Wave-propagation based generalized Lagrangian method for hyperbolic balance laws

Application to inviscid compressible flow

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Objective

Describe simple Lagrangian-like moving grid approach for numerical resolution of nonlinear hyperbolic balance laws of the form

$$\frac{\partial}{\partial t} q\left(\vec{x},\ t\right) + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} f_j\left(q,\ \vec{x}\right) = \psi\left(q,\ \vec{x}\right)$$

with discontinuous initial data in general $N \geq 1$ rectangular or non-rectangular geometry

- $\vec{x} = (x_1, x_2, \ldots, x_N)$: spatial vector, t: time
- $q \in \mathbb{R}^m$: vector of m state quantities
- $f_j \in \mathbb{R}^m$: flux vector, $j = 1, 2, \ldots, N$, $\psi \in \mathbb{R}^m$: source term
- Model equation is hyperbolic if $\sum_{j=1}^{N_d} \alpha_j \left(\partial f_j / \partial q\right)$ is diagonalizable with real eigenvalues, $\alpha_j \in \mathbb{R}$

Outline

- Mathematical model for general balance laws
  - Eulerian formulation
  - Generalized Lagrangian formulation
  - Example to single component compressible flow
- Wave-propagation based finite volume methods
  - Generalized Riemann problem & approximate solver
  - Flux-based wave decomposition method
- Sample numerical examples
- Extension to compressible two-phase flow
- Future work

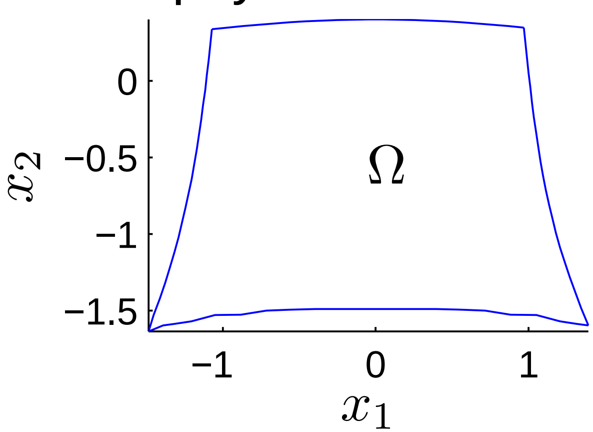
Mathematical Model

To begin with, consider a general non-rectangular domain Ω ($N = 2$ shown below) & introduce coordinate change $(\vec{x},\ t) \mapsto (\vec{\xi},\ \tau)$ via

$$\vec{\xi} = (\xi_1, \xi_2, \ldots, \xi_N)\,, \qquad \xi_j = \xi_j(\vec{x}, t), \qquad \tau = t,$$

that maps a physical domain Ω to a logical one $\hat{\Omega}$

physical domain

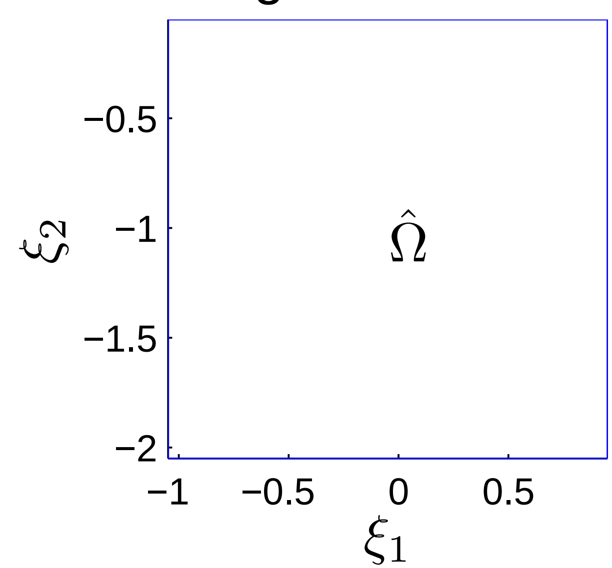


mapping

$$\longrightarrow$$

$$\xi_1 = \xi_1(x_1, x_2)$$
$$\xi_2 = \xi_2(x_1, x_2)$$

logical domain



Mathematical Model (Cont.)

To derive hyperbolic balance laws in this generalized coordinate $(\vec{\xi},\ \tau)$, using chain rule of partial differentiation, derivatives in physical space become

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \sum_{i=1}^{N} \frac{\partial \xi_i}{\partial t} \frac{\partial}{\partial \xi_i}, \qquad \frac{\partial}{\partial x_j} = \sum_{i=1}^{N} \frac{\partial \xi_i}{\partial x_j} \frac{\partial}{\partial \xi_i} \quad \text{for } j = 1, 2, \ldots, N,$$

yielding the equation

$$\frac{\partial q}{\partial \tau} + \sum_{j=1}^{N} \left(\frac{\partial \xi_j}{\partial t} \frac{\partial q}{\partial \xi_j} + \sum_{i=1}^{N} \frac{\partial \xi_i}{\partial x_j} \frac{\partial f_j}{\partial \xi_i} \right) = \psi(q)$$

Note this is not in divergence form, and hence is not conservative, in case the source term ψ is ignored.

Mathematical Model (Cont.)

To obtain a strong conservation-law form as

$$\frac{\partial \tilde{q}}{\partial \tau} + \sum_{j=1}^{N} \frac{\partial \tilde{f}_j}{\partial \xi_j} = \tilde{\psi}$$

for some $\tilde{q}$, $\tilde{f}_j$, & $\tilde{\psi}$, we first multiply $J = \det\left(\partial \vec{\xi} / \partial \vec{x}\right)^{-1}$ to the aforementioned non-conservative equations, and have

$$J \frac{\partial q}{\partial \tau} + \sum_{j=1}^{N} J \left(\frac{\partial \xi_j}{\partial t} \frac{\partial q}{\partial \xi_j} + \sum_{i=1}^{N} \frac{\partial \xi_i}{\partial x_j} \frac{\partial f_j}{\partial \xi_i} \right) = J \psi(q)$$

Then use differentiation by parts, $u\, dv = d(uv) - v\, du$, yielding

$$\frac{\partial \tilde{q}}{\partial \tau} + \sum_{j=1}^{N} \frac{\partial \tilde{f}_j}{\partial \xi_j} = \tilde{\psi} + \mathcal{G}$$

with $\tilde{q} = Jq$, $\quad \tilde{f}_j = J \left(q \frac{\partial \xi_j}{\partial t} + \sum_{k=1}^{N} f_k \frac{\partial \xi_j}{\partial x_k} \right)$, $\quad \tilde{\psi} = J\psi$, $\quad$ & $\mathcal{G}$ (see next)

Mathematical Model (Cont.)

Here we have

$$\mathcal{G} = q \left[\frac{\partial J}{\partial \tau} + \sum_{j=1}^{N} \frac{\partial}{\partial \xi_j} \left(J \frac{\partial \xi_j}{\partial t} \right) \right] + \sum_{j=1}^{N} f_j \left[\sum_{k=1}^{N} \frac{\partial}{\partial \xi_k} \left(J \frac{\partial \xi_k}{\partial x_j} \right) \right]$$

Note with the use of basic grid-metric relations (see next), it is known that

$$\frac{\partial J}{\partial \tau} + \sum_{j=1}^{N} \frac{\partial}{\partial \xi_j} \left(J \frac{\partial \xi_j}{\partial t} \right) = 0 \quad \text{(geometrical conservation law)}$$

$$\sum_{k=1}^{N} \frac{\partial}{\partial \xi_k} \left(J \frac{\partial \xi_k}{\partial x_j} \right) = 0 \quad \forall\, j = 1, 2, \ldots, N \quad \text{(compatibility condition)}$$

and hence $\mathcal{G} = 0$

Basic Grid-Metric Relations

Assume existence of inverse transformation

$$t = \tau, \qquad x_j = x_j(\vec{\xi}, t) \qquad \text{for} \quad j = 1, 2, \ldots, N,$$

To find basic grid-metric relations between different coordinates, employ elementary differential rule

$$\frac{\partial(\tau, \vec{\xi})}{\partial(t, \vec{x})} = \frac{\partial(t, \vec{x})}{\partial(\tau, \vec{\xi})}^{-1}$$

yielding in $N = 3$ case, for example, as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \partial_t \xi_1 & \partial_{x_1} \xi_1 & \partial_{x_2} \xi_1 & \partial_{x_3} \xi_1 \\ \partial_t \xi_2 & \partial_{x_1} \xi_2 & \partial_{x_2} \xi_2 & \partial_{x_3} \xi_2 \\ \partial_t \xi_3 & \partial_{x_1} \xi_3 & \partial_{x_2} \xi_3 & \partial_{x_3} \xi_3 \end{pmatrix} = \frac{1}{J} \begin{pmatrix} J & 0 & 0 & 0 \\ J_{01} & J_{11} & J_{21} & J_{31} \\ J_{02} & J_{12} & J_{22} & J_{32} \\ J_{03} & J_{13} & J_{23} & J_{33} \end{pmatrix}$$

Grid-Metric Relations (Cont.)

Here

$$J = \left| \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} \right| = \text{det}\left(\frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)} \right),$$

$$J_{11} = \left| \frac{\partial(x_2, x_3)}{\partial(\xi_2, \xi_3)} \right|, \quad J_{21} = \left| \frac{\partial(x_1, x_3)}{\partial(\xi_3, \xi_2)} \right|, \quad J_{31} = \left| \frac{\partial(x_1, x_2)}{\partial(\xi_2, \xi_3)} \right|,$$

$$J_{12} = \left| \frac{\partial(x_2, x_3)}{\partial(\xi_3, \xi_1)} \right|, \quad J_{22} = \left| \frac{\partial(x_1, x_3)}{\partial(\xi_1, \xi_3)} \right|, \quad J_{32} = \left| \frac{\partial(x_1, x_2)}{\partial(\xi_3, \xi_1)} \right|,$$

$$J_{13} = \left| \frac{\partial(x_2, x_3)}{\partial(\xi_1, \xi_2)} \right|, \quad J_{23} = \left| \frac{\partial(x_1, x_3)}{\partial(\xi_2, \xi_1)} \right|, \quad J_{33} = \left| \frac{\partial(x_1, x_2)}{\partial(\xi_1, \xi_2)} \right|,$$

$$J_{0j} = -\sum_{i=1}^{N_d} J_{ij} \partial_\tau x_i, \qquad j = 1, 2, 3,$$

and so grid-metric relations between different coordinates

$$\nabla \xi_j = (\partial_t \xi_j,\ \nabla_{\vec{x}} \xi_j) = (\partial_t \xi_j,\ \partial_{x_1} \xi_j,\ \partial_{x_2} \xi_j,\ \partial_{x_3} \xi_j) = \frac{1}{J} (J_{0j},\ J_{1j},\ J_{2j},\ J_{3j})$$

Grid-Metric Relations (Cont.)

Note in two dimensions $N = 2$, we have

$$\left(\frac{\partial \xi_1}{\partial t}, \frac{\partial \xi_1}{\partial x_1}, \frac{\partial \xi_1}{\partial x_2}\right) = \frac{1}{J}\left(-\frac{\partial x_1}{\partial \tau}\frac{\partial x_2}{\partial \xi_2} + \frac{\partial x_2}{\partial \tau}\frac{\partial x_1}{\partial \xi_2}, \frac{\partial x_2}{\partial \xi_2}, -\frac{\partial x_1}{\partial \xi_2}\right)$$

$$\left(\frac{\partial \xi_2}{\partial t}, \frac{\partial \xi_2}{\partial x_1}, \frac{\partial \xi_2}{\partial x_2}\right) = \frac{1}{J}\left(\frac{\partial x_1}{\partial \tau}\frac{\partial x_2}{\partial \xi_1} - \frac{\partial x_2}{\partial \tau}\frac{\partial x_1}{\partial \xi_1}, -\frac{\partial x_2}{\partial \xi_1}, \frac{\partial x_1}{\partial \xi_1}\right)$$

$$J = \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_1}$$

Thus to have $\mathcal{G} = 0$ fulfilled, grid-metrics should obey

$$\frac{\partial J}{\partial \tau} + \frac{\partial}{\partial \xi_1}\left(J\frac{\partial \xi_1}{\partial t}\right) + \frac{\partial}{\partial \xi_2}\left(J\frac{\partial \xi_2}{\partial t}\right) = 0$$

$$\frac{\partial}{\partial \xi_1}\left(J\frac{\partial \xi_1}{\partial x_1}\right) + \frac{\partial}{\partial \xi_2}\left(J\frac{\partial \xi_2}{\partial x_1}\right) = \frac{\partial}{\partial \xi_1}\left(\frac{\partial x_2}{\partial \xi_2}\right) + \frac{\partial}{\partial \xi_2}\left(-\frac{\partial x_2}{\partial \xi_1}\right) = 0$$

$$\frac{\partial}{\partial \xi_1}\left(J\frac{\partial \xi_1}{\partial x_2}\right) + \frac{\partial}{\partial \xi_2}\left(J\frac{\partial \xi_2}{\partial x_2}\right) = \frac{\partial}{\partial \xi_1}\left(\frac{-\partial x_1}{\partial \xi_2}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{\partial x_1}{\partial \xi_1}\right) = 0$$

Mathematical Model (Cont.)

