Adaptive moving-mesh relaxation scheme for compressible two-phase barotropic flow with cavitation

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Outline

- Introduce simple relaxation model for compressible 2-phase barotropic flow with & without cavitation
- Describe finite-volume method for solving proposed model numerically on fixed mapped (body-fitted) grids
- Discuss adaptive moving mesh approach for efficient improvement of numerical resolution
- Show sample results & discuss future extensions

Conventional barotropic 2-phase model

Assume homogeneous equilibrium barotropic flow with single pressure p & velocity $\vec{u}$ in 2-phase mixture region

- Equations of motion

$$\partial_t(\alpha_1\rho_1) + \nabla\cdot(\alpha_1\rho_1\vec{u}) = 0 \qquad (\text{Continuity } \alpha_1\rho_1)$$

$$\partial_t(\alpha_2\rho_2) + \nabla\cdot(\alpha_2\rho_2\vec{u}) = 0 \qquad (\text{Continuity } \alpha_2\rho_2)$$

$$\partial_t(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \nabla p = 0 \qquad (\text{Momentum } \rho\vec{u})$$

- Saturation condition for volume fraction α_k, $k = 1, 2$

$$\alpha_1 + \alpha_2 = 1 \quad \Longrightarrow \quad \frac{\alpha_1\rho_1}{\rho_1(p)} + \frac{\alpha_2\rho_2}{\rho_2(p)} = 1$$

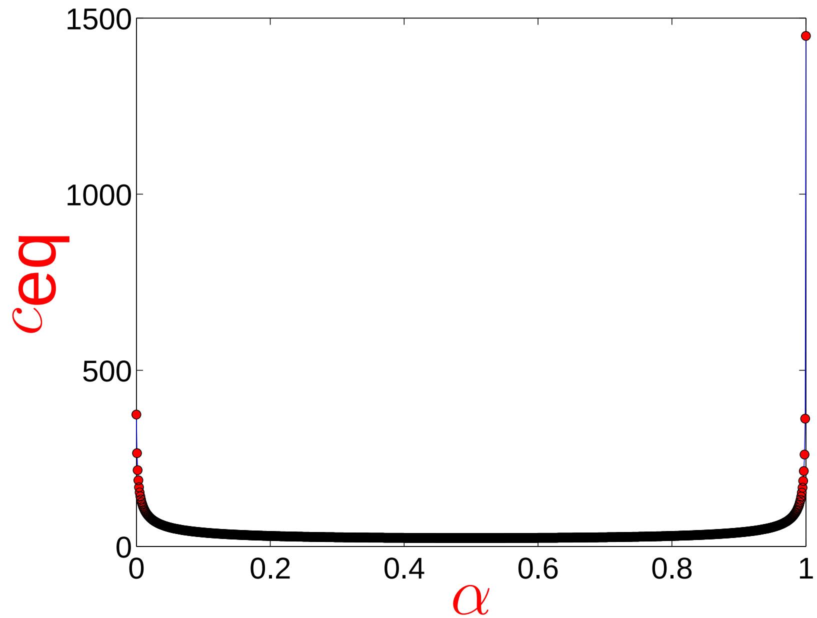
yielding scalar nonlinear equation for equilibrium p

Equilibrium speed of sound

Non-monotonic sound speed c_{eq} defined as

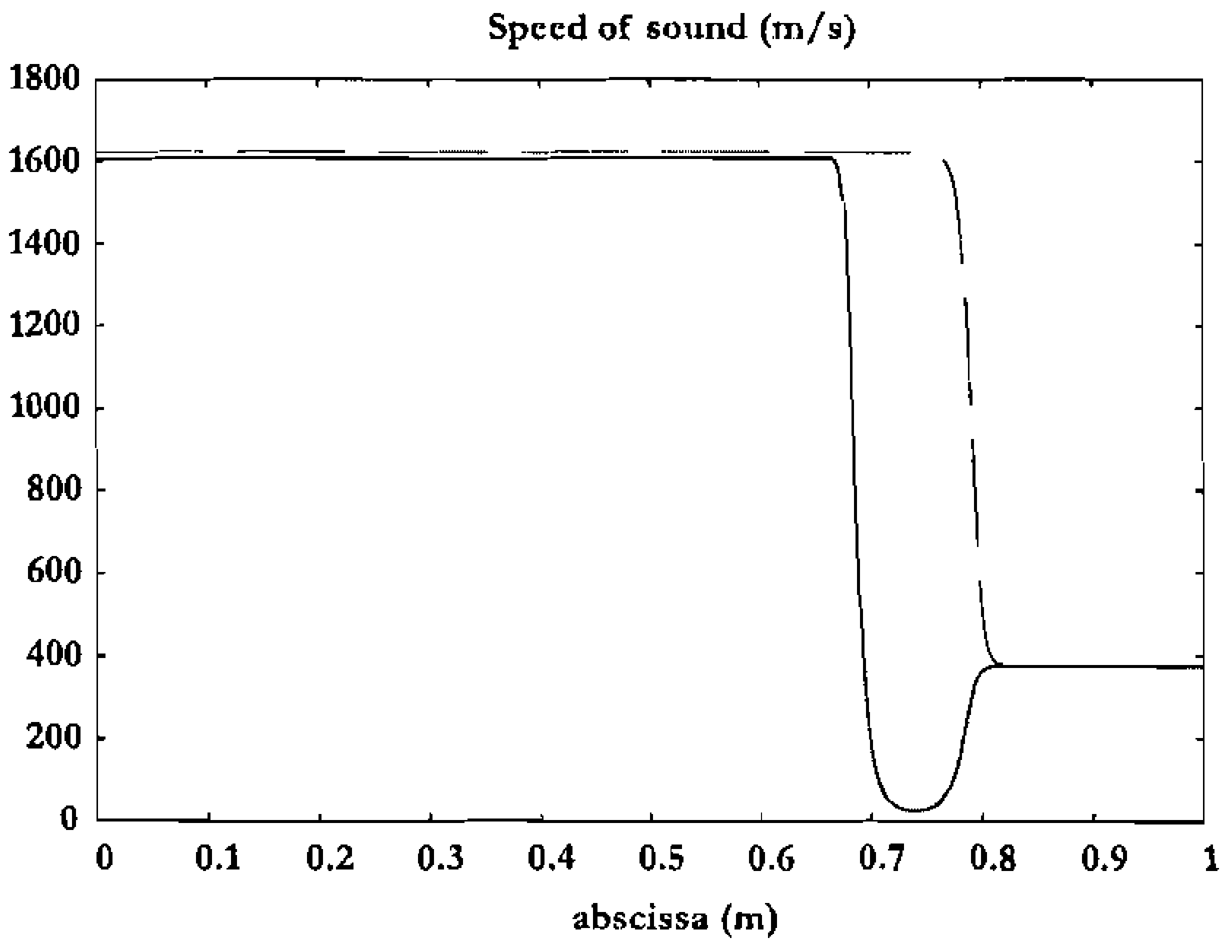
$$\frac{1}{\rho c_{\text{eq}}^2} = \frac{\alpha_1}{\rho_1 c_1^2} + \frac{\alpha_2}{\rho_2 c_2^2} \qquad \text{(Wood's formula)}$$

exists in 2-phase region, air-water case shown below; yielding numerical difficulty such as inaccurate wave transmission across diffused interface



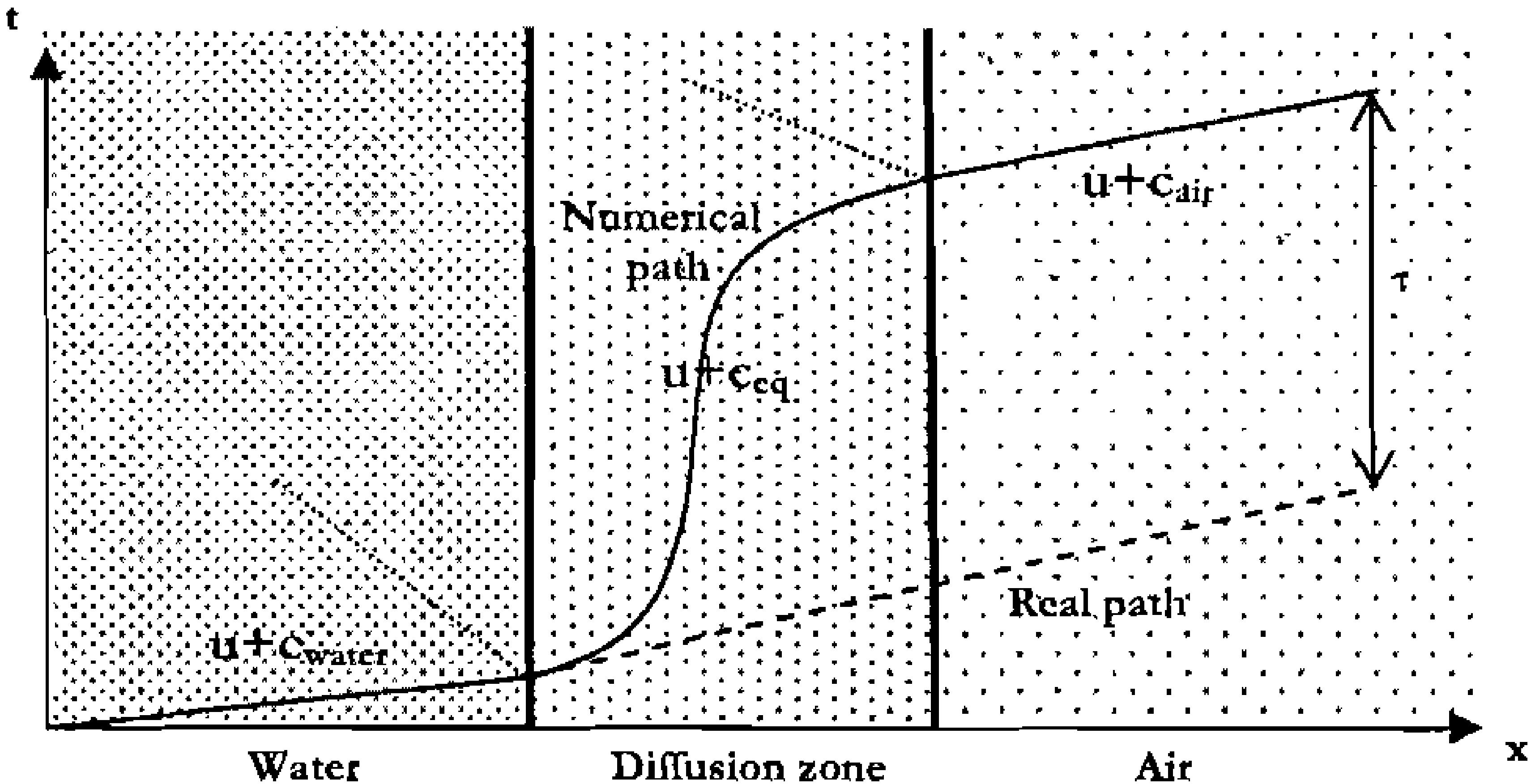
Equilibrium vs. frozen sound speed

Equilibrium (solid) & frozen (dashed) sound speeds, $c_{\mathsf{f}}^2 = \sum Y_k c_k^2$, in case of passive advection of air-water interface



Equilibrium vs. frozen sound speed

When acoustic wave interacts with numerical diffusion zone, sound speeds difference leads to time delay τ of transmitted waves through interface



Relaxation barotropic 2-phase model

To overcome these difficulties, we consider a relaxation model proposed by Caro, Coquel, Jamet, Kokh, Saurel *et al.*

$$\partial_t (\alpha_1 \rho_1) + \nabla \cdot (\alpha_1 \rho_1 \vec{u}) = 0$$

$$\partial_t (\alpha_2 \rho_2) + \nabla \cdot (\alpha_2 \rho_2 \vec{u}) = 0$$

$$\partial_t (\rho \vec{u}) + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) + \nabla (\alpha_1 p_1 + \alpha_2 p_2) = 0$$

$$\partial_t \alpha_1 + \vec{u} \cdot \nabla \alpha_1 = \mu \left(p_1(\rho_1) - p_2(\rho_2)\right) \quad (\text{Transport } \alpha_1)$$

When taking infinite pressure relaxation $\mu \to \infty$, we have

$$p_1(\rho_1) = p_2(\rho_2) \quad \Longrightarrow \quad p_1 \left(\frac{\alpha_1 \rho_1}{\alpha_1}\right) - p_2 \left(\frac{\alpha_2 \rho_2}{1 - \alpha_1}\right) = 0$$

yielding scalar nonlinear equation for volume fraction α_1

Relaxation barotropic model: Remarks

- This model can be viewed as isentropic version of relaxation model proposed by Saurel, Petitpas, Berry (JCP 2009, see below) for 2-phase flow
- This model is hyperbolic & has monotonic sound speed $c_f^2 = \sum Y_k c_k^2$
- Cavitation is modeled as a simplified mechanical relaxation process, occurring at infinite rate & not as a mass transfer process

  i.e., cavitation pockets appear as volume fraction increases for a small amount of gas present initially

Relaxation 2-phase model

Saurel, Petitpas, Berry (JCP 2009)

$$\partial_t(\alpha_1\rho_1) + \nabla\cdot(\alpha_1\rho_1\vec{u}) = 0$$

$$\partial_t(\alpha_2\rho_2) + \nabla\cdot(\alpha_2\rho_2\vec{u}) = 0$$

$$\partial_t(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \nabla(\alpha_1 p_1 + \alpha_2 p_2) = 0$$

$$\partial_t(\alpha_1\rho_1 e_1) + \nabla\cdot(\alpha_1\rho_1 e_1\vec{u}) + \alpha_1 p_1\nabla\cdot\vec{u} = -\bar{p}\mu(p_1 - p_2)$$

$$\partial_t(\alpha_2\rho_2 e_2) + \nabla\cdot(\alpha_2\rho_2 e_2\vec{u}) + \alpha_2 p_2\nabla\cdot\vec{u} = \bar{p}\mu(p_1 - p_2)$$

$$\partial_t\alpha_1 + \vec{u}\cdot\nabla\alpha_1 = \mu(p_1 - p_2)$$

$$\bar{p} = (p_1 Z_2 + p_2 Z_1)/(Z_1 + Z_2); \qquad Z_k = \rho_k c_k$$

Model agrees with reduced 2-phase model of Kapila, Menikoff, Bdzil, Son, Stewart (Phys. Fluid 2001) formally

1-phase barotropic cavitation models

- Cutoff model

$$p = \begin{cases} p(\rho) & \text{if} \quad \rho \geq \rho_{sat} \\ p_{sat} & \text{if} \quad \rho < \rho_{sat} \end{cases}$$

- Schmidt model

$$p = \begin{cases} p(\rho) & \text{if} \quad \rho \geq \rho_{sat} \\ p_{sat} + p_{gl} \ln \left[\frac{\rho_g c_g^2 \rho_l c_l^2 (\rho_l + \alpha(\rho_g - \rho_l))}{\rho_l \left(\rho_g c_g^2 - \alpha \left(\rho_g c_g^2 - \rho_l c_l^2 \right) \right)} \right] & \text{if} \quad \rho < \rho_{sat} \end{cases}$$

with $$p_{gl} = \frac{\rho_g c_g^2 \rho_l c_l^2 \left(\rho_g - \rho_l \right)}{\rho_g^2 c_g^2 - \rho_l^2 c_l^2} \quad \& \quad \rho = \alpha \rho_g + (1 - \alpha) \rho_l$$

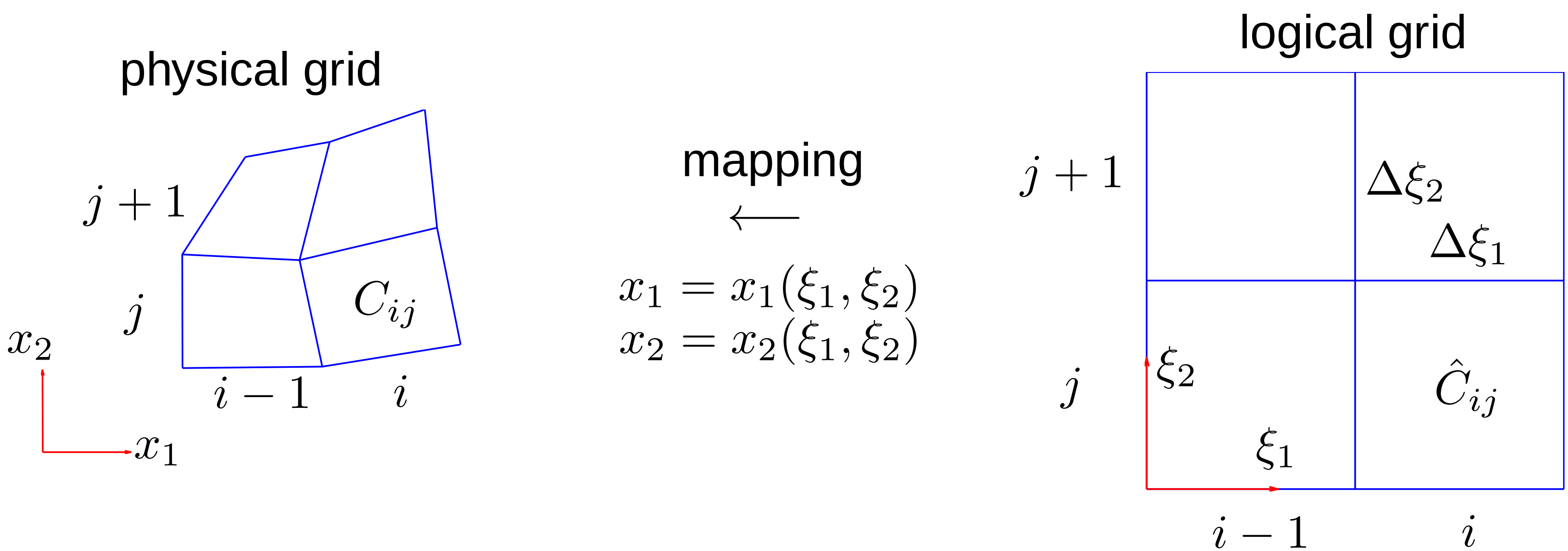
- Modified Schmidt model & its variant

Mapped grid method

We want to use finite-volume mapped grid approach to solve proposed relaxation model in complex geometry

Assume mapped grids are logically rectangular & will review method for hyperbolic system of conservation laws

$$\partial_t \mathbf{q} + \nabla \cdot \mathbf{f}(\mathbf{q}) = 0$$



Mapped grid methods

On a curvilinear grid, a finite volume method takes

$$Q_{ij}^{n+1} = Q_{ij}^{n} - \frac{\Delta t}{\kappa_{ij}\Delta\xi_1}\left(F^1_{i+\frac{1}{2},j} - F^1_{i-\frac{1}{2},j}\right) - \frac{\Delta t}{\kappa_{ij}\Delta\xi_2}\left(F^2_{i,j+\frac{1}{2}} - F^2_{i,j-\frac{1}{2}}\right)$$

$\Delta\xi_1$, $\Delta\xi_2$ denote mesh sizes in ξ_1- & ξ_2-directions

$\kappa_{ij} = \mathcal{M}(C_{ij})/\Delta\xi_1\Delta\xi_2$ is area ratio between areas of grid cell in physical & logical spaces

$F^1_{i-\frac{1}{2},j} = \gamma_{i-\frac{1}{2},j}\breve{F}_{i-\frac{1}{2},j}$, $F^2_{i,j-\frac{1}{2}} = \gamma_{i,j-\frac{1}{2}}\breve{F}_{i,j-\frac{1}{2}}$ are normal fluxes per unit length in logical space with $\gamma_{i-\frac{1}{2},j} = h_{i-\frac{1}{2},j}/\Delta\xi_1$ & $\gamma_{i,j-\frac{1}{2}} = h_{i,j-\frac{1}{2}}/\Delta\xi_2$ representing length ratios

Wave propagation method

First order wave propagation method devised by LeVeque on mapped grid is a Godunov-type finite volume method

$$Q_{ij}^{n+1} = Q_{ij}^{n} - \frac{\Delta t}{\kappa_{ij}\Delta\xi_1}\left(\mathcal{A}_1^{+}\Delta Q_{i-\frac{1}{2},j} + \mathcal{A}_1^{-}\Delta Q_{i+\frac{1}{2},j}\right) - \frac{\Delta t}{\kappa_{ij}\Delta\xi_2}\left(\mathcal{A}_2^{+}\Delta Q_{i,j-\frac{1}{2}} + \mathcal{A}_2^{-}\Delta Q_{i,j+\frac{1}{2}}\right)$$

with right-, left-, up-, & down-moving fluctuations $\mathcal{A}_1^{+}\Delta Q_{i-\frac{1}{2},j}$, $\mathcal{A}_1^{-}\Delta Q_{i+\frac{1}{2},j}$, $\mathcal{A}_2^{+}\Delta Q_{i,j-\frac{1}{2}}$, & $\mathcal{A}_2^{-}\Delta Q_{i,j+\frac{1}{2}}$ that are entering into grid cell

To determine these fluctuations, one-dimensional Riemann problems in direction normal to cell edges are solved

Wave propagation method

Speeds & limited versions of waves are used to calculate second order correction terms as

$$Q_{ij}^{n+1} := Q_{ij}^{n+1} - \frac{1}{\kappa_{ij}} \frac{\Delta t}{\Delta \xi_1} \left(\widetilde{\mathcal{F}}^1_{i+\frac{1}{2},j} - \widetilde{\mathcal{F}}^1_{i-\frac{1}{2},j} \right) - \frac{1}{\kappa_{ij}} \frac{\Delta t}{\Delta \xi_2} \left(\widetilde{\mathcal{F}}^2_{i,j+\frac{1}{2}} - \widetilde{\mathcal{F}}^2_{i,j-\frac{1}{2}} \right)$$

For example, at cell edge $(i - \frac{1}{2}, j)$ correction flux takes

$$\widetilde{\mathcal{F}}^1_{i-\frac{1}{2},j} = \frac{1}{2} \sum_{k=1}^{N_w} \left| \lambda^{1,k}_{i-\frac{1}{2},j} \right| \left(1 - \frac{\Delta t}{\kappa_{i-\frac{1}{2},j} \Delta \xi_1} \left| \lambda^{1,k}_{i-\frac{1}{2},j} \right| \right) \widetilde{\mathcal{W}}^{1,k}_{i-\frac{1}{2},j}$$

$\kappa_{i-\frac{1}{2},j} = (\kappa_{i-1,j} + \kappa_{ij})/2$. To aviod oscillations near discontinuities, a wave limiter is applied leading to limited waves $\widetilde{\mathcal{W}}$

Wave propagation method

Transverse wave propagation is included to ensure second order accuracy & also improve stability that $\mathcal{A}_1^{\pm}\Delta Q_{i-\frac{1}{2},j}$ are each split into two transverse fluctuations: up- & down-going $\mathcal{A}_2^{\pm}\mathcal{A}_1^{+}\Delta Q_{i-\frac{1}{2},j}$ & $\mathcal{A}_2^{\pm}\mathcal{A}_1^{-}\Delta Q_{i-\frac{1}{2},j}$, at each cell edge

This method can be shown to be conservative & stable under a variant of CFL (Courant-Friedrichs-Lewy) condition of form

$$\nu = \Delta t \max_{i,j,k} \left(\frac{\left|\lambda^{1,k}_{i-\frac{1}{2},j}\right|}{\kappa_{i_p,j}\Delta\xi_1}, \frac{\left|\lambda^{2,k}_{i,j-\frac{1}{2}}\right|}{\kappa_{i,j_p}\Delta\xi_2} \right) \le 1,$$

$$i_p = i \quad \text{if } \lambda^{1,k}_{i-\frac{1}{2},j} > 0 \quad \& \quad i-1 \quad \text{if } \lambda^{1,k}_{i-\frac{1}{2},j} < 0$$

Extension to moving mesh

To extend mapped grid method to solution adaptive moving grid method one simple way is to take approach proposed by

- H. Tang & T. Tang, Adaptive mesh methods for one- and two-dimensional hyperbolic conservation laws, SIAM J. Numer. Anal., 2003

In each time step, this moving mesh method consists of three basic steps:

(1) Mesh redistribution

(2) Conservative interpolation of solution states

(3) Solution update on a fixed mapped grid

Mesh redistribution scheme

- Winslow's approach (1981)

  Solve $\nabla \cdot (D\nabla\xi_j) = 0, \quad j = 1, \ldots, N_d$

  for $\xi(\mathbf{x})$. Coefficient D is a positive definite matrix which may depend on solution gradient

- Variational approach (Tang & many others)

  Solve $\nabla_\xi \cdot (D\nabla_\xi x_j) = 0, \quad j = 1, \ldots, N_d$

  for $\mathbf{x}(\xi)$ that minimizes "energy" functional

$$\mathcal{E}(\mathbf{x}(\xi)) = \frac{1}{2}\int_\Omega \sum_{j=1}^{N_d} \nabla_\xi^T D\nabla x_j d\xi$$

- Lagrangian (ALE)-type approach (*e.g.*, CAVEAT code)

Conservative interpolation

Numerical solutions need to be updated conservatively, *i.e.*

$$\sum \mathcal{M}\left(C^{k+1}\right) Q^{k+1} = \sum \mathcal{M}\left(C^{k}\right) Q^{k}$$

after each redistribution iterate k. This can be done by

- Finite-volume approach

$$\mathcal{M}\left(C^{k+1}\right) Q^{k+1} = \mathcal{M}\left(C^{k}\right) Q^{k} - \sum_{j=1}^{N_s} h_j \breve{G}_j, \quad \breve{G} = (\dot{\mathbf{x}} \cdot \mathbf{n}) Q$$

- Geometric approach

$$\left[\sum_S \mathcal{M}\left(C_p^{k+1} \cap S_p^k\right)\right] Q_C^{k+1} = \sum_S \mathcal{M}\left(C_p^{k+1} \cap S_p^k\right) Q_S^k$$

C_p, S_p are polygonal regions occupied by cells C & S

Interpolation-free moving mesh

If we want to derive an interpolation-free moving mesh method, one may first consider coordinate change of equations via $(\mathbf{x},\ t) \mapsto (\xi,\ t)$, yielding transformed conservation law as

$$\partial_t \tilde{\mathbf{q}} + \nabla_\xi \cdot \tilde{\mathbf{f}} = \mathcal{G}$$

$$\tilde{q} = Jq, \qquad \tilde{f}_j = J\left(q\,\partial_t \xi_j + \nabla \xi_j \cdot \mathbf{f}\right), \qquad J = \mathrm{det}\left(\partial \xi / \partial \mathbf{x}\right)^{-1}$$

$$\mathcal{G} = q\left[\partial_t J + \nabla_\xi \cdot (J \partial_t \xi_j)\right] + \sum_{j=1}^{N} f_j \nabla_\xi \cdot \left(J \partial_{x_j} \xi_k\right)$$

$$= 0 \quad (\text{if GCL \& SCL are satisfied})$$

Numerical method can be devised easily to solve these equations

Relaxation solver on moving meshes

In each time step, our numerical method for solving barotropic 2-phase flows on a moving mesh consists of following steps:

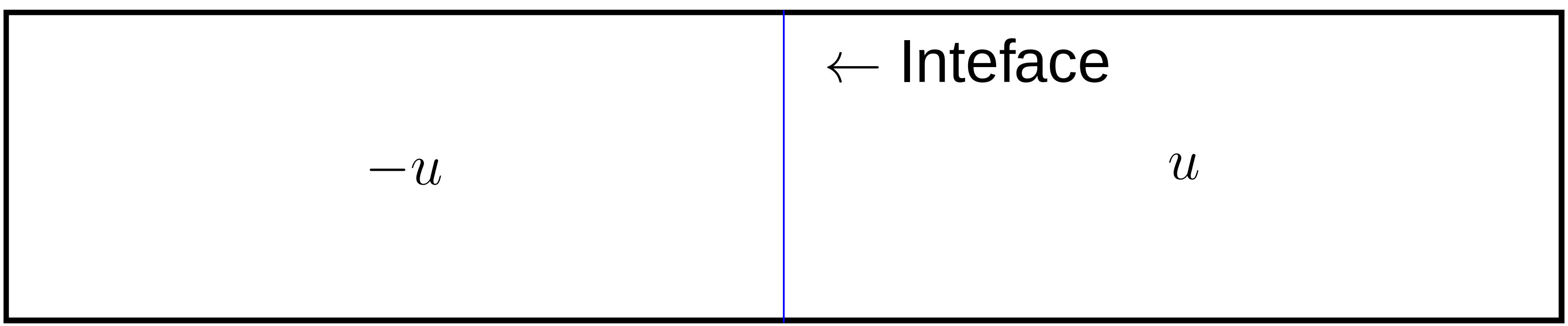
(1) Moving mesh step
Determine cell-interface velocity & cell-interface location in physical space over a time step

(2) Frozen step $\mu \to 0$
Solve homogeneous part of relaxation model on a moving mapped grid over same time step as in step 1

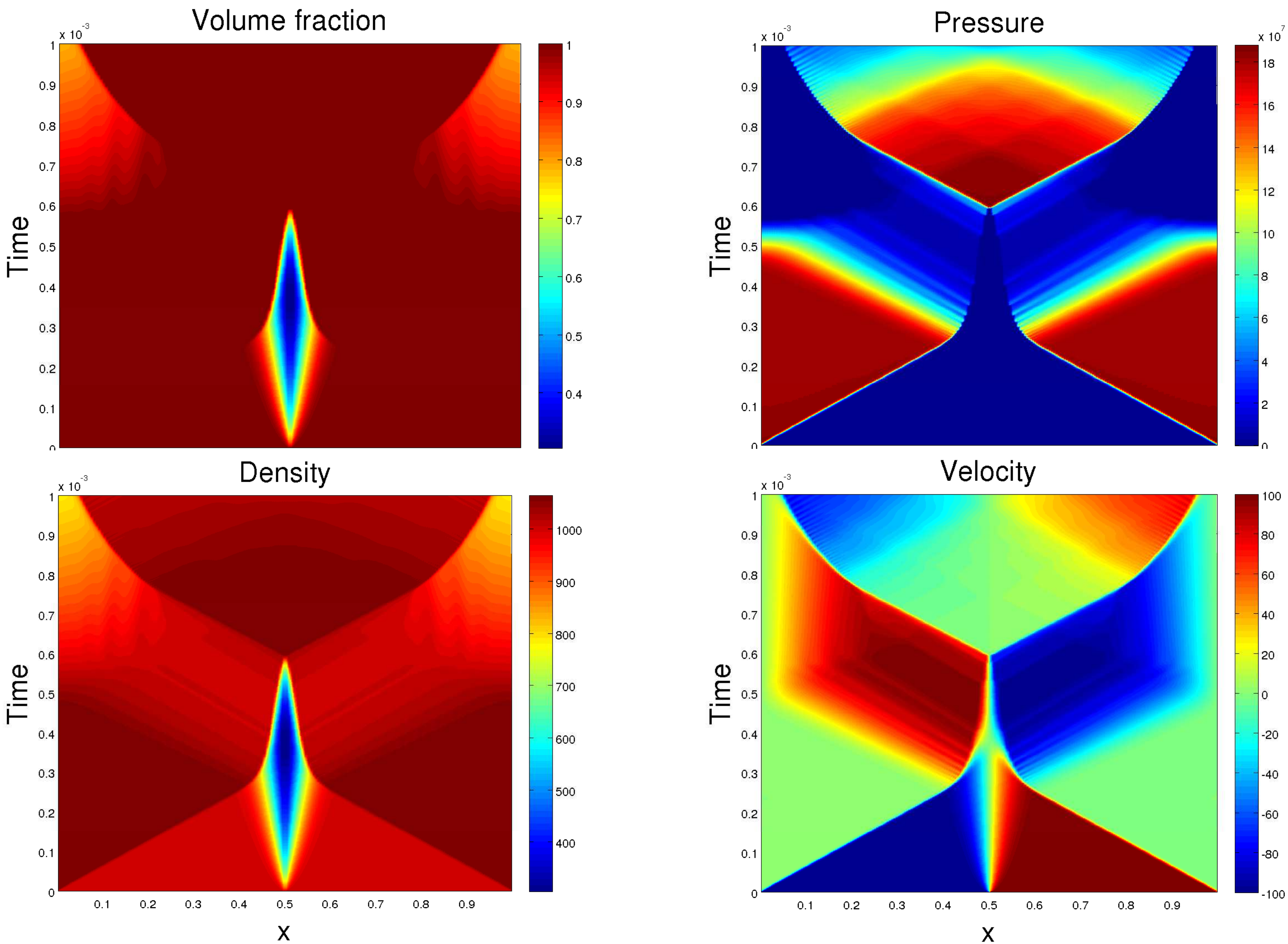
(3) Relaxation step $\mu \to \infty$
Solve model system with only source terms in infinite relaxation limit

Water-vapor cavitation

- Initially, in closed shock tube, flow is homogeneous (contains $\alpha = 10^{-6}$ gas in bulk liquid) at standard atmospheric condition & exists interface separating flow with opposite motion ($u = 100$ m/s)
- Result in pressure drop & formation of cavitation zone in middle; shocks form also from both ends
- Eventually, shock-cavitation collision occurs

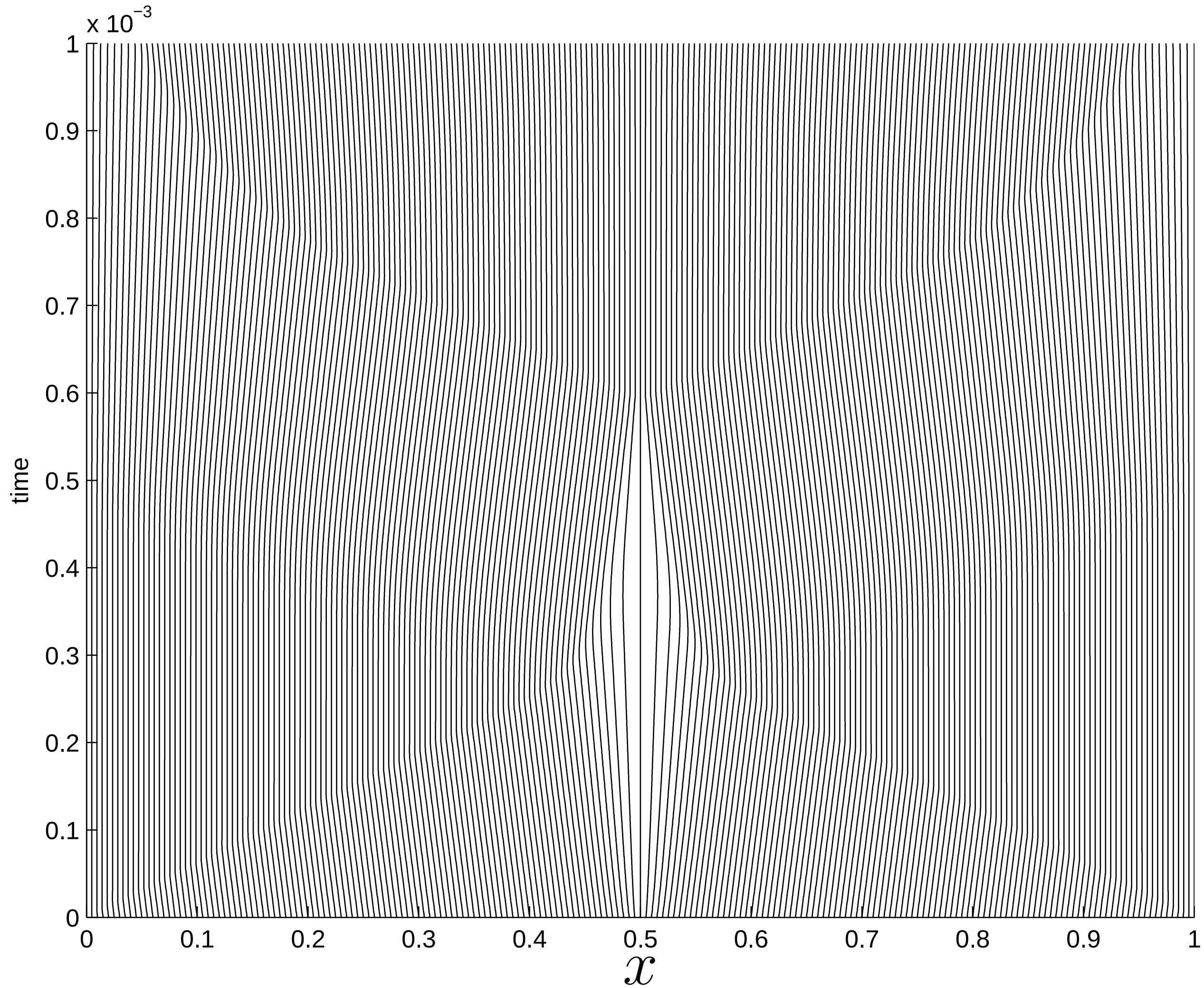


Water-vapor cavitation



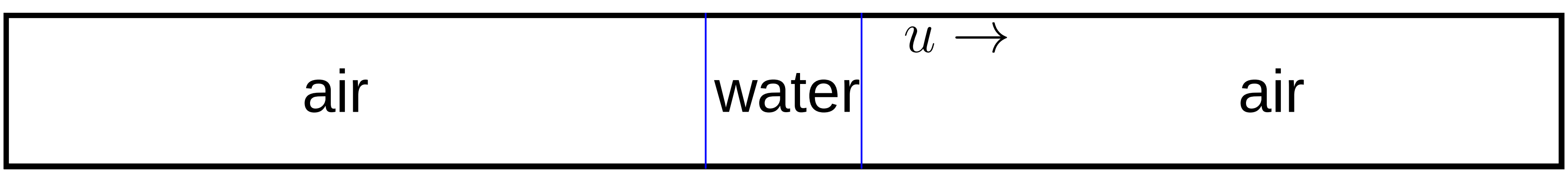
Water-vapor cavitation

Physical grid in x-t plane



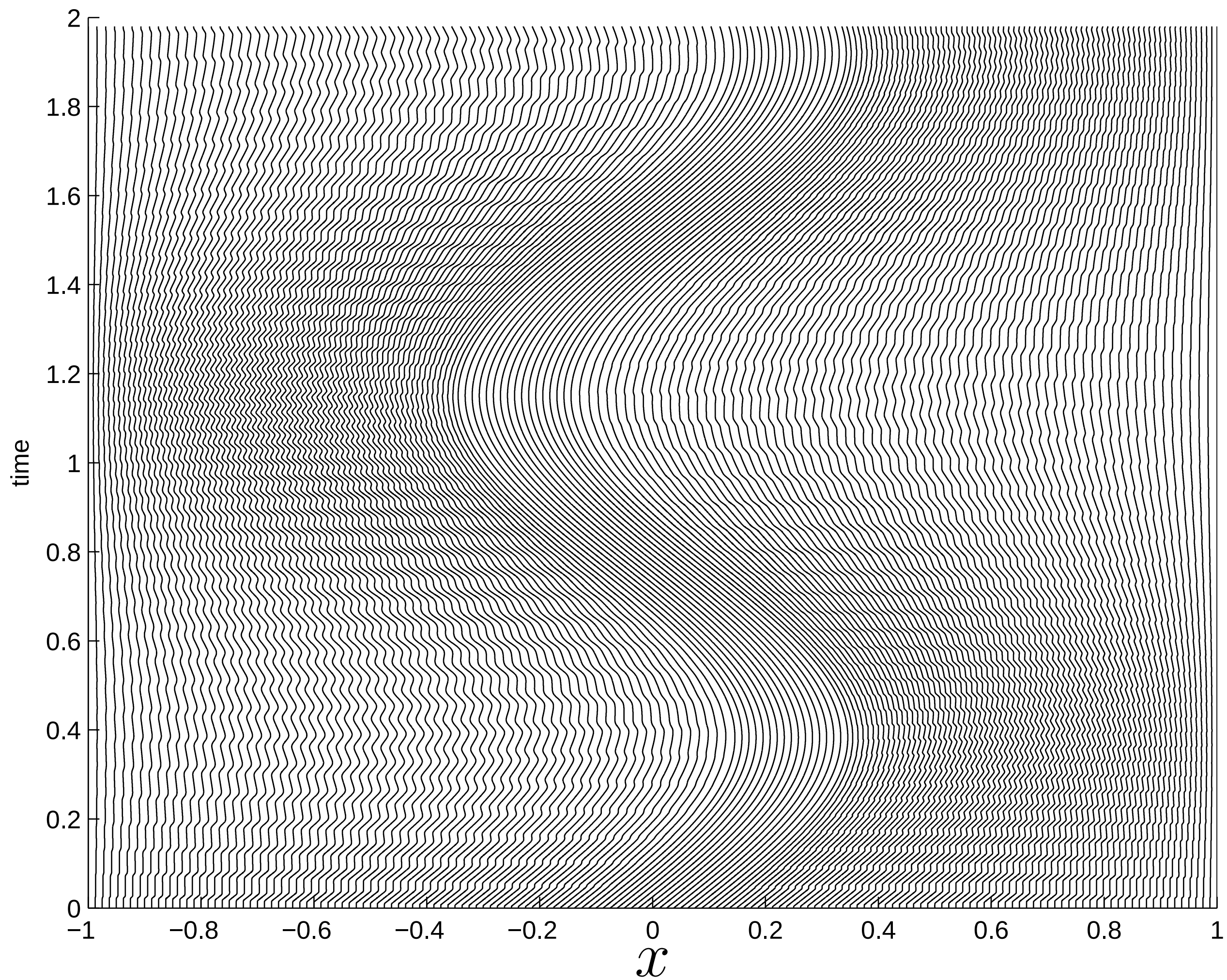
Oscillating water column

- Initially, in closed shock tube, water column moves at $u = 1$ from left to right, yielding air compression at right & air expansion at left
- Subsequently, pressure difference built up across water column resulting deceleration of column of water to right, makes a stop, & then acceleration to left; a reverse pressure difference built up across water column redirecting flow from left to right again
- Eventually, water column starts to oscillate

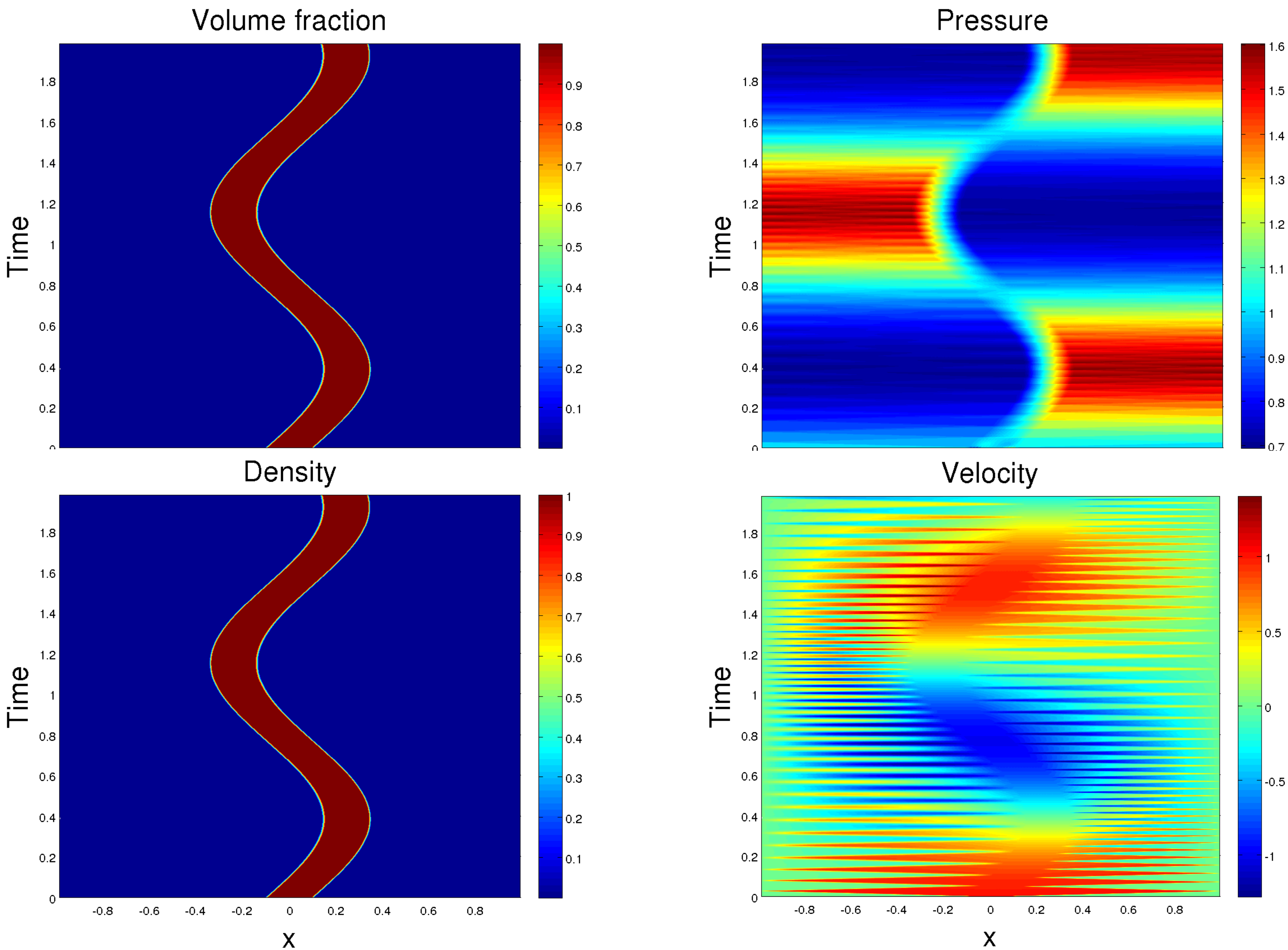


Oscillating water column

Physical grid in x-t plane

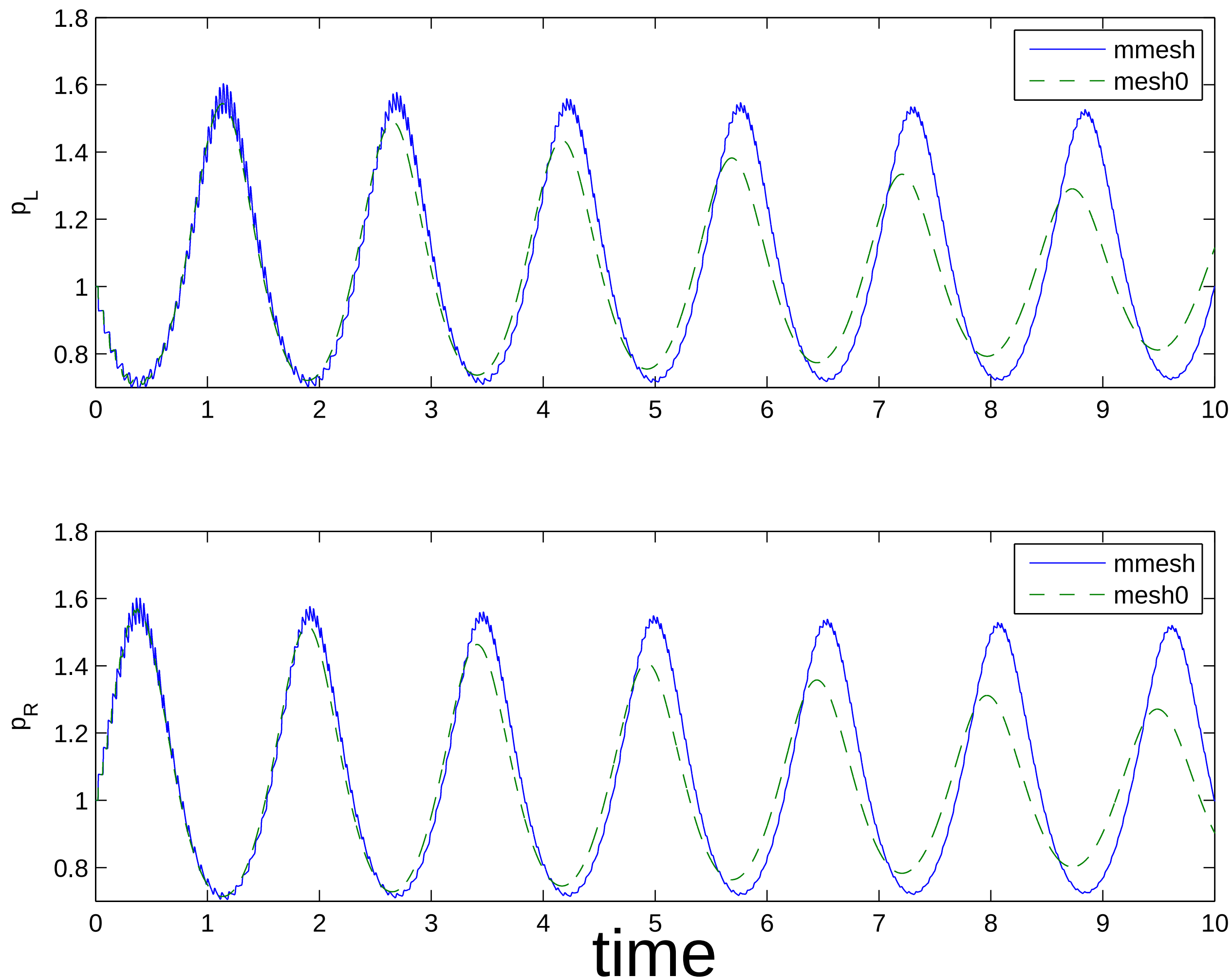


Oscillating water column



Oscillating water column

Time evolution of pressure at left & right boundaries



Tait equation of state & parameters

Each fluid phase (liquid & gas) satisfies Tait equation of state

$$p_k(\rho) = (p_{0k} + \mathcal{B}_k)\left(\frac{\rho}{\rho_{0k}}\right)^{\gamma_k} - \mathcal{B}_k \qquad \text{for } k = 1, 2.$$

with parameters for liquid phase as

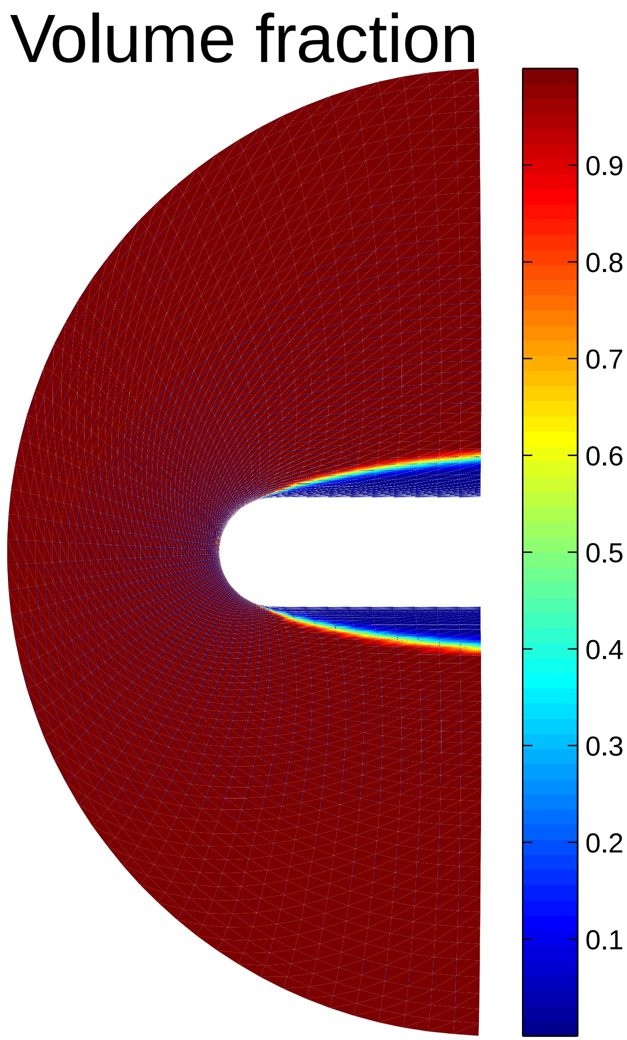
$$(\gamma, \mathcal{B}, \rho_0, p_0)_1 = \left(7, 3000\ \text{bar}, 10^3\ \text{kg/m}^3, 1\ \text{bar}\right)$$

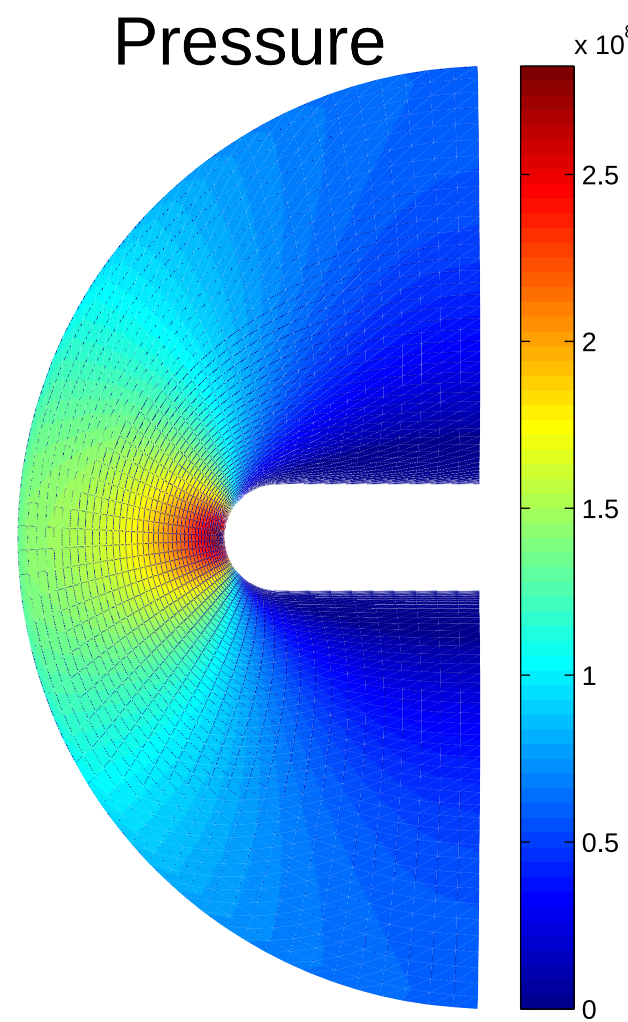
while parameters for gas phase as

$$(\gamma, \mathcal{B}, \rho_0, p_0)_2 = \left(1.4, 0, 1\ \text{kg/m}^3, 1\ \text{bar}\right)$$

Supersonic flow over a bluntbody

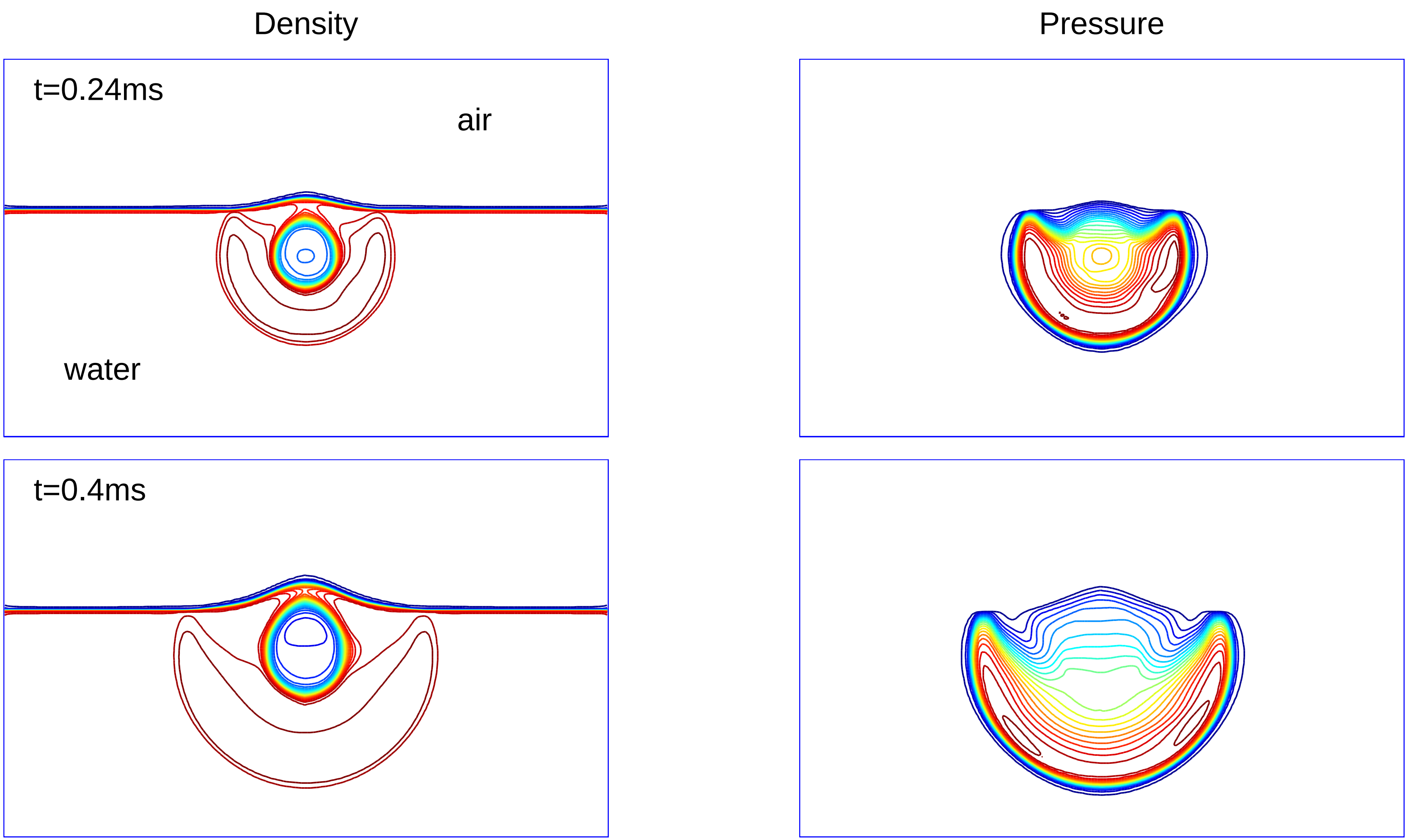
Formation of cavitation zone





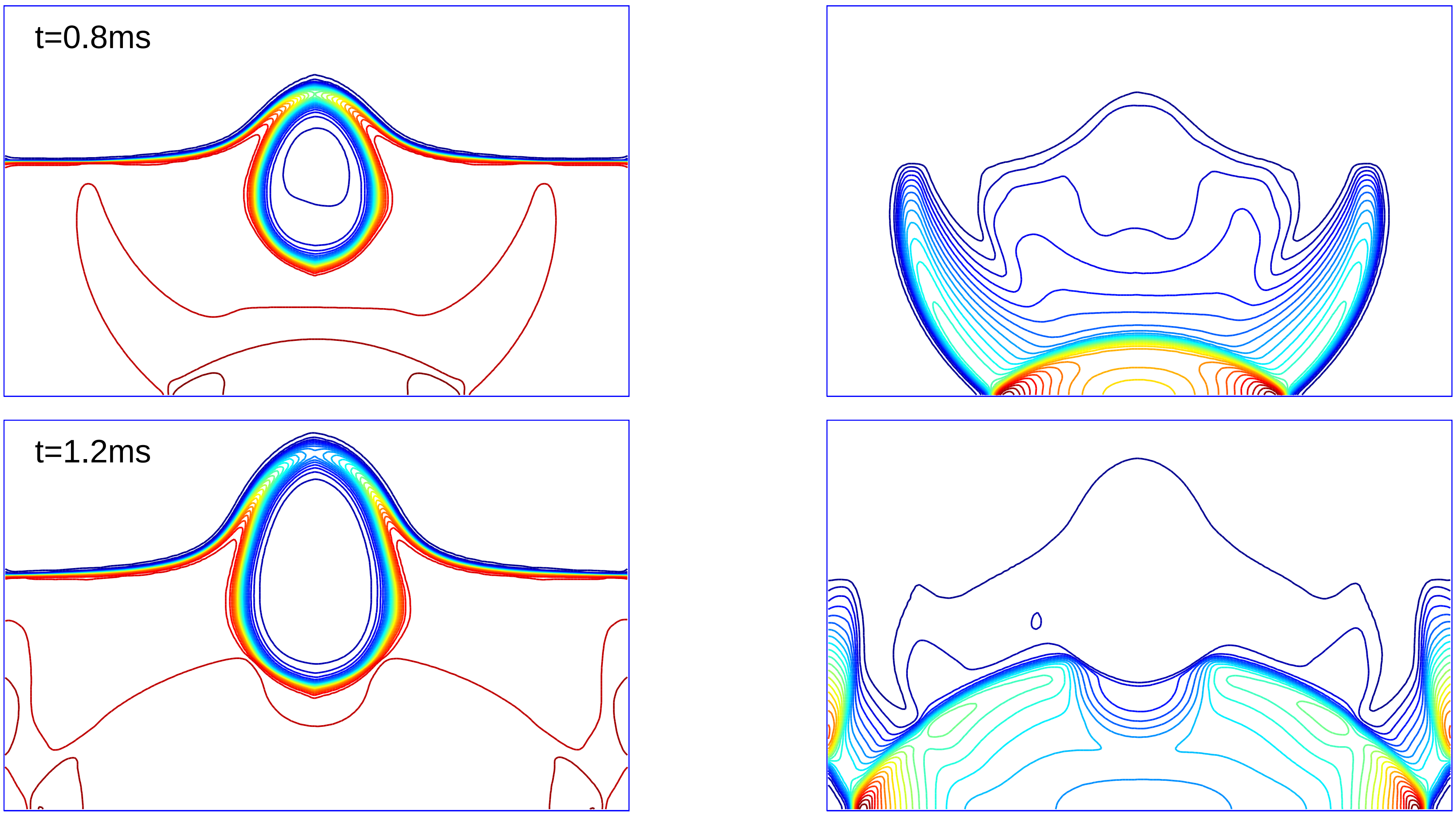
Underwater explosion

Contours of density and pressure at selected times



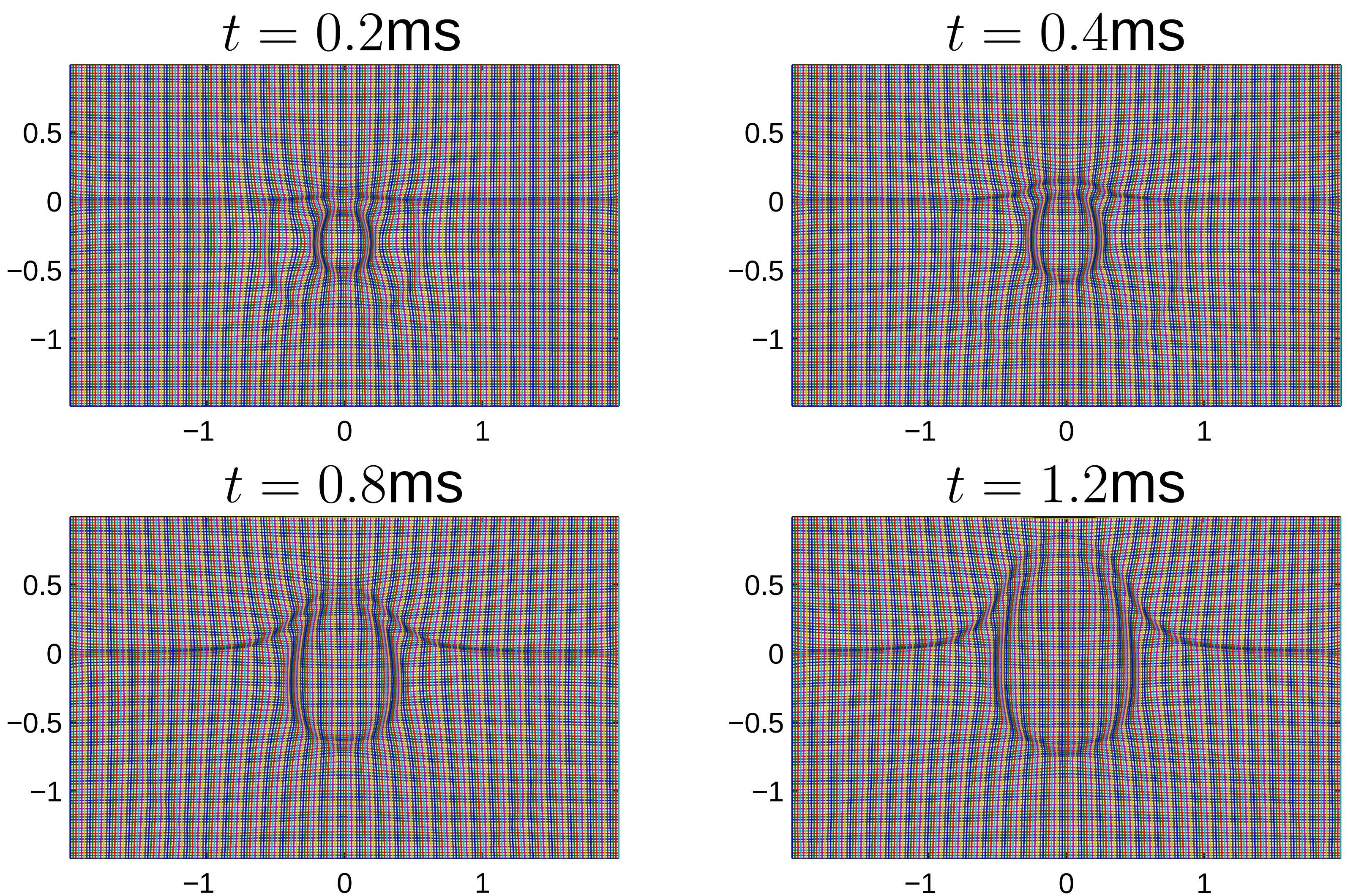
Underwater explosion

Contours of density and pressure at selected times



Underwater explosion

Physical grid at selected times



Final remarks

- Show preliminary results obtained using relaxation moving mesh methods for barotropic 2-phase flow problem
- Cavitation problems are often occurred in low Mach scenario & so suitable fixed up such as preconditioning & others are necessary for solution accuracy improvement
- Extension to non-barotropc cavitation with phase transition
- ...

Thank you

Reduced 2-phase flow model

Reduced 2-phase flow model of Kapila *et al.* is zero-order approximation of Baer-Nunziato equations with stiff mechanical relaxation that takes

$$\partial_t \left(\alpha_1 \rho_1\right) + \nabla \cdot \left(\alpha_1 \rho_1 \vec{u}\right) = 0$$

$$\partial_t \left(\alpha_2 \rho_2\right) + \nabla \cdot \left(\alpha_2 \rho_2 \vec{u}\right) = 0$$

$$\partial_t \left(\rho \vec{u}\right) + \nabla \cdot \left(\rho \vec{u} \otimes \vec{u}\right) + \nabla p = 0$$

$$\partial_t \left(\rho E\right) + \nabla \cdot \left(\rho E \vec{u} + p \vec{u}\right) = 0$$

$$\partial_t \alpha_1 + \vec{u} \cdot \nabla \alpha_1 = \frac{\rho_2 c_2^2 - \rho_1 c_1^2}{\sum_{k=1}^{2} \rho_k c_k^2 / \alpha_k} \nabla \cdot \vec{u}$$

Baer-Nunziato Two-Phase Flow Model

- Baer & Nunziato (J. Multiphase Flow 1986)

$$(\alpha_1\rho_1)_t + \nabla\cdot(\alpha_1\rho_1\vec{u}_1) = 0$$

$$(\alpha_1\rho_1\vec{u}_1)_t + \nabla\cdot(\alpha_1\rho_1\vec{u}_1\otimes\vec{u}_1) + \nabla(\alpha_1 p_1) = p_0\nabla\alpha_1 + \lambda(\vec{u}_2 - \vec{u}_1)$$

$$(\alpha_1\rho_1 E_1)_t + \nabla\cdot(\alpha_1\rho_1 E_1\vec{u}_1 + \alpha_1 p_1\vec{u}_1) = p_0\,(\alpha_2)_t + \lambda\vec{u}_0\cdot(\vec{u}_2 - \vec{u}_1)$$

$$(\alpha_2\rho_2)_t + \nabla\cdot(\alpha_2\rho_2\vec{u}_2) = 0$$

$$(\alpha_2\rho_2\vec{u}_2)_t + \nabla\cdot(\alpha_2\rho_2\vec{u}_2\otimes\vec{u}_2) + \nabla(\alpha_2 p_2) = p_0\nabla\alpha_2 - \lambda(\vec{u}_2 - \vec{u}_1)$$

$$(\alpha_2\rho_2 E_2)_t + \nabla\cdot(\alpha_2\rho_2 E_2\vec{u}_2 + \alpha_2 p_2\vec{u}_2) = -p_0\,(\alpha_2)_t - \lambda\vec{u}_0\cdot(\vec{u}_2 - \vec{u}_1)$$

$$(\alpha_2)_t + \vec{u}_0\cdot\nabla\alpha_2 = \mu\,(p_2 - p_1)$$

$\alpha_k = V_k/V$: volume fraction ($\alpha_1 + \alpha_2 = 1$), ρ_k: density, $\vec{u}_k$: velocity, p_k: pressure, $E_k = e_k + \vec{u}_k^2/2$: specific total energy, e_k: specific internal energy, $k = 1, 2$

Baer-Nunziato Model (Cont.)

p_0 & $\vec{u}_0$: interfacial pressure & velocity

- Baer & Nunziato (1986)
  - $p_0 = p_2, \quad \vec{u}_0 = \vec{u}_1$
- Saurel & Abgrall (1999)
  - $p_0 = \sum_{k=1}^{2} \alpha_k p_k, \quad \vec{u}_0 = \sum_{k=1}^{2} \alpha_k \rho_k \vec{u}_k \bigg/ \sum_{k=1}^{2} \alpha_k \rho_k$

λ & μ (> 0): relaxation parameters that determine rates at which velocities and pressures of two phases reach equilibrium