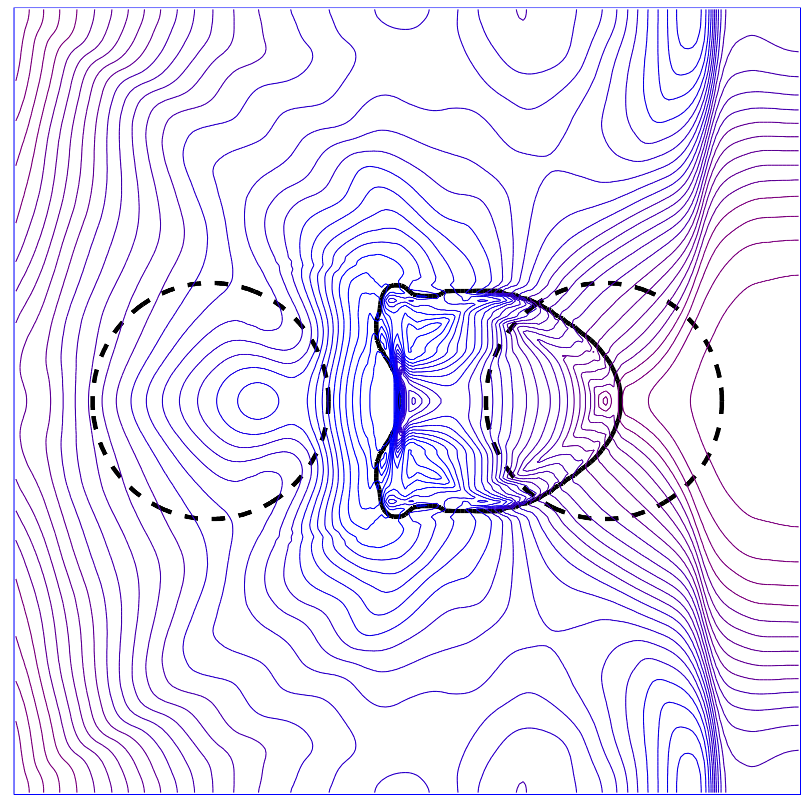
Vapor mass fraction

t=14ms



Mixture pressure

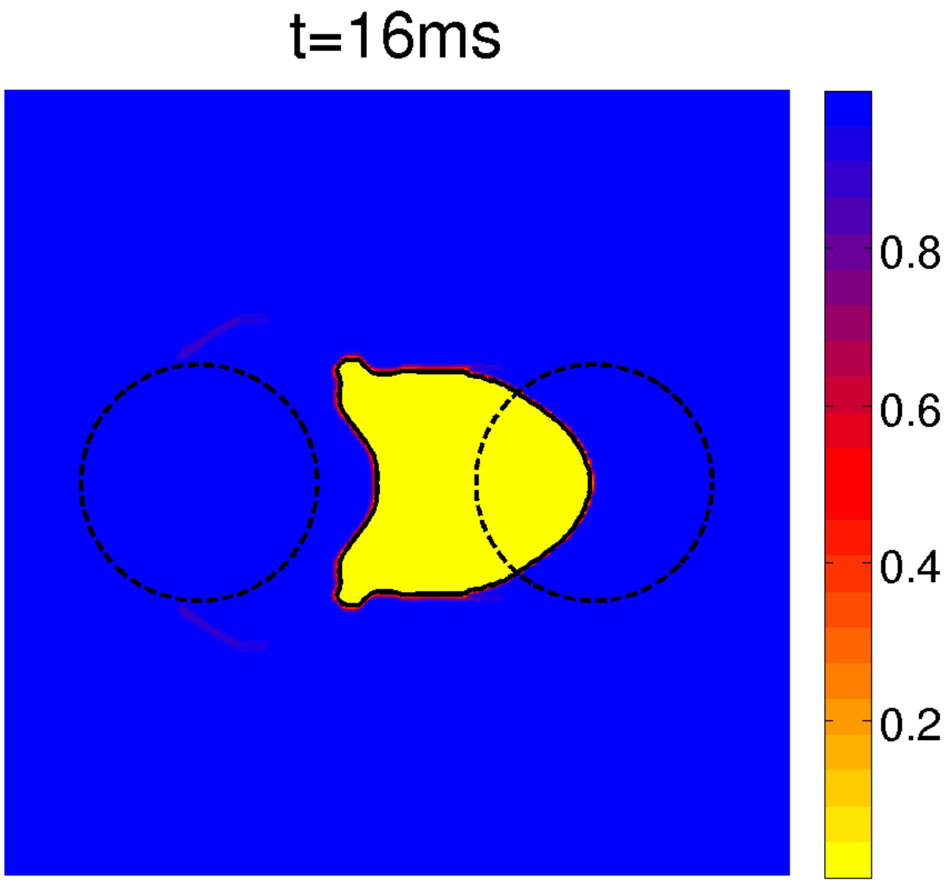
t=14ms

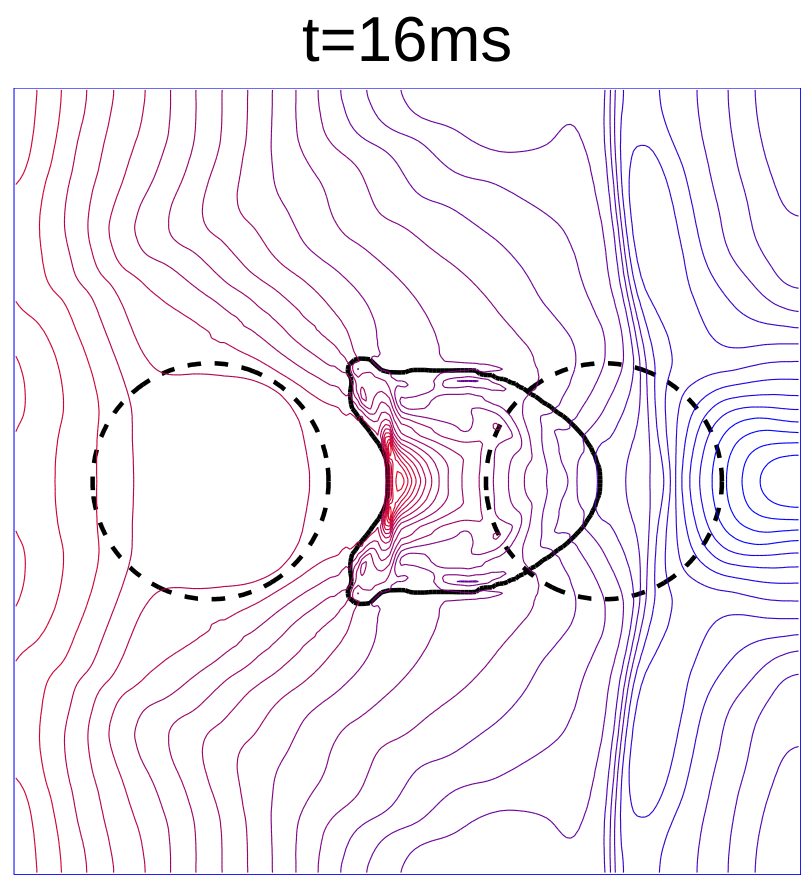


Liquid drops depression (cont.)

Vapor mass fraction

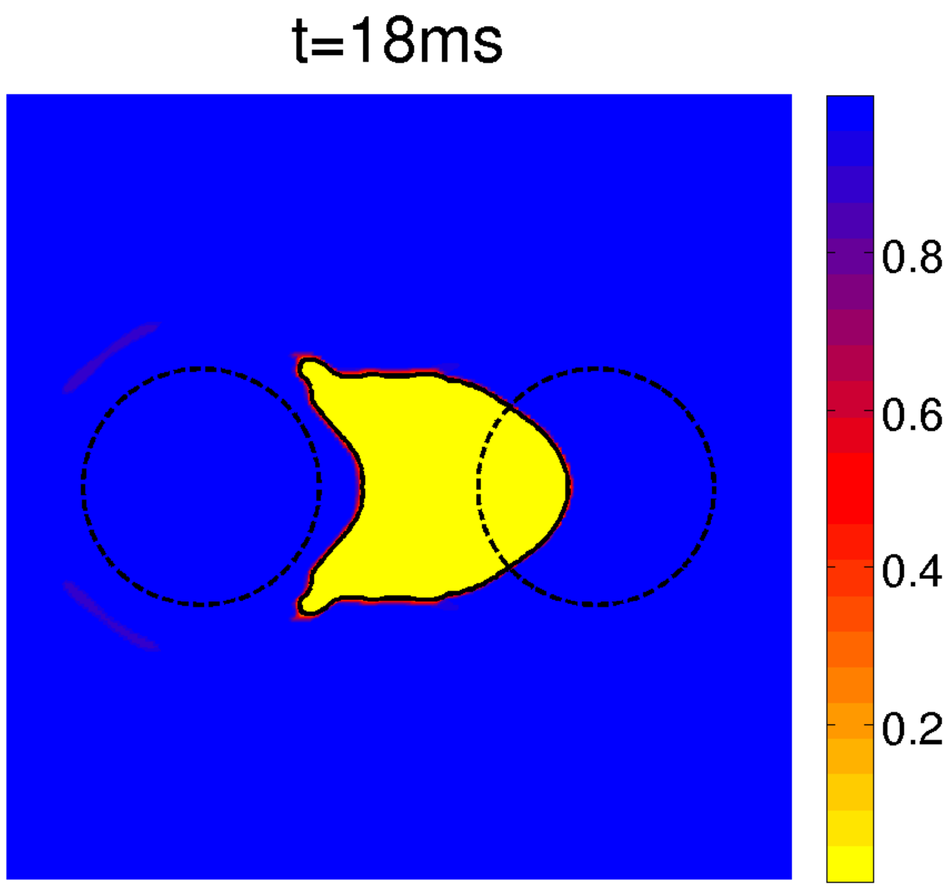
Mixture pressure



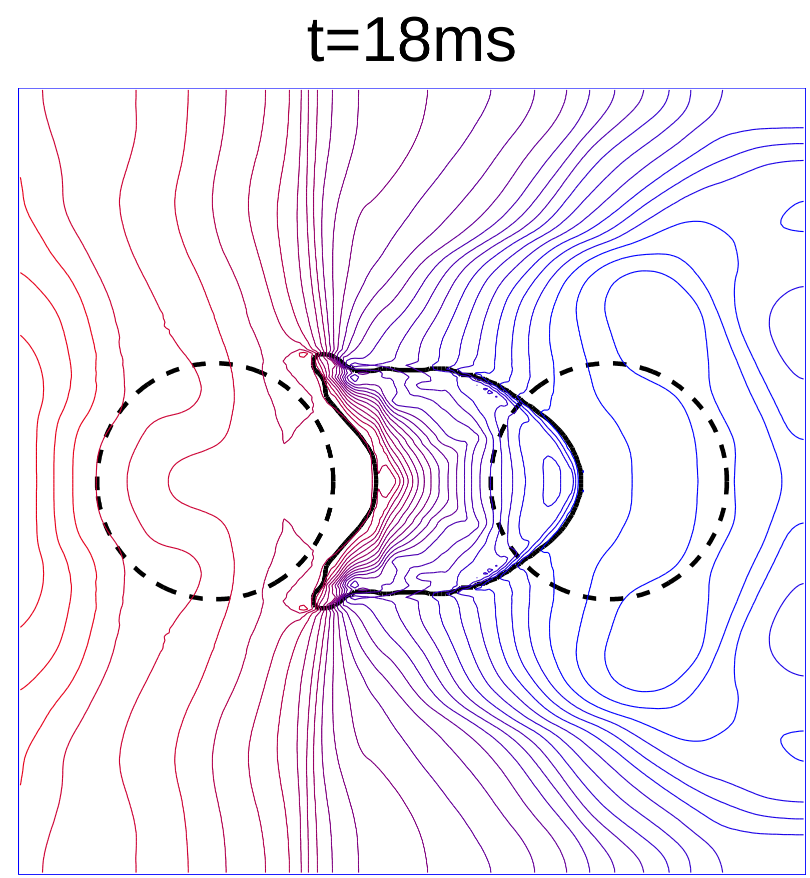


Liquid drops depression (cont.)

Vapor mass fraction

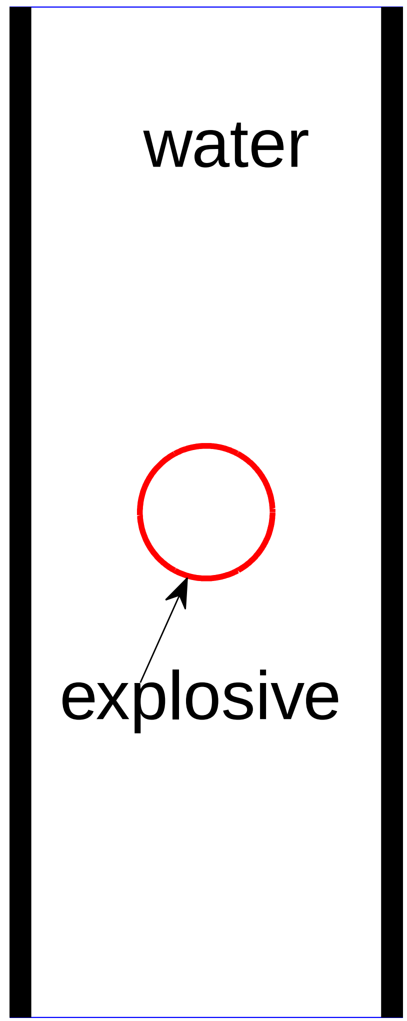


Mixture pressure

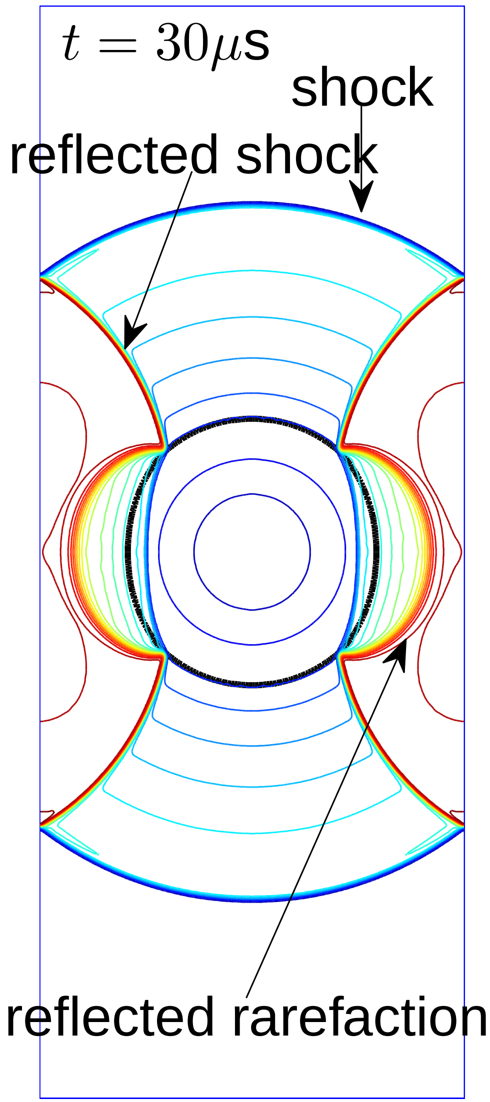


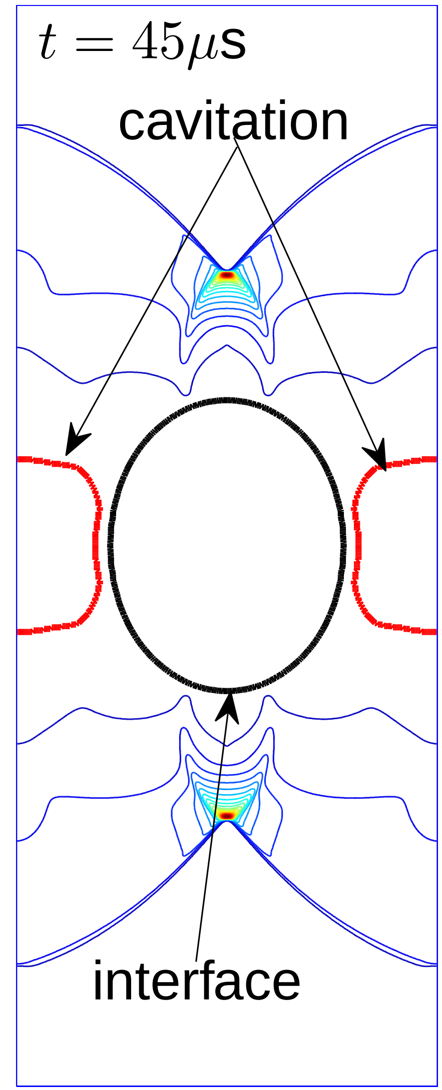
Underwater explosions: cylindrical wall

High pressure gaseous explosive in water (cylindrical case)

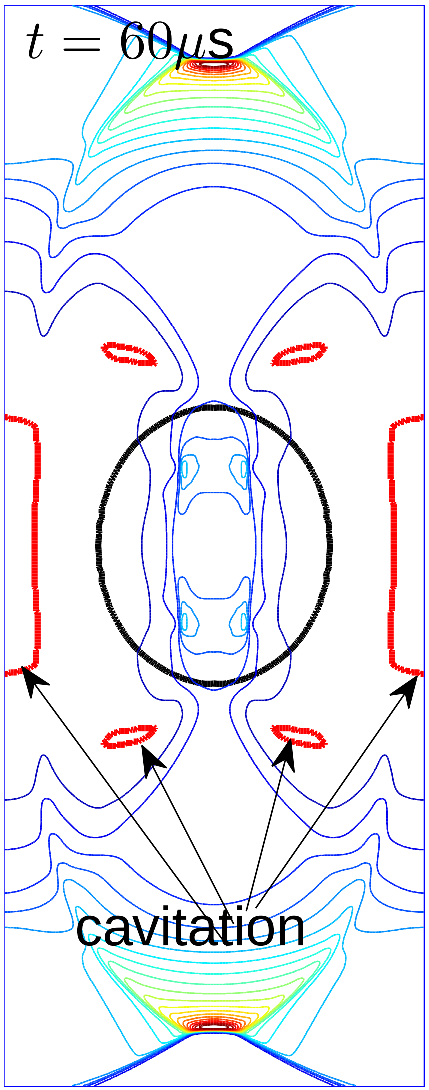


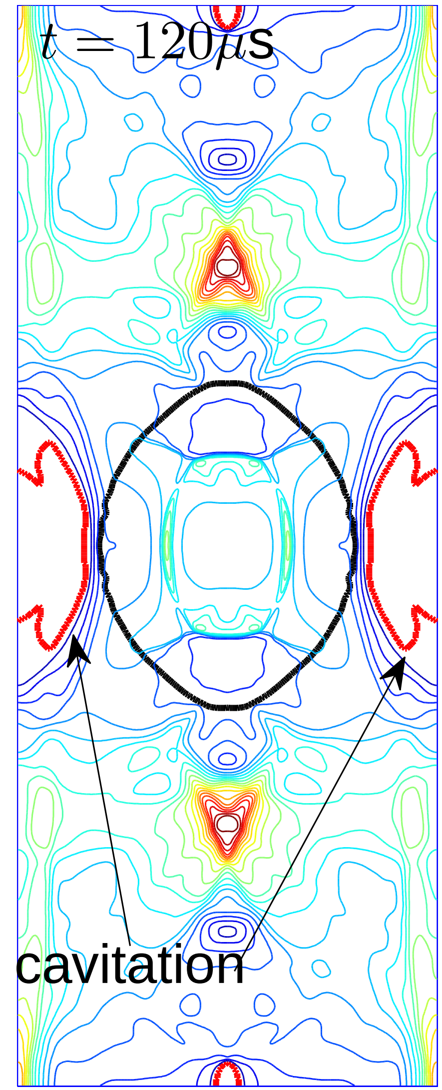
UNDEX: cylindrical wall (cont.)



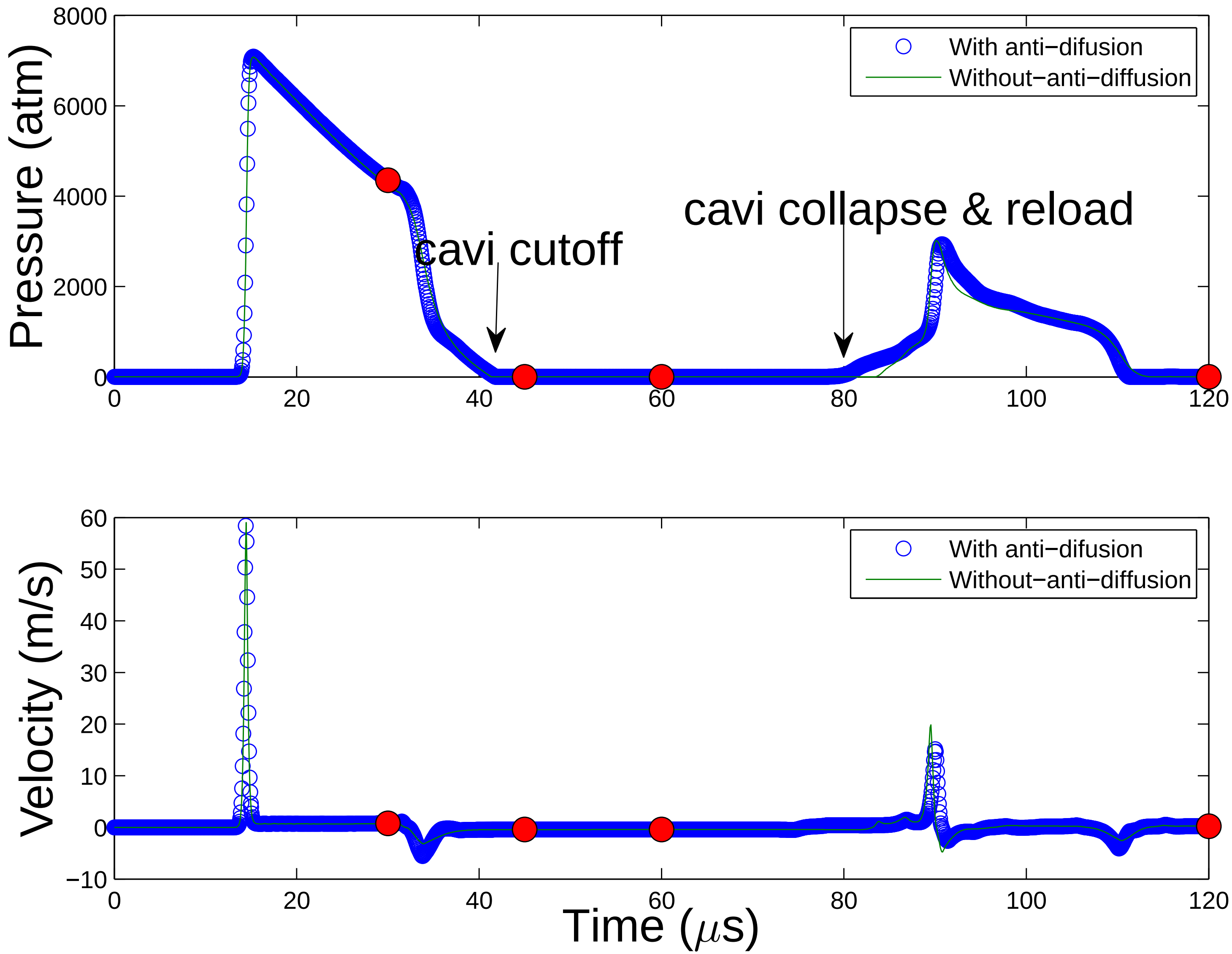


UNDEX: cylindrical wall (cont.)





UNDEX: cylindrical wall (cont.)



Algebraic-based interface sharpening

Incompressible flow: Algebraic-based interface sharpening

- CICSAM (Compressive Interface Capturing Scheme for Arbitrary Meshes): Ubbink & Issa JCP 1999
- THINC (Tangent of Hyperbola for INterface Capturing): Xiao, Honma & Kono Int. J. Numer. Meth. Fluids 2005
- Improved THINC

Compressible flow: THINC-type interface sharpening

1. Cell edge solution reconstruction: THINC
2. Solution update: MUSCL or semi-discretize scheme

THINC interface sharpening

THINC reconstruction assumes

$$\alpha_i(x) = \alpha_{\min} + \frac{\alpha_{\max} - \alpha_{\min}}{2}\left[1 + \gamma \tanh\left(\beta \frac{x - x_{i-1/2}}{\Delta x} - \bar{x}_i\right)\right]$$

$$\alpha_{\min} = \min\left(\alpha_{i-1}, \alpha_{i+1}\right),\ \alpha_{\max} = \max\left(\alpha_{i-1}, \alpha_{i+1}\right),\ \gamma = \text{sign}\left(\alpha_{i+1} - \alpha_{i-1}\right)$$

β measures sharpeness (given constant) & $\bar{x}_i$ chosen by

$$\alpha_i = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \alpha_i(x)\, dx$$

Cell edges determined by

$$\alpha_{i+1/2,L} = \alpha_{\min} + \frac{\alpha_{\max} - \alpha_{\min}}{2}\left(1 + \gamma \frac{\tanh\beta + C}{1 + C\tanh\beta}\right)$$

$$\alpha_{i-1/2,R} = \alpha_{\min} + \frac{\alpha_{\max} - \alpha_{\min}}{2}\left(1 + \gamma C\right)$$ (C not shown)

Homogeneous-equilibrium THINC

With $\alpha_{i+1/2,L}$ & $\alpha_{i-1/2,R}$ obtained by THINC reconstruction

Homogeneous-equilibrium reconstruction (analgously to α-based 5-equation anti-diffusion model)

$$
\begin{aligned}
(\alpha_1\rho_1)_{i+1/2,L} &= \rho_{1,i}\left(\alpha_{1,i+1/2,L} - \alpha_{1,i}\right) \\
(\alpha_2\rho_2)_{i+1/2,L} &= \rho_{2,i}\left(\alpha_{2,i+1/2,L} - \alpha_{2,i}\right) \\
\rho_{i+1/2,L} &= (\alpha_1\rho_1)_{i+1/2,L} + (\alpha_2\rho_2)_{i+1/2,L} \\
(\rho u)_{i+1/2,L} &= u_i\rho_{i+1/2,L} \\
E_{i+1/2,L} &= K_i\rho_{i+1/2,L} + \sum_{k=1}^{2}(\rho_k e_k)_i\left(\alpha_{k,i+1/2,L} - \alpha_{k,i}\right)
\end{aligned}
$$

$z_{i-1/2,R}$ can be made in a similar manner

6-equation anti-diffusion model

Anti-diffusion to 6-equation model (Pelanti & Shyue 2012)

$$\partial_t \left(\alpha_1 \rho_1\right) + \nabla \cdot \left(\alpha_1 \rho_1 \vec{u}\right) = \frac{1}{\mu} \mathcal{D}_{\alpha_1 \rho_1}$$

$$\partial_t \left(\alpha_2 \rho_2\right) + \nabla \cdot \left(\alpha_2 \rho_2 \vec{u}\right) = \frac{1}{\mu} \mathcal{D}_{\alpha_2 \rho_2}$$

$$\partial_t (\rho \vec{u}) + \nabla \cdot \left(\rho \vec{u} \otimes \vec{u}\right) + \nabla \left(\alpha_1 p_1 + \alpha_2 p_2\right) = \frac{1}{\mu} \mathcal{D}_{\rho u}$$

$$\partial_t \left(\alpha_1 E_1\right) + \nabla \cdot \left(\alpha_1 E_1 \vec{u} + \alpha_1 p_1 \vec{u}\right) + w\left(q, \nabla q\right) = -\nu p_I \left(p_1 - p_2\right) + \frac{1}{\mu} \mathcal{D}_{\alpha_1 E_1}$$

$$\partial_t \left(\alpha_2 E_2\right) + \nabla \cdot \left(\alpha_2 E_2 \vec{u} + \alpha_2 p_2 \vec{u}\right) - w\left(q, \nabla q\right) = \nu p_I \left(p_1 - p_2\right) + \frac{1}{\mu} \mathcal{D}_{\alpha_2 E_2}$$

$$\partial_t \alpha_1 + \vec{u} \cdot \nabla \alpha_1 = \nu \left(p_1 - p_2\right) + \frac{1}{\mu} \mathcal{D}_{\alpha_1}$$

$$w = -\vec{u}\left(\left(Y_2 p_1 + Y_1 p_2\right) \nabla \alpha_1 + \alpha_1 Y_2 \nabla p_1 - \alpha_2 Y_1 \nabla p_2\right)$$

Future perspective

- Shape-preserving interface shapening on mapped grid
- Interface shapening to flow with physical sources (usefulness ?)

$$\partial_t q + \nabla \cdot f(q) + B\nabla q = \psi(q) + \frac{1}{\mu}\mathcal{D}_q$$

- Extension to low Mach weakly compressible flow
- Hybrid interface-sharpening & WENO towards high order for compressible turbulent flow

Thank you