As an example, with gravity effect included, Euler equations for single component compressible gas flow take

- Cartesian coordinate case

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho u_i \\ E\end{pmatrix}+\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\begin{pmatrix}\rho u_j \\ \rho u_i u_j + p\delta_{ij} \\ E u_j + p u_j\end{pmatrix}=\begin{pmatrix}0 \\ -\rho\frac{\partial \phi}{\partial x_i} \\ -\rho\vec{u}\cdot\nabla\phi\end{pmatrix}, \quad i=1,\ldots,N$$

- Generalized coordinate case

$$\frac{\partial}{\partial \tau}\begin{pmatrix}\rho J \\ \rho J u_i \\ J E\end{pmatrix}+\sum_{j=1}^{N}\frac{\partial}{\partial \xi_j}J\begin{pmatrix}\rho U_j \\ \rho u_i U_j + p\frac{\partial \xi_j}{\partial x_i} \\ E U_j + p U_j - p\frac{\partial \xi_j}{\partial t}\end{pmatrix}=\begin{pmatrix}0 \\ -\rho J\frac{\partial \phi}{\partial x_i} \\ -\rho J\vec{u}\cdot\nabla\phi\end{pmatrix}$$

ρ: density, $p=p(\rho,e)$: pressure , e: internal energy

$E=\rho e+\rho\sum_{j=1}^{N}u_j^2/2$: total energy, ϕ: gravitational potential

$U_j=\partial_t\xi_j+\sum_{i=1}^{N}u_i\partial_{x_i}\xi_j$: contravariant velocity in ξ_j-direction

Mathematical Model (Cont.)

Note that to complete the model, we must

1. make clear the transformation $(\vec{x},\ t) \mapsto (\vec{\xi},\ \tau)$ initially
   - Depending on how complex the geometry is, this can be done by various numerical means

Mathematical Model (Cont.)

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Mathematical Model (Cont.)

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2. choose a moving grid strategy for $\partial_\tau \vec{x}$
   - When $\partial_\tau \vec{x} = 0$ or $\partial_\tau \vec{x} = \vec{u}_b(t)$ (rigid-body motion)
     - $\partial_t \vec{\xi}$ & $\nabla_{\vec{x}} \vec{\xi}$ are time-independent; no need to have more additional condition

Mathematical Model (Cont.)

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     - $\partial_t \vec{\xi}$ & $\nabla_{\vec{x}} \vec{\xi}$ are time-independent; no need to have more additional condition
   - While $\partial_\tau \vec{x} \neq 0$ (flow-dependent motion) (see next)
     - $\partial_t \vec{\xi}$ & $\nabla_{\vec{x}} \vec{\xi}$ would be time-dependent; require additional conditions to determine $\nabla_{\vec{\xi}} \vec{x}$ (N^2 of them in total) over time (see below)

Lagrangian-Like Moving Grid

For compressible flow, to improve numerical resolution of interfaces (material or slip lines), it is popular to take $\partial_\tau \vec{x}$ as

- Lagrangian case: $\partial_\tau \vec{x} = \vec{u}$ (flow velocity)
- Lagrangian-like case: $\partial_\tau \vec{x} = h_0 \vec{u}$ (pseudo velocity)
  - $h_0 \in [0, 1]$ (fixed piecewise const.)
- Unified coordinate case: $\partial_\tau \vec{x} = h \vec{u}$
  - $h \in [0, 1]$ but is determined from a PDE constraint arising from such as grid-angle or grid-Jacobian preserving condition
- ALE-like case: $\partial_\tau \vec{x} = \vec{\mathcal{U}}$ (arbitrary velocity)

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Here we will focus on the simple Lagrangian-like case

Unified Coordinate System

Consider $N = 2$ case, for example, and use simplified notation $\vec{x} = (x,\ y)$, $\vec{\xi} = (\xi,\ \eta)$. At given time instance, free parameter h can be chosen based on

- Grid-angle preserving condition (Hui *et al.* JCP 1999)

$$
\begin{aligned}
\frac{\partial}{\partial \tau} \cos^{-1}\left(\frac{\nabla \xi}{|\nabla \xi|} \cdot \frac{\nabla \eta}{|\nabla \eta|}\right) &= \frac{\partial}{\partial \tau} \cos^{-1}\left(\frac{-y_\eta x_\eta - y_\xi x_\xi}{\sqrt{y_\xi^2 + y_\eta^2}\sqrt{x_\xi^2 + x_\eta^2}}\right) \\
&= \cdots \\
&= \mathcal{A} h_\xi + \mathcal{B} h_\eta + \mathcal{C} h = 0 \qquad (\text{1st order PDE})
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{A} &= \sqrt{x_\eta^2 + y_\eta^2}\,(v x_\xi - u y_\xi)\,, \quad \mathcal{B} = \sqrt{x_\xi^2 + y_\xi^2}\,(u y_\eta - v x_\eta) \\
\mathcal{C} &= \sqrt{x_\xi^2 + y_\xi^2}\,(u_\eta y_\eta - v_\eta x_\eta) - \sqrt{x_\eta^2 + y_\eta^2}\,(u_\xi y_\xi - v_\xi x_\xi)
\end{aligned}
$$

Unified Coordinate System

Consider $N = 2$ case, for example, and use simplified notation $\vec{x} = (x,\ y)$, $\vec{\xi} = (\xi,\ \eta)$. Or alternatively, based on

- Grid-Jacobian preserving condition

$$\begin{aligned}
\frac{\partial J}{\partial \tau} &= \frac{\partial}{\partial \tau}\left(x_\xi y_\eta - x_\eta y_\xi\right) \\
&= x_{\xi\tau}\, y_\eta + x_\xi\, y_{\eta\tau} - x_{\eta\tau}\, y_\xi - x_\eta\, y_{\xi\tau} \\
&= \cdots \\
&= \mathcal{A} h_\xi + \mathcal{B} h_\eta + \mathcal{C} h = 0 \qquad (\text{1st order PDE })
\end{aligned}$$

with

$$\mathcal{A} = u y_\eta - v x_\eta, \quad \mathcal{B} = v x_\xi - u y_\xi, \quad \mathcal{C} = u_\xi y_\eta + v_\eta x_\xi - u_\eta y_\xi - v_\xi x_\eta$$

Lagrangian-Like Grid (Cont.)

Now with the temporal motion of the coordinate system governed by $\partial_\tau \vec{x} = h_0 \vec{u}$. We should impose conditions on grid metrics $\partial_t \vec{\xi}$ & $\nabla_{\vec{x}} \vec{\xi}$ to have the fulfillment of geometrical conservation law

$$\frac{\partial J}{\partial \tau} + \sum_{j=1}^{N} \frac{\partial}{\partial \xi_j} \left(J \frac{\partial \xi_j}{\partial t} \right) = 0$$

Here we are interested in an approach that is based on the compatibility condition of $\partial_\tau \partial_{\xi_j} x_i$ & $\partial_{\xi_j} \partial_\tau x_i$, *i.e.*,

$$\frac{\partial}{\partial \tau} \left(\frac{\partial x_i}{\partial \xi_j} \right) + \frac{\partial}{\partial \xi_j} \left(-\frac{\partial x_i}{\partial \tau} \right) = 0 \qquad \text{for } i, j = 1, 2, \ldots, N.$$

for unknowns $\partial x_i / \partial \xi_j$, yielding easy computation of J & $\nabla \xi_j$

Mathematical Model (Cont.)

In summary, our Lagrangina-like model system for single component compressible flow problems consists of

- Physical balance laws

$$\frac{\partial}{\partial \tau}\begin{pmatrix} \rho J \\ \rho J u_i \\ J E \end{pmatrix} + \sum_{j=1}^{N} \frac{\partial}{\partial \xi_j} J \begin{pmatrix} \rho U_j \\ \rho u_i U_j + p\frac{\partial \xi_j}{\partial x_i} \\ E U_j + p U_j - p\frac{\partial \xi_j}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 \\ -\rho J \frac{\partial \phi}{\partial x_i} \\ -\rho J \vec{u} \cdot \nabla \phi \end{pmatrix}$$

- Geometrical conservation laws

$$\frac{\partial}{\partial \tau}\left(\frac{\partial x_i}{\partial \xi_j}\right) + \frac{\partial}{\partial \xi_j}\left(-\frac{\partial x_i}{\partial \tau}\right) = 0 \qquad \text{for } i, j = 1, 2, \dots, N.$$

- Moving grid condition $\partial_\tau \vec{x} = h_0 \vec{u}$ & pressure law $p(\rho, e)$

Axisymmetric Compressible Flow

- Physical balance laws (ξ_1: axisymmetric direction)

$$\frac{\partial}{\partial \tau}\begin{pmatrix}\rho J\\ \rho J u_i\\ J E\end{pmatrix} + \sum_{j=1}^{2}\frac{\partial}{\partial \xi_j} J \begin{pmatrix}\rho U_j\\ \rho u_i U_j + p\frac{\partial \xi_j}{\partial x_i}\\ E U_j + p U_j - p\frac{\partial \xi_j}{\partial t}\end{pmatrix} =$$

$$\begin{pmatrix}-\frac{1}{x}\rho J u_1\\ -\frac{1}{x}\rho J u_i u_j - \rho J\frac{\partial \phi}{\partial x_i}\\ -\frac{1}{x}J(E+p)u_1 - \rho J\vec{u}\cdot\nabla\phi\end{pmatrix} \qquad \text{for } i = 1,2$$

- Geometrical conservation laws

$$\frac{\partial}{\partial \tau}\left(\frac{\partial x_i}{\partial \xi_j}\right) + \frac{\partial}{\partial \xi_j}\left(-\frac{\partial x_i}{\partial \tau}\right) = 0 \qquad \text{for } i, j = 1, 2.$$

- Moving grid condition $\partial_\tau \vec{x} = h_0 \vec{u}$ & pressure law $p(\rho, e)$

Mathematical Models: Remarks

For single component compressible flow model mentioned above, it is known that under some thermodynamic stability conditions

- when $h_0 = 0$ (Eulerian case), the model is hyperbolic
- when $h_0 = 1$ (Lagrangian case), the model is weakly hyperbolic (do not possess complete eigenvectors)
- when $h_0 \in (0, 1)$ (Lagrangian-like case), the model is hyperbolic

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- when $h_0 \in (0, 1)$ (Lagrangian-like case), the model is hyperbolic

If a prescribed velocity $\vec{u}_b$ for a rigid body motion is included in the formulation *i.e.*, with $\partial_\tau \vec{x} = h_0 \vec{u} + \vec{u}_b$, we should be able to use the model to solve some moving body problems as well.

Review of Previous Work

The work presented here is related to

- W.H. Hui *et al.* (JCP 1999, 2001): Unified coordinated system for Euler equations
- W.H. Hui (Comm. Phys. Sci. 2007): Unified coordinate system in CFD
- C. Jin & K. Xu (JCP 2007): Moving grid gas-kinetic method for viscous flow
- P. Jia *et al.* (Computers and Fluids 2006) Unified coordinated system for compressible milti-material flow
- Z. Chen *et al.* (Int J. Numer. Meth Fluids 2007): Wave speed based moving coordinates for compressible flow equations

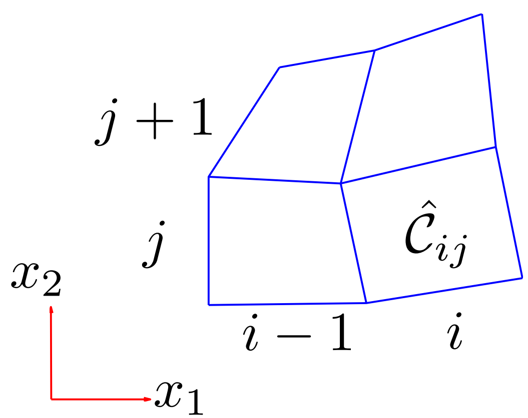
Numerical Methods

Employ finite volume formulation of numerical solution

$$Q_{ijk}^n \approx \frac{1}{\Delta\xi_1 \Delta\xi_2 \Delta\xi_3} \int_{C_{ijk}} q(\xi_1, \xi_2, \xi_3, \tau_n)\, dV$$

that gives approximate value of cell average of solution q over cell C_{ijk} at time τ_n (sample case in $2D$ shown below)

physical domain

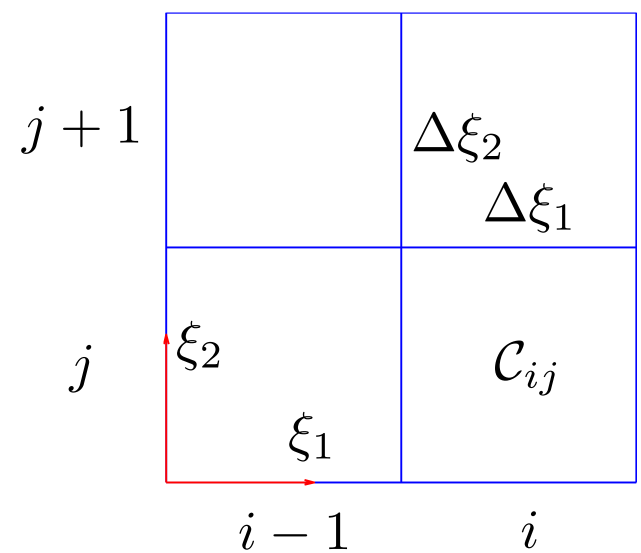


mapping

$\longleftarrow$

$$x_1 = x_1(\xi_1, \xi_2)$$
$$x_2 = x_2(\xi_1, \xi_2)$$

logical domain



Numerical Methods (Cont.)

In three dimensions $N = 3$, equations to be solved take

$$\frac{\partial}{\partial \tau} q\left(\vec{\xi},\ \tau\right) + \sum_{j=1}^{N} \frac{\partial}{\partial \xi_j} f_j\left(q,\ \vec{\xi}\right) = \psi\left(q,\ \vec{\xi}\right)$$

A simple dimensional-splitting method based on f-wave approach of LeVeque *et al.* is used for approximation, *i.e.*,

- Solve one-dimensional generalized Riemann problem (defined below) at each cell interfaces
- Use resulting jumps of fluxes (decomposed into each wave family) of Riemann solution to update cell averages
- Introduce limited jumps of fluxes to achieve high resolution

Numerical Methods (cont.)

Basic steps of a dimensional-splitting scheme

- ξ_1-sweeps: solve

$$\frac{\partial q}{\partial \tau} + f_1\left(\frac{\partial}{\partial \xi}, q, \nabla\vec{\xi}\right) = 0 \quad \text{updating } Q^n_{ijk} \text{ to } Q^*_{ijk}$$

- ξ_2-sweeps: solve

$$\frac{\partial q}{\partial \tau} + f_2\left(\frac{\partial}{\partial \xi_2}, q, \nabla\vec{\xi}\right) = 0 \quad \text{updating } Q^*_{ijk} \text{ to } Q^{**}_{ijk}$$

- ξ_3-sweeps: solve

$$\frac{\partial q}{\partial \tau} + f_3\left(\frac{\partial}{\partial \xi_3}, q, \nabla\vec{\xi}\right) = 0 \quad \text{updating } Q^{**}_{ijk} \text{ to } Q^{n+1}_{ijk}$$

Numerical Methods (cont.)

Consider ξ_1-sweeps, for example,

- First order update is

$$Q^*_{ijk} = Q^n_{ijk} - \frac{\Delta\tau}{\Delta\xi_1}\left[\left(\mathcal{A}_1^+\Delta Q\right)^n_{i-1/2,jk} + \left(\mathcal{A}_1^-\Delta Q\right)^n_{i+1/2,jk}\right]$$

  with the fluctuations

$$(\mathcal{A}_1^+\Delta Q)^n_{i-1/2,jk} = \sum_{m:(\lambda_{1,m})^n_{i-1/2,jk}>0} (\mathcal{Z}_{1,m})^n_{i-1/2,jk}$$

  and

$$(\mathcal{A}_1^-\Delta Q)^n_{i+1/2,jk} = \sum_{m:(\lambda_{1,m})^n_{i+1/2,jk}<0} (\mathcal{Z}_{1,m})^n_{i+1/2,jk}$$

  $(\lambda_{1,m})^n_{\iota-1/2,jk}$ & $(\mathcal{Z}_{1,m})^n_{\iota-1/2,jk}$ are in turn wave speed and f-waves for the mth family of the 1D Riemann problem solutions

Numerical Methods (cont.)

- High resolution correction is

$$Q^{*}_{ijk} := Q^{*}_{ijk} - \frac{\Delta\tau}{\Delta\xi_1}\left[\left(\tilde{\mathcal{F}}_1\right)^n_{i+1/2,jk} - \left(\tilde{\mathcal{F}}_1\right)^n_{i-1/2,jk}\right]$$

with $\quad (\tilde{\mathcal{F}}_1)^n_{i-1/2,jk} = \frac{1}{2}\sum_{m=1}^{m_w}\left[\text{sign}\left(\lambda_{1,m}\right)\left(1 - \frac{\Delta\tau}{\Delta\xi_1}\left|\lambda_{1,m}\right|\right)\tilde{\mathcal{Z}}_{1,m}\right]^n_{i-1/2,jk}$

$\tilde{\mathcal{Z}}_{\iota,m}$ is a limited value of $\mathcal{Z}_{\iota,m}$

It is clear that this method belongs to a class of upwind schemes, and is stable when the typical CFL (Courant-Friedrichs-Lewy) condition:

$$\nu = \frac{\Delta\tau\,\max_m\left(\lambda_{1,m}, \lambda_{2,m}, \lambda_{3,m}\right)}{\min\left(\Delta\xi_1, \Delta\xi_2, \Delta\xi_3\right)} \leq 1,$$

Generalized Riemann Problem

Generalized Riemann problem for our model equations at cell interface $\xi_{i-1/2}$ consists of the equation

$$\begin{cases} \dfrac{\partial q_{i-1,jk}}{\partial \tau} + f_1\left(\dfrac{\partial}{\partial \xi_1}, q_{i-1,jk}\right) = 0 & \text{if } \xi_1 < (\xi_1)_{i-1/2}, \\ \dfrac{\partial q_{ijk}}{\partial \tau} + f_1\left(\dfrac{\partial}{\partial \xi_1}, q_{ijk}\right) = 0 & \text{if } \xi_1 > (\xi_1)_{i-1/2}, \end{cases}$$

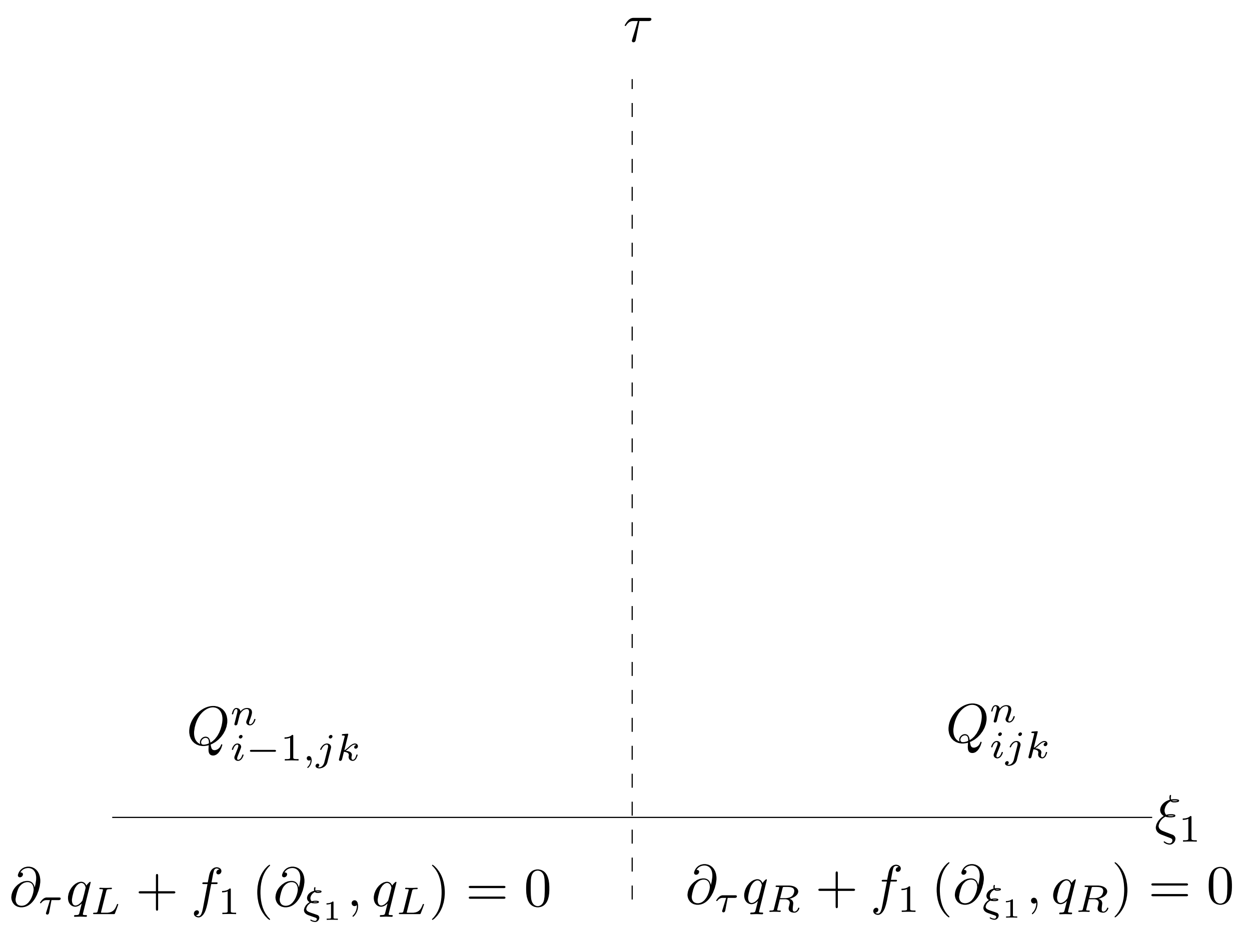
together with piecewise constant initial data

$$q(\xi_1, 0) = \begin{cases} Q^n_{i-1,jk} & \text{for} \quad \xi < \xi_{i-1/2} \\ Q^n_{ijk} & \text{for} \quad \xi > \xi_{i-1/2} \end{cases}$$

$$q_{ijk} = q\big|_{(\partial_{\xi_2}\vec{x},\, \partial_{\xi_3}\vec{x})_{ijk}} \quad \& \quad f_1(\partial_{\xi_1}, q_{ijk}) = f_1(\partial_{\xi_1}, q)\big|_{(\partial_{\xi_2}\vec{x},\, \partial_{\xi_3}\vec{x})_{ijk}}$$

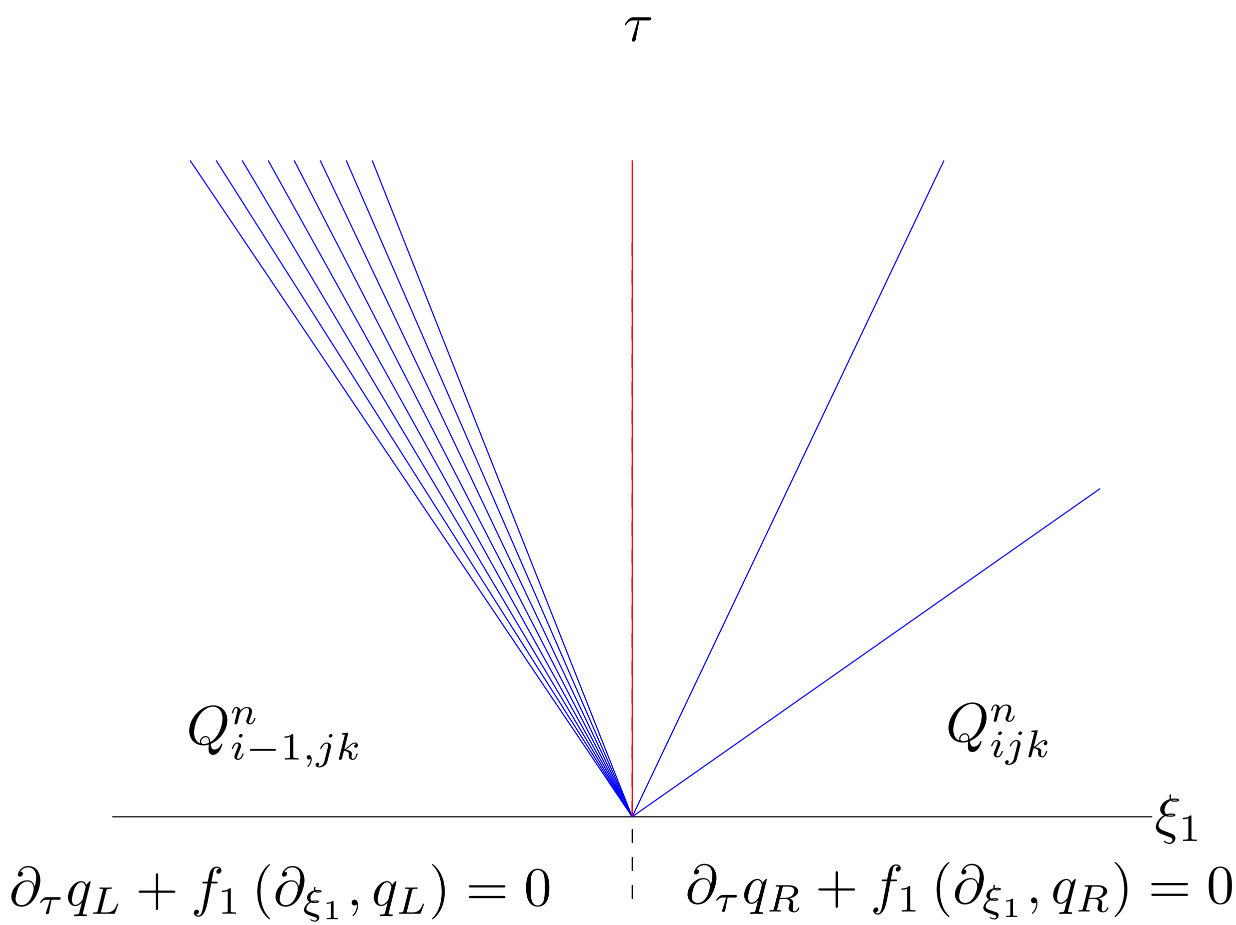
Generalized Riemann Problem

Generalized Riemann problem at time $\tau = 0$



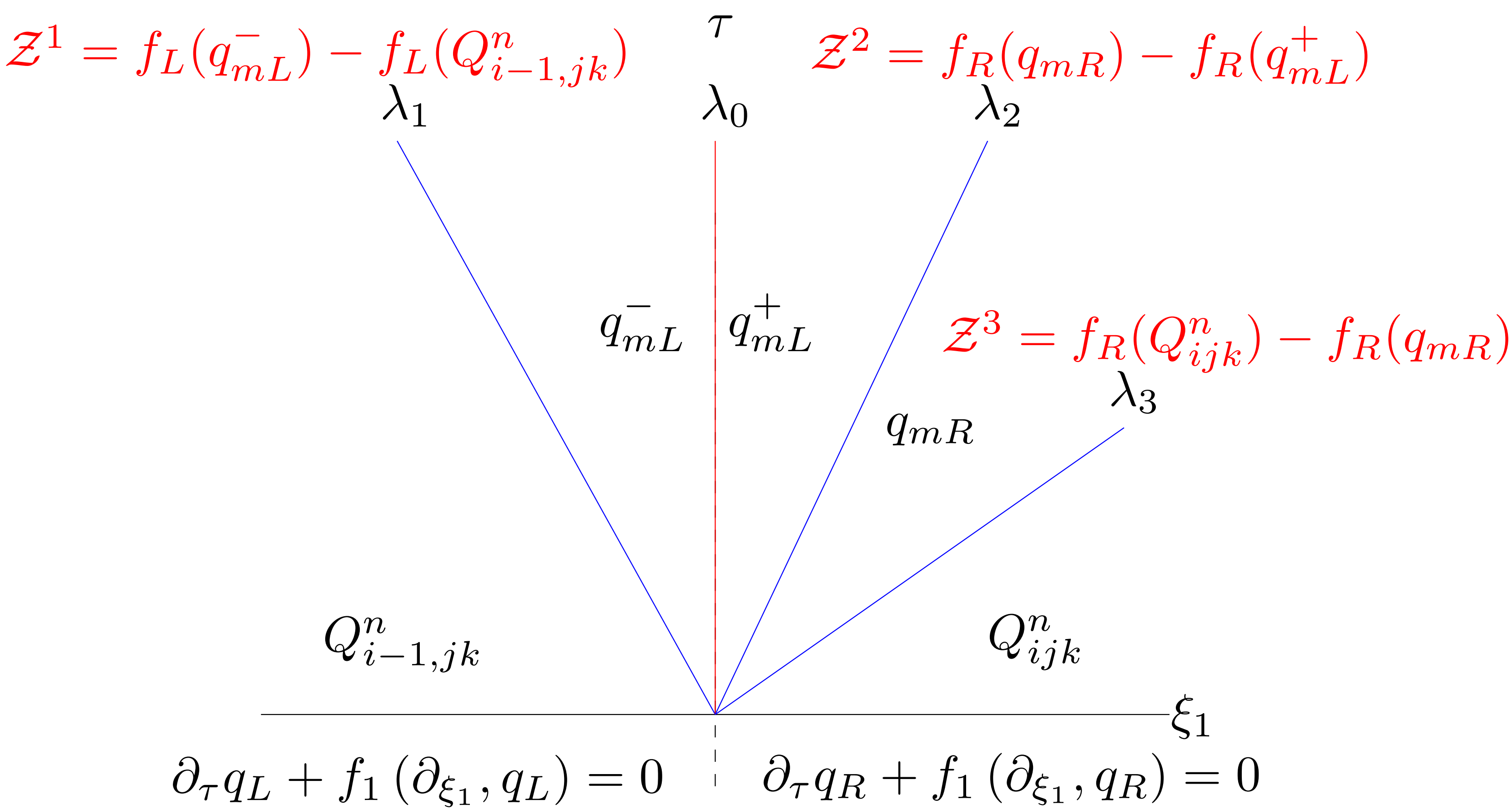
Generalized Riemann Problem

Exact generalized Riemann solution: basic structure



Generalized Riemann Problem

Shock-only approximate Riemann solution: basic structure



Shock-only Riemann Solver

- Rotate velocity vector in Riemann data normal to each cell interface
- Find midstate velocity υ_m and pressure p_m by solving

$$\phi(p_m) = \upsilon_{mR}(p_m) - \upsilon_{mL}(p_m) = 0$$

derived from Rankine-Hugoniot relation iteratively, where

$$\upsilon_{mL}(p) = \upsilon_L - \frac{p - p_L}{M_L(p)}, \qquad \upsilon_{mR}(p) = \upsilon_R + \frac{p - p_R}{M_R(p)}$$

- Propagation speed of each moving discontinuity is determined by

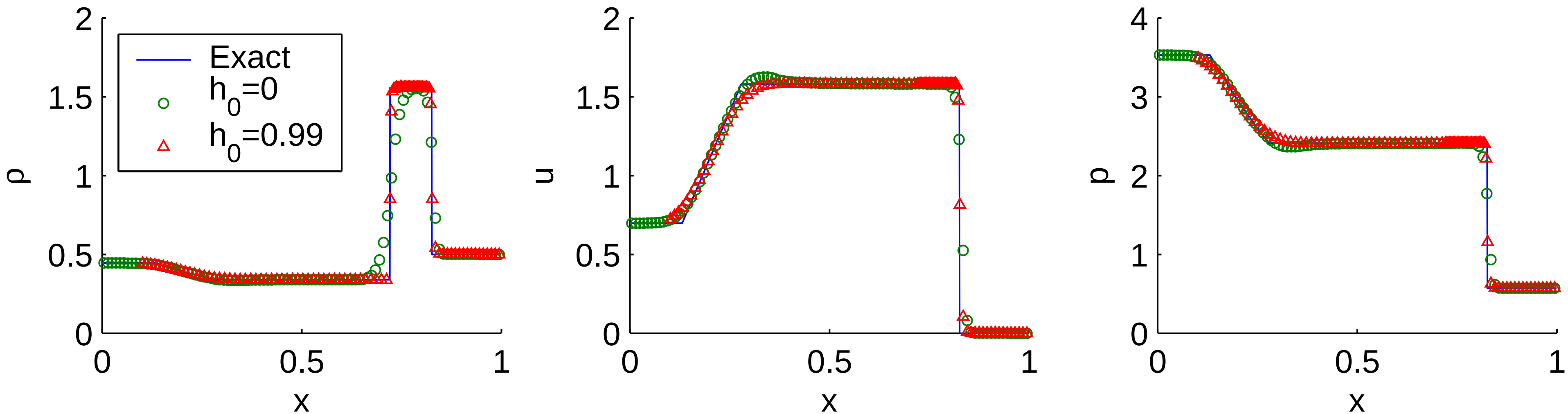
$$(\lambda_{1,1})_{i-1/2,jk} = \left[(1-h_0)\upsilon_m - \frac{M_L(p_m)}{\rho_{mL}(p_m)}\right] \left|\nabla_{\vec{X}}\xi_1\right|_{i-1/2,jk}$$

$$(\lambda_{1,2})_{i-1/2,jk} = (1-h_0)\upsilon_m \left|\nabla_{\vec{X}}\xi_1\right|_{i-1/2,jk}$$

$$(\lambda_{1,3})_{i-1/2,jk} = \left[(1-h_0)\upsilon_m + \frac{M_R(p_m)}{\rho_{mR}(p_m)}\right] \left|\nabla_{\vec{X}}\xi_1\right|_{i-1/2,jk}$$

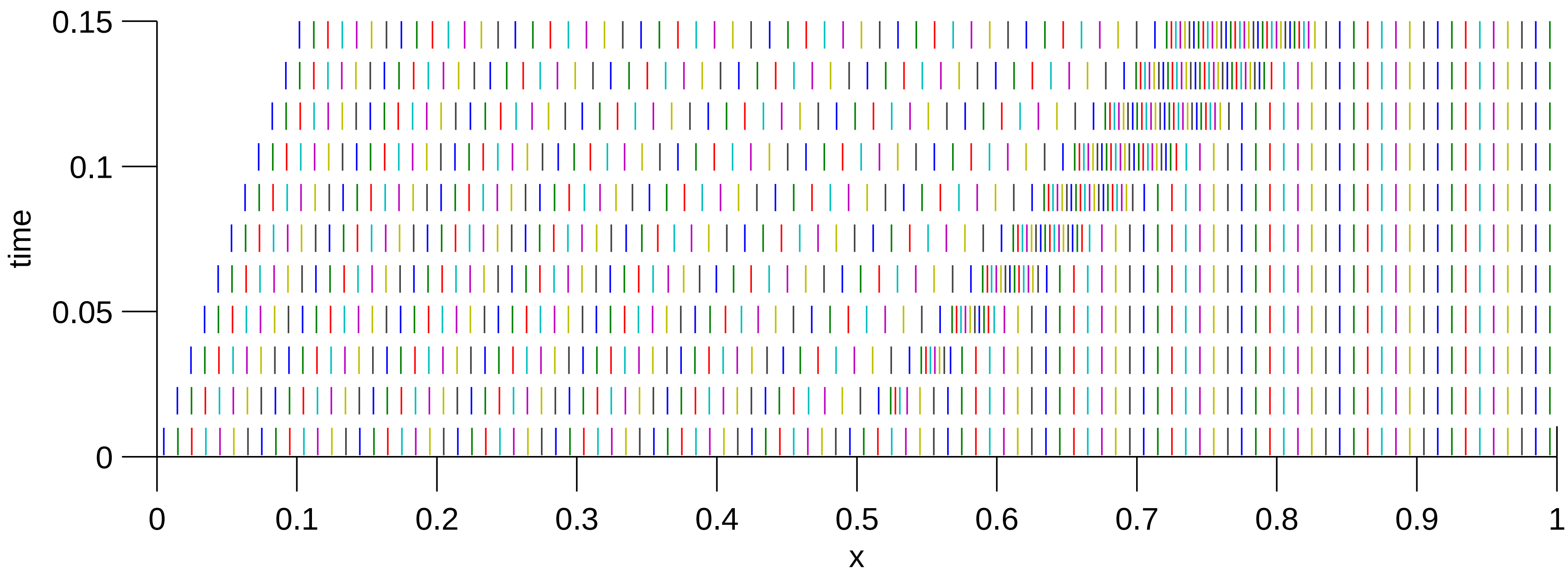
Lax's Riemann Problem

- Ideal gas EOS with $\gamma = 1.4$
- $h_0 = 0$ Eulerian result
- $h_0 = 0.99$ Lagrangian-like result
  - sharper resolution for contact discontinuity



Lax's Riemann Problem

- Physical grid coordinates at selected times
  - Each little dashed line gives a cell-center location of the proposed Lagrange-like grid system



Woodward-Colella's Problem

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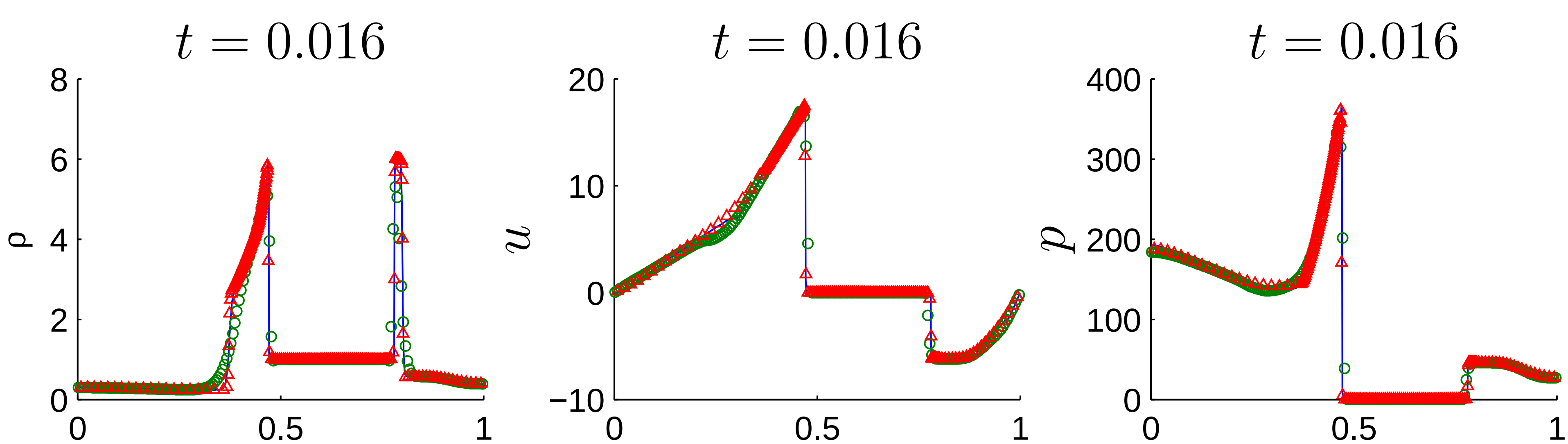

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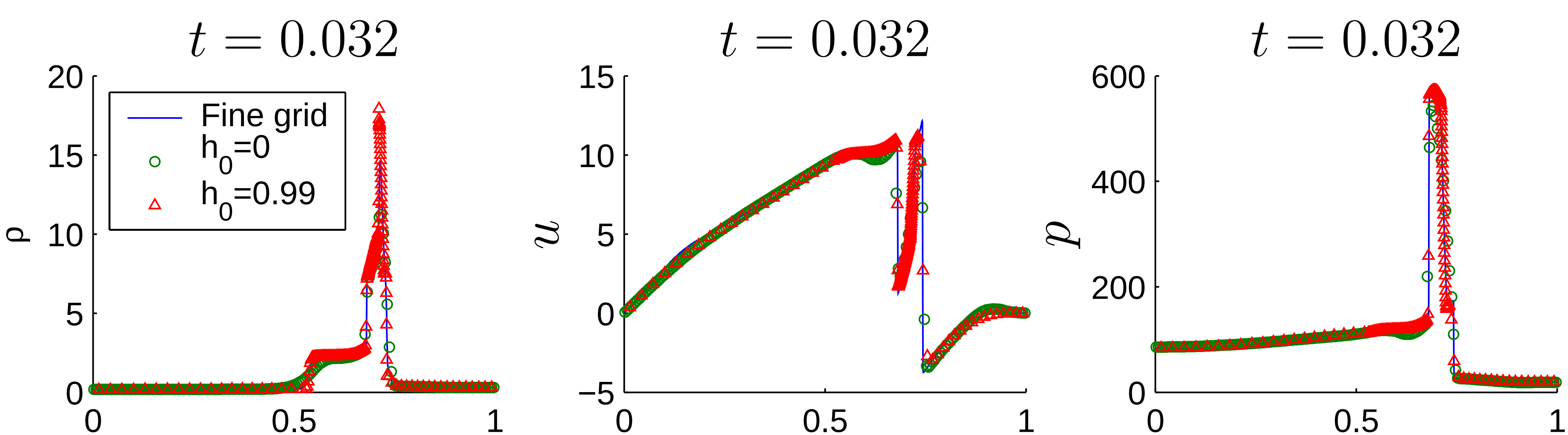

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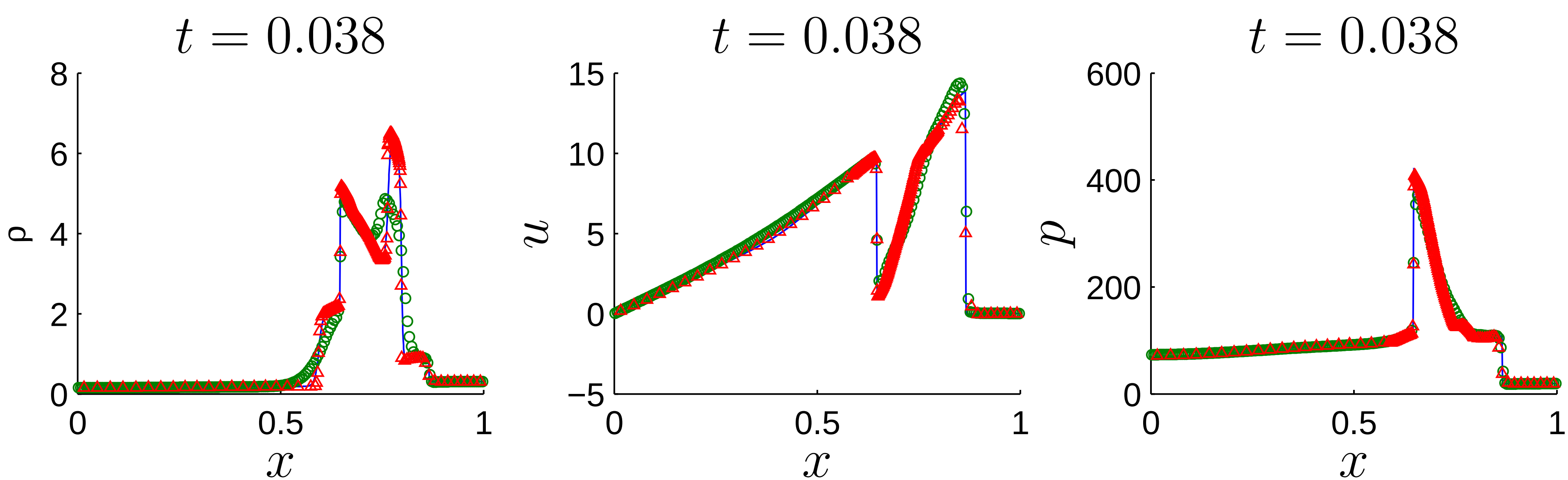

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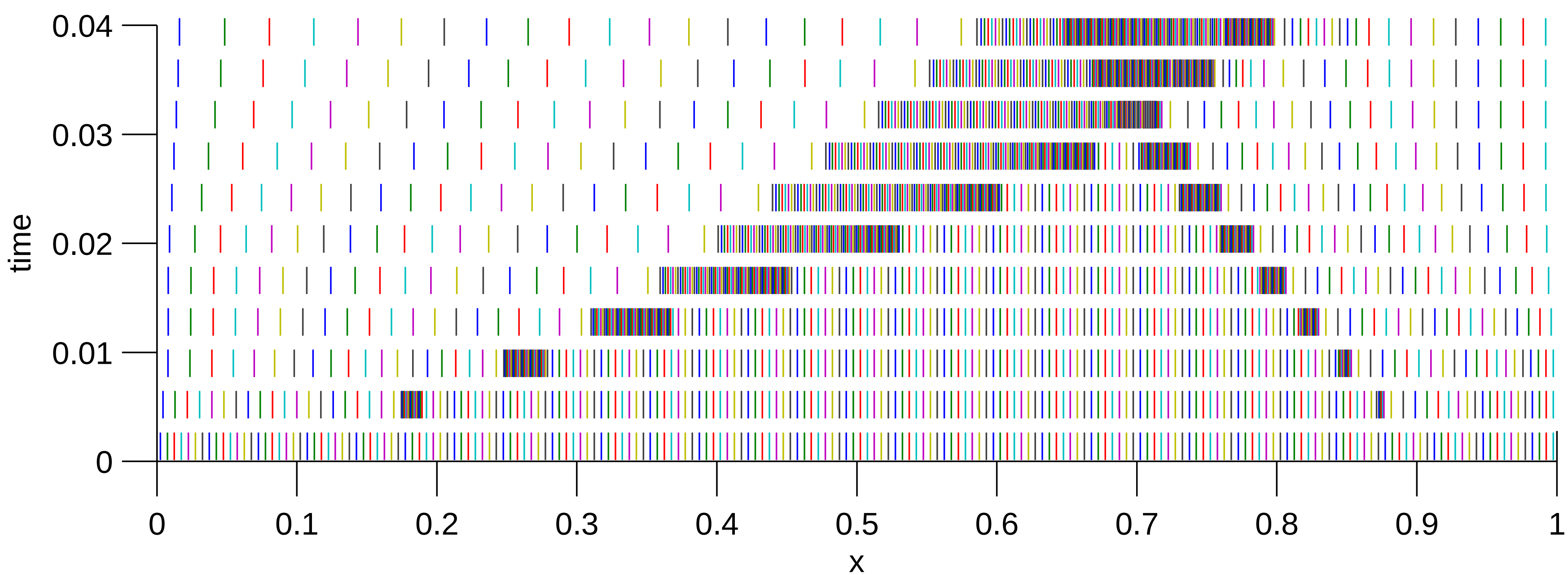
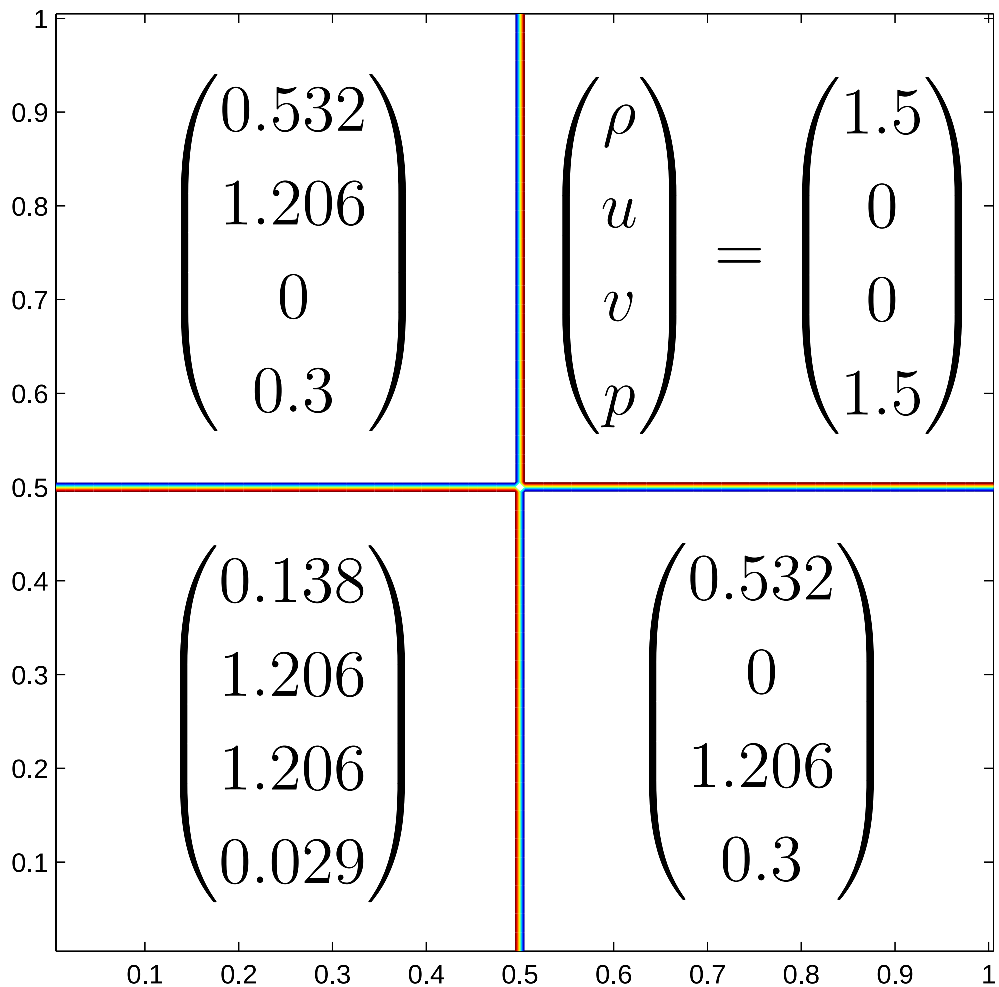


2D Riemann Problem

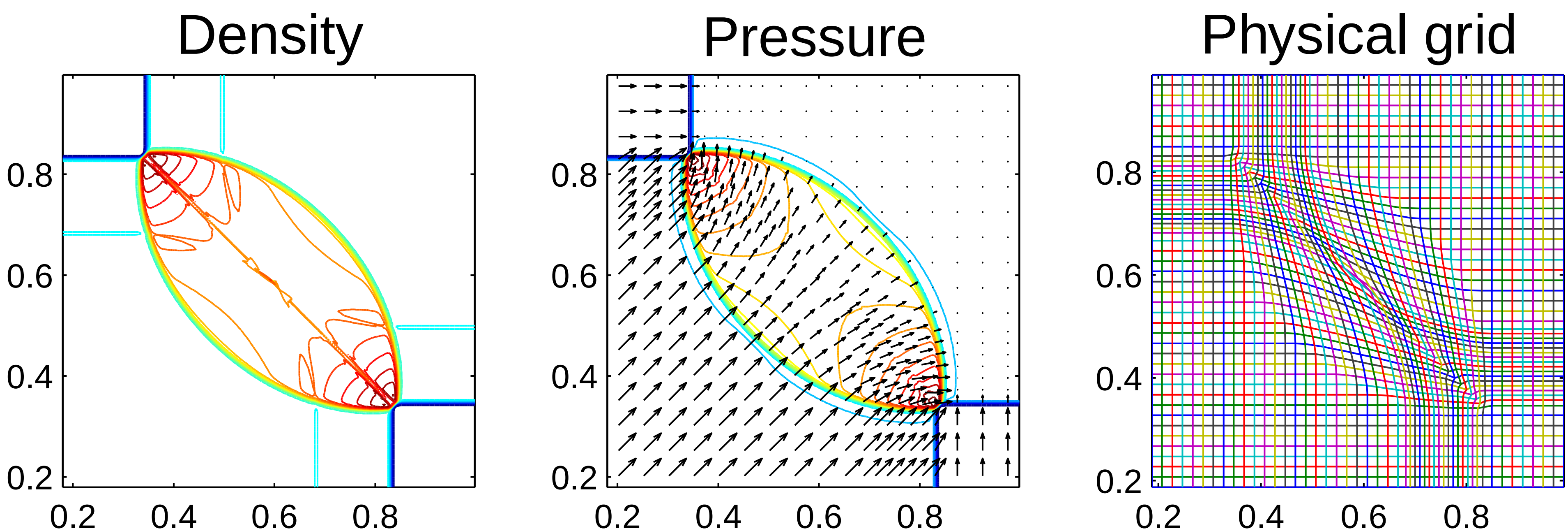
With initial 4-shock wave pattern



2D Riemann Problem

With initial 4-shock wave pattern

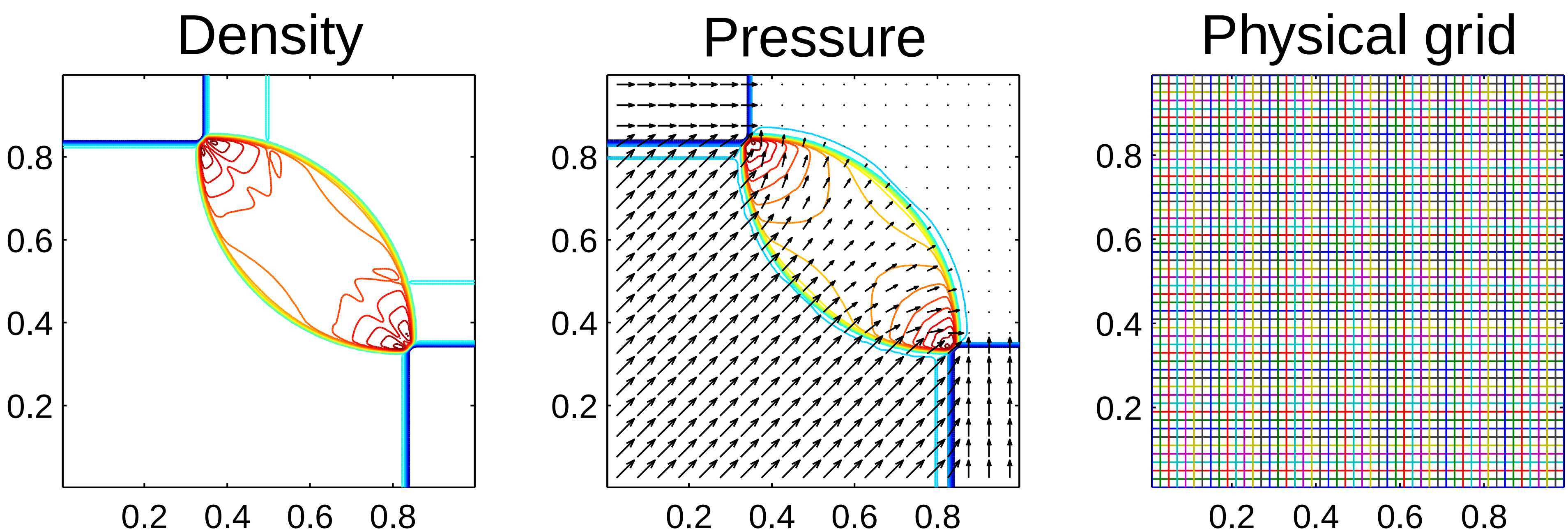
- Lagrangian-like result
  - Occurrence of simple Mach reflection



2D Riemann Problem

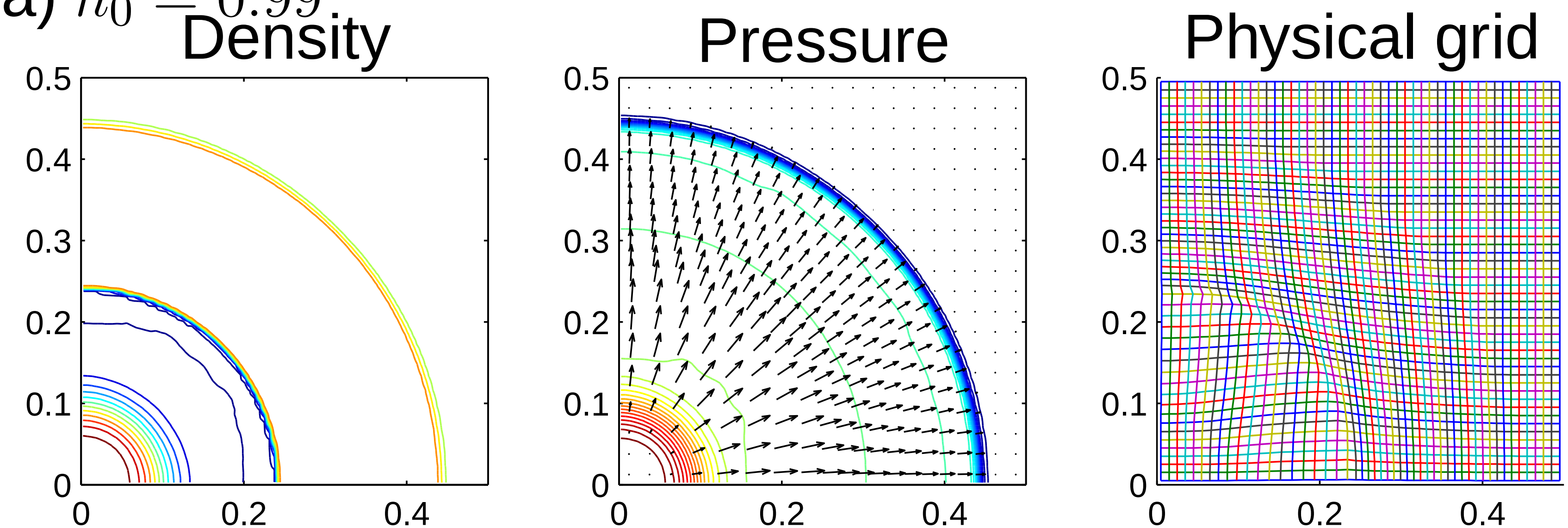
With initial 4-shock wave pattern

- Eulerian result
  - Poor resolution around simple Mach reflection

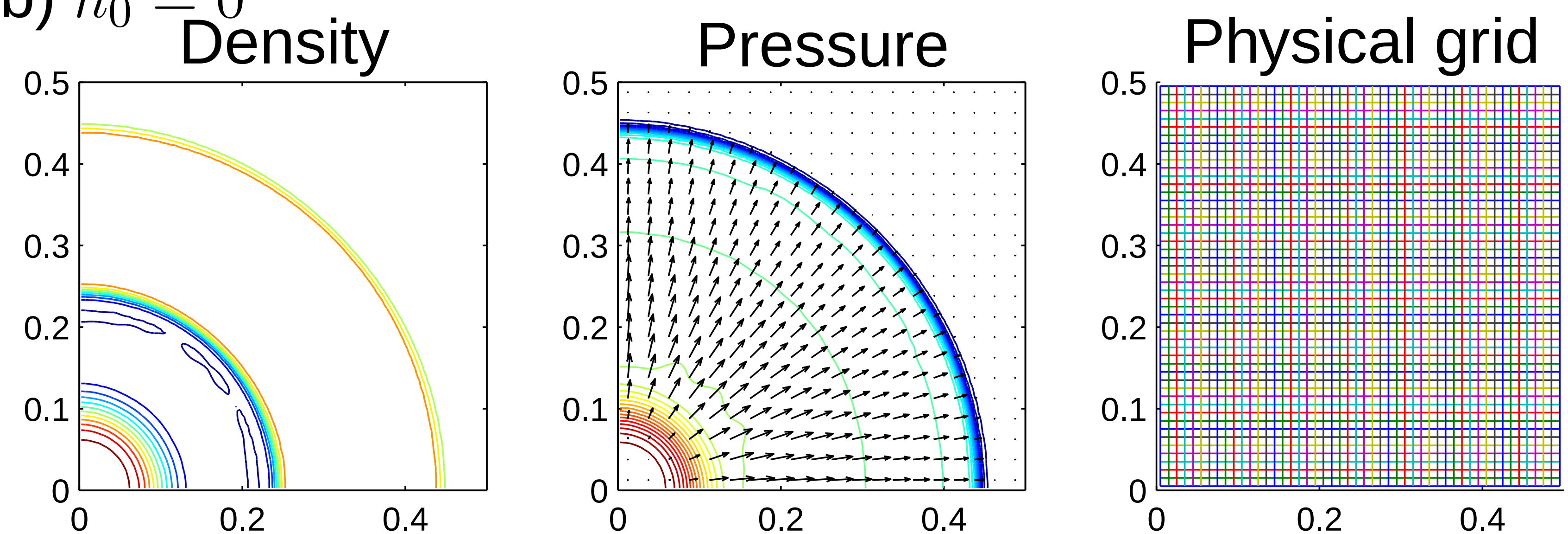


Radially Symmetric Problem

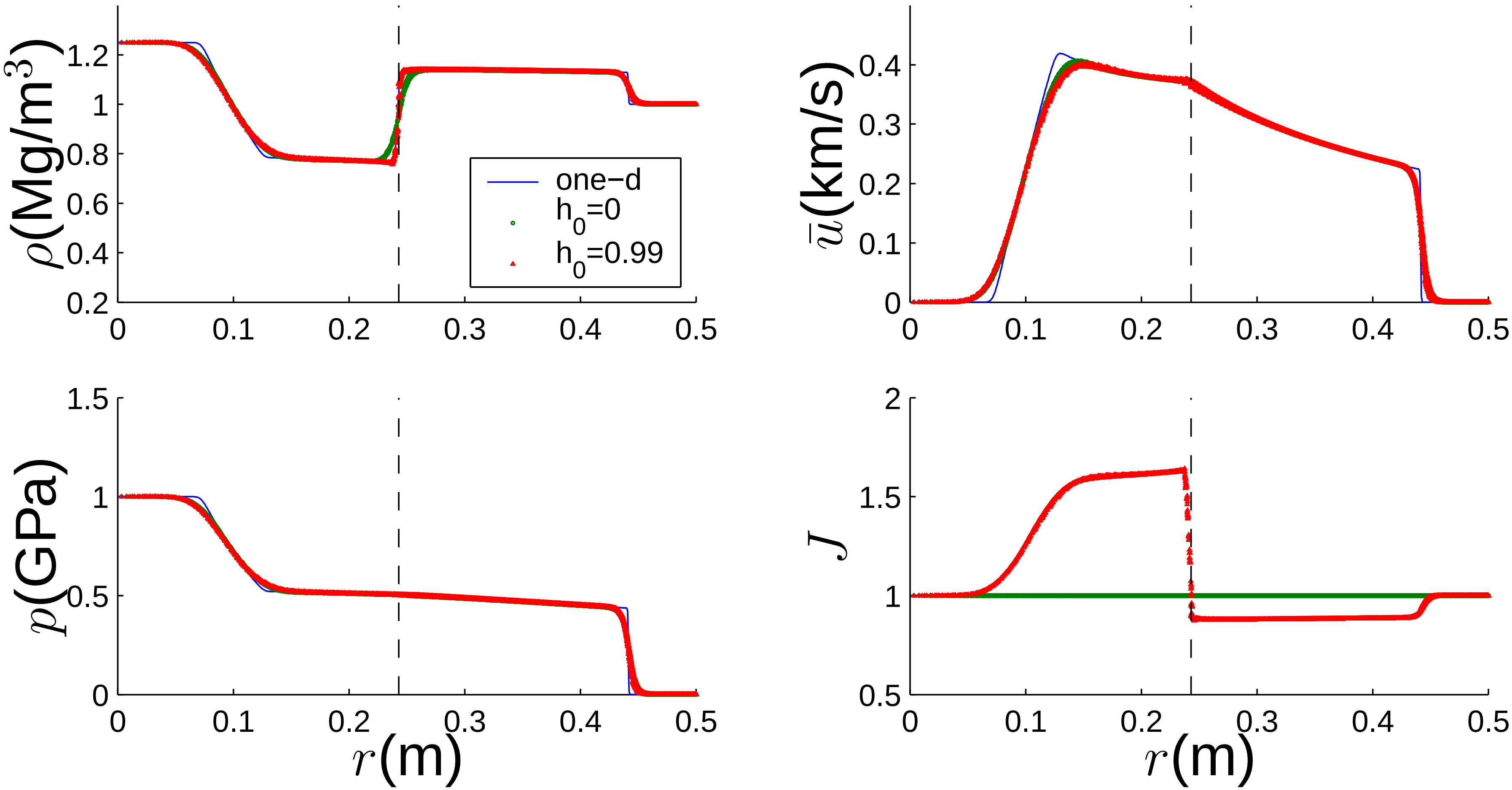
a) $h_0 = 0.99$



b) $h_0 = 0$

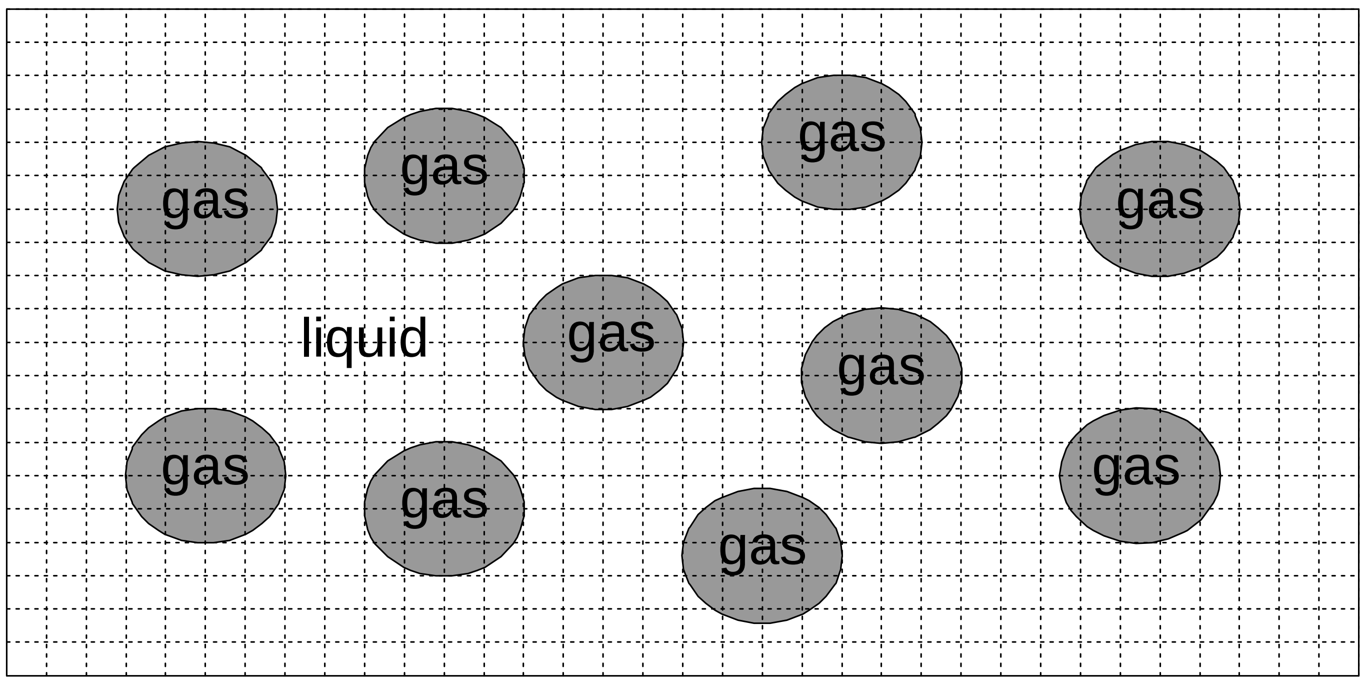


Radially Symmetric Prob. (Cont.)



Extension to Multifluid

- Assume homogeneous (1-pressure & 1-velocity) flow; *i.e.*, across interfaces $p_\iota = p$ & $\vec{u}_\iota = \vec{u}$, $\forall$ fluid phase ι



α-based Multifluid Model

- Physical balance laws

$$\frac{\partial}{\partial \tau}\begin{pmatrix} \rho J \\ \rho J u_i \\ J E \end{pmatrix} + \sum_{j=1}^{N} \frac{\partial}{\partial \xi_j} J \begin{pmatrix} \rho U_j \\ \rho u_i U_j + p \frac{\partial \xi_j}{\partial x_i} \\ E U_j + p U_j - p \frac{\partial \xi_j}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 \\ -\rho J \frac{\partial \phi}{\partial x_i} \\ -\rho J \vec{u} \cdot \nabla \phi \end{pmatrix}$$

- Geometrical conservation laws

$$\frac{\partial}{\partial \tau}\left(\frac{\partial x_i}{\partial \xi_j}\right) + \frac{\partial}{\partial \xi_j}\left(-\frac{\partial x_i}{\partial \tau}\right) = 0 \qquad \text{for } i, j = 1, 2, \ldots, N.$$

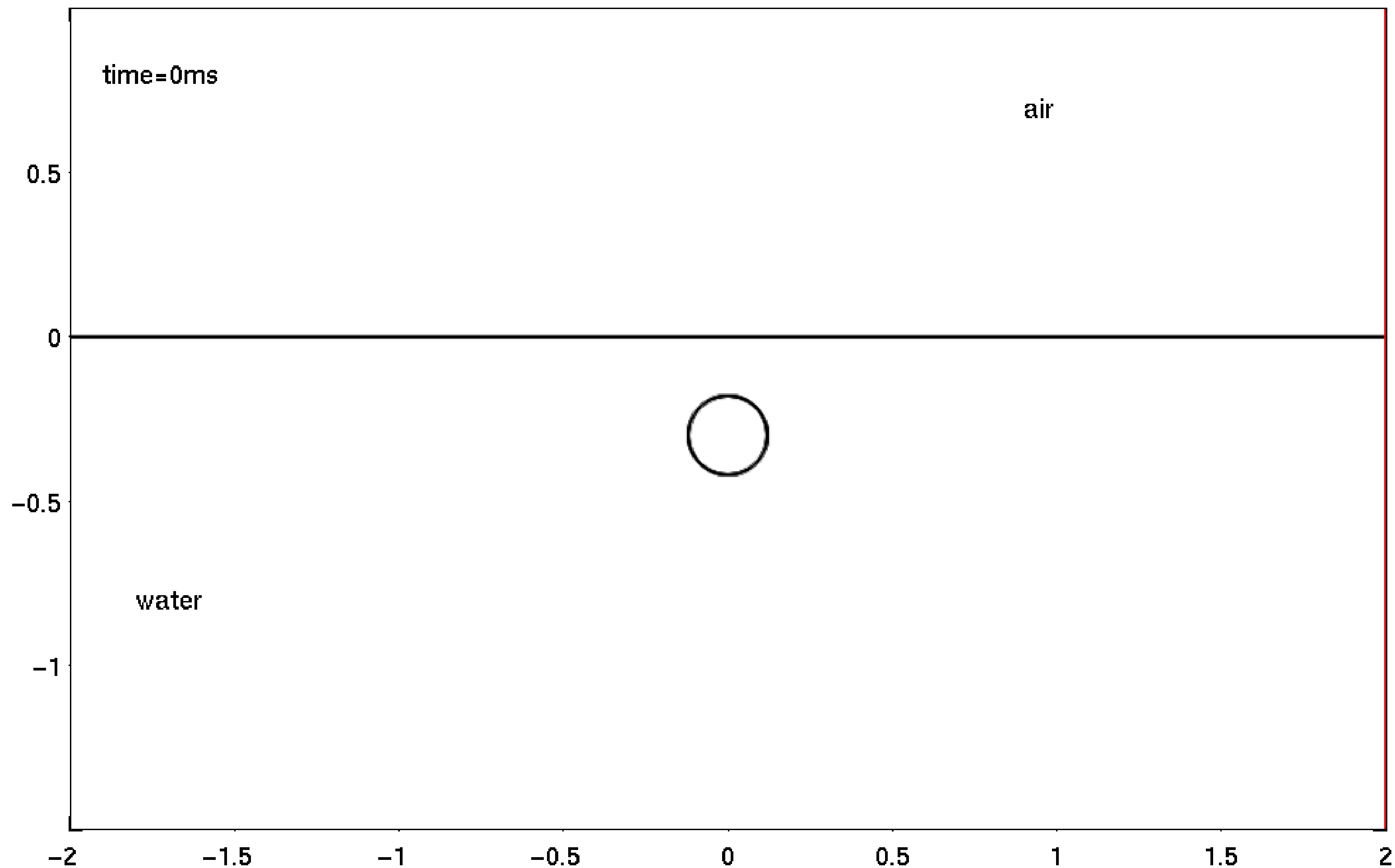
- Volume fraction transport equation

$$\frac{\partial \alpha}{\partial \tau} + \sum_{j=1}^{N} U_j \frac{\partial \alpha}{\partial \xi_j} = 0$$

- Moving grid condition $\partial_\tau \vec{x} = h_0 \vec{u}$ & pressure law $p(\rho, e, \alpha)$

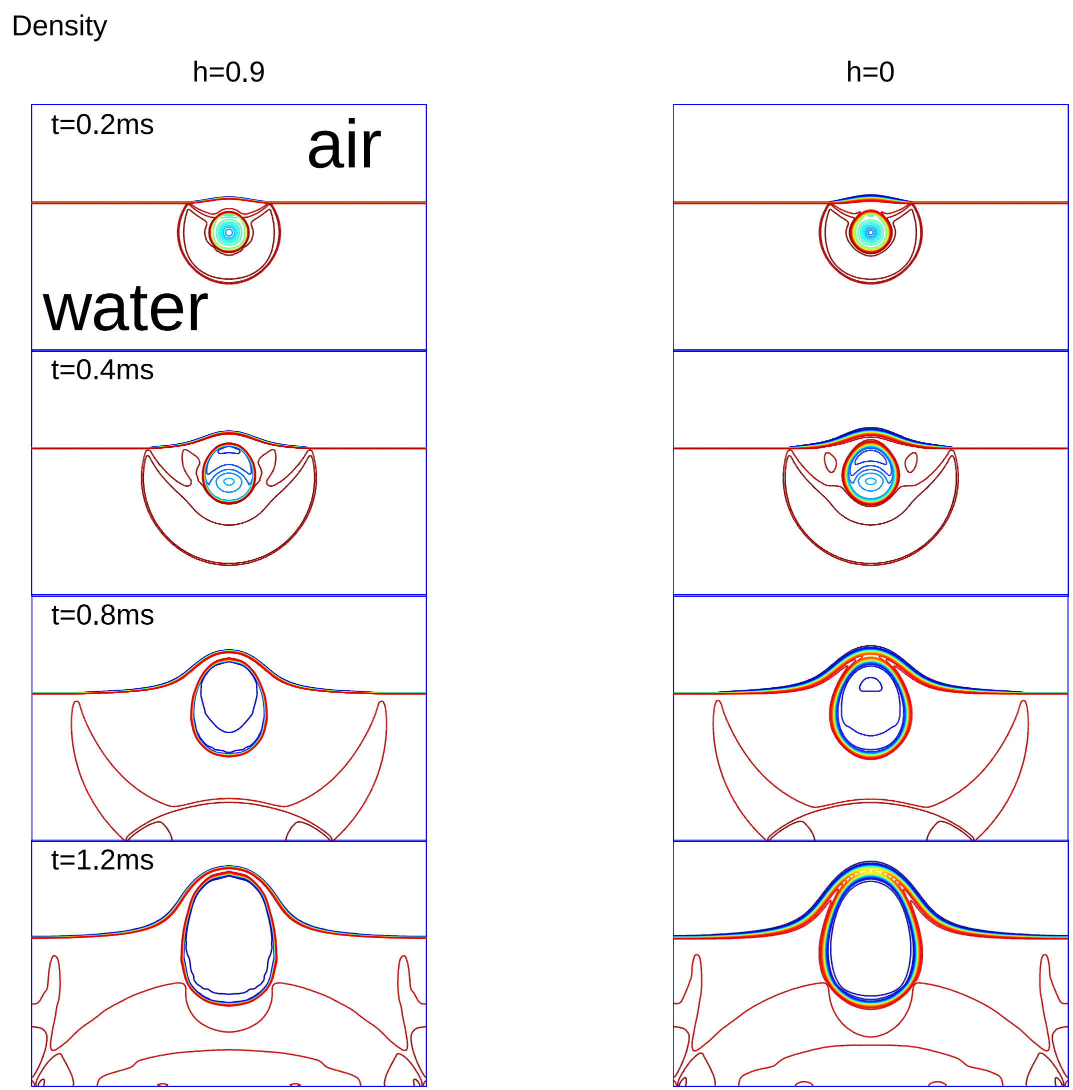
Underwater Explosions

- Solution with $h_0 = 0.9$, 800×500 grid



Underwater Explosions

- Solution Comparison between $h_0 = 0.9$ & $h_0 = 0$



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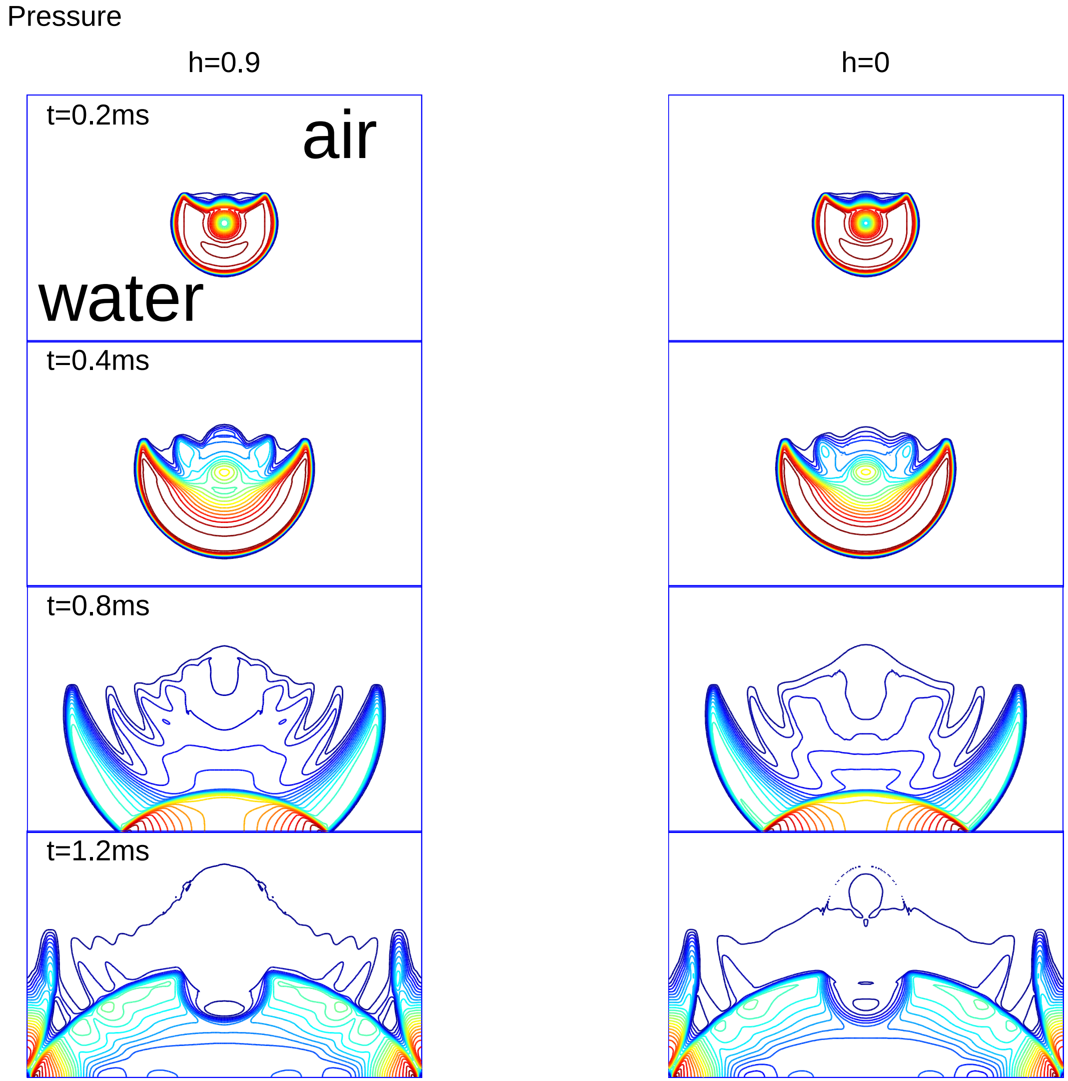
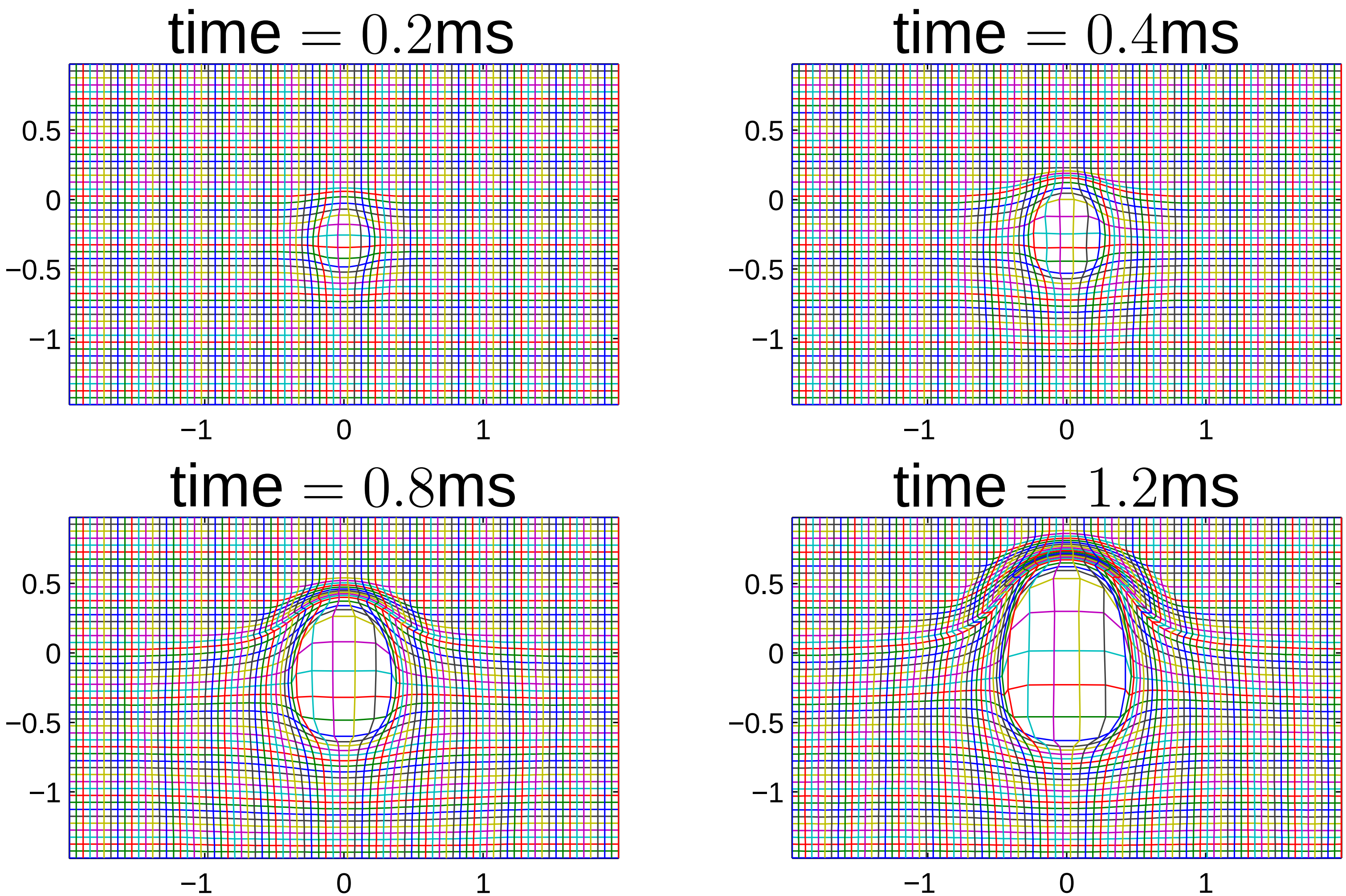


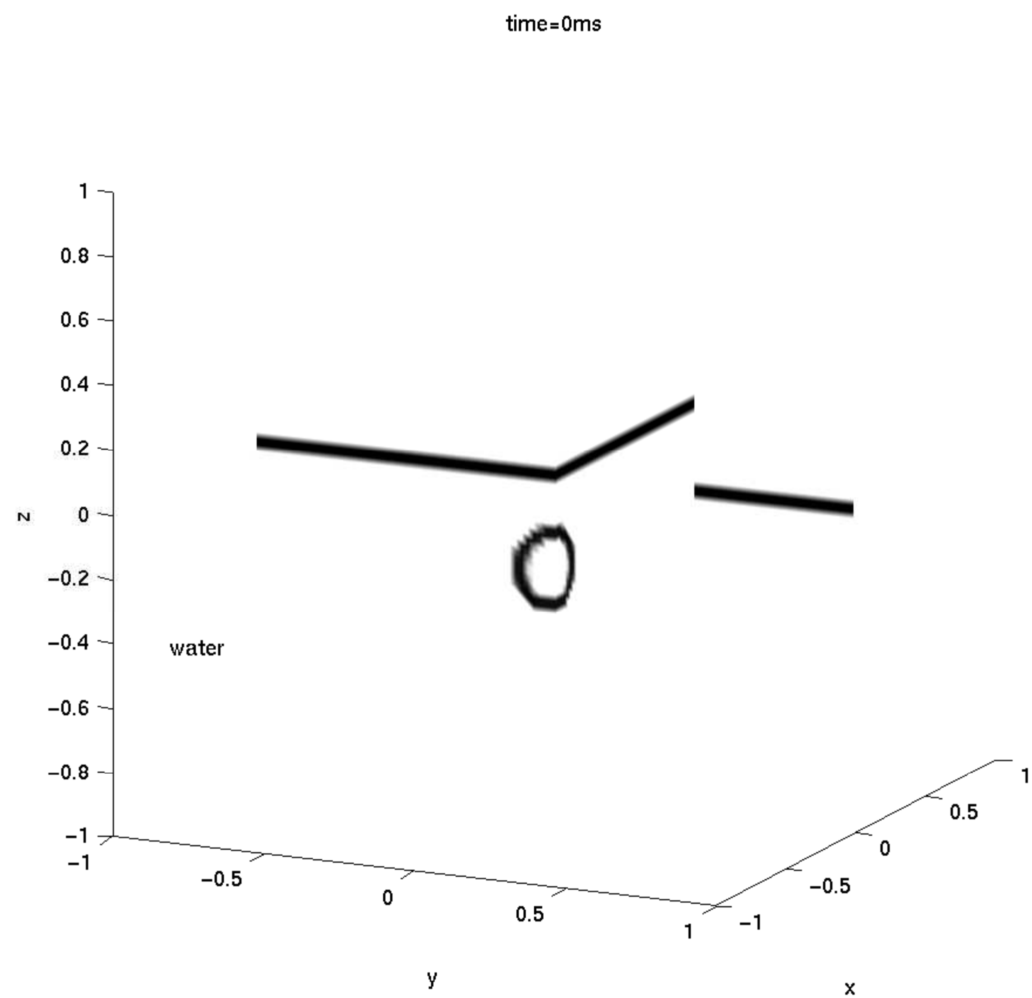
Underwater Explosions (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.9$



3D Underwater Explosions

- Numerical schlieren images $h_0 = 0.6$, 100^3 grid



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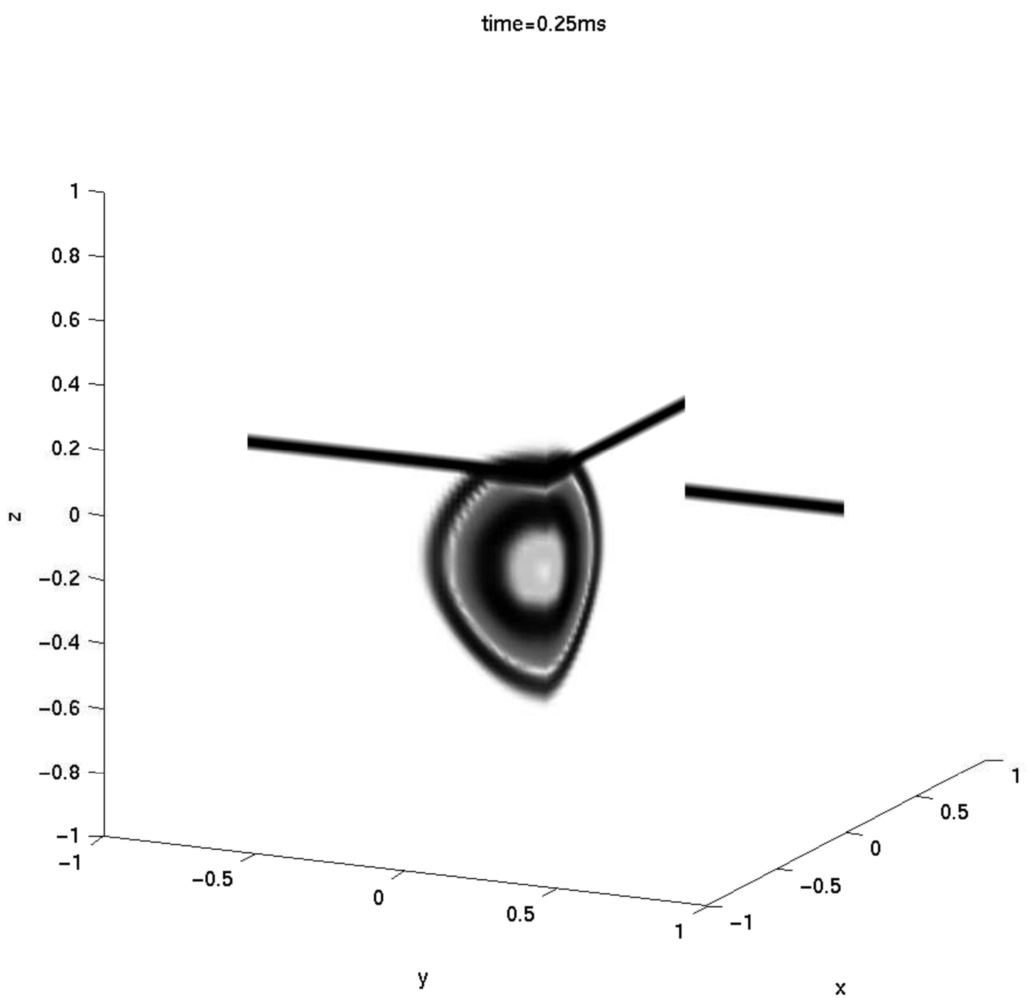

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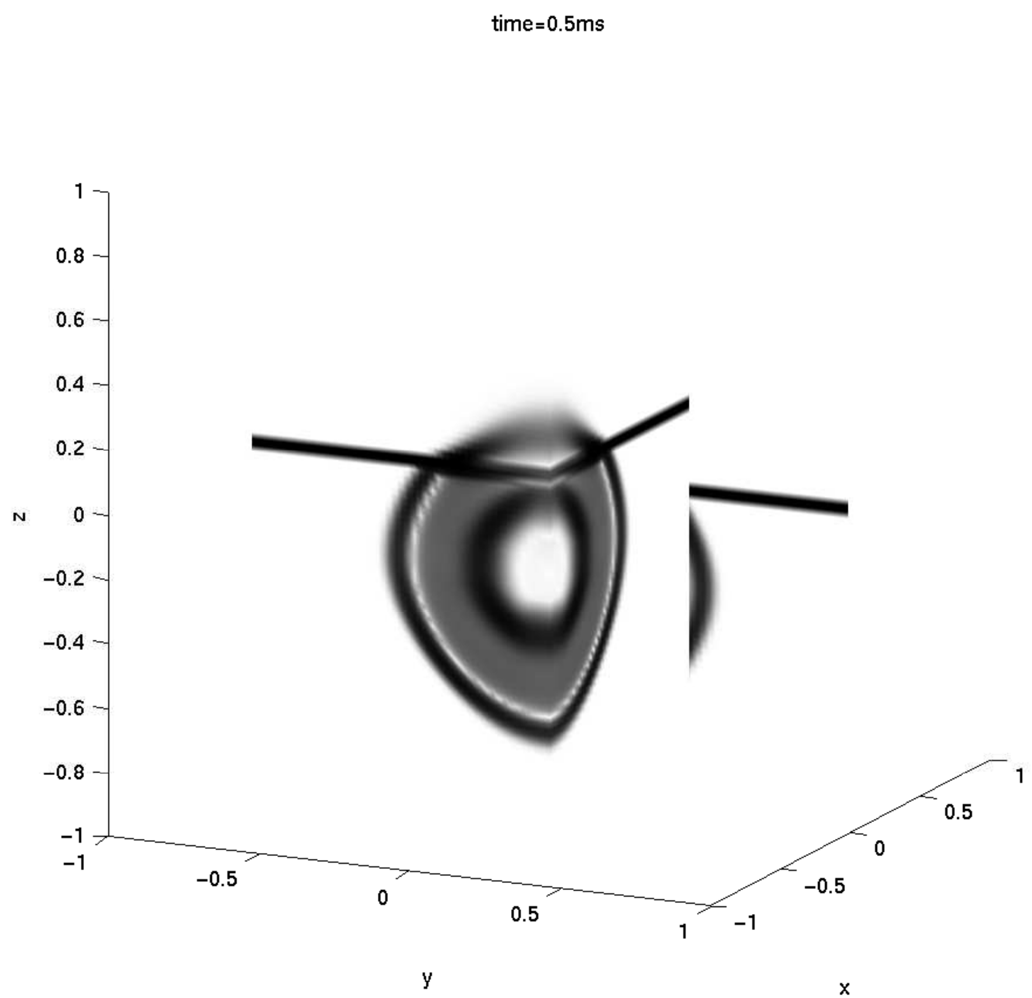

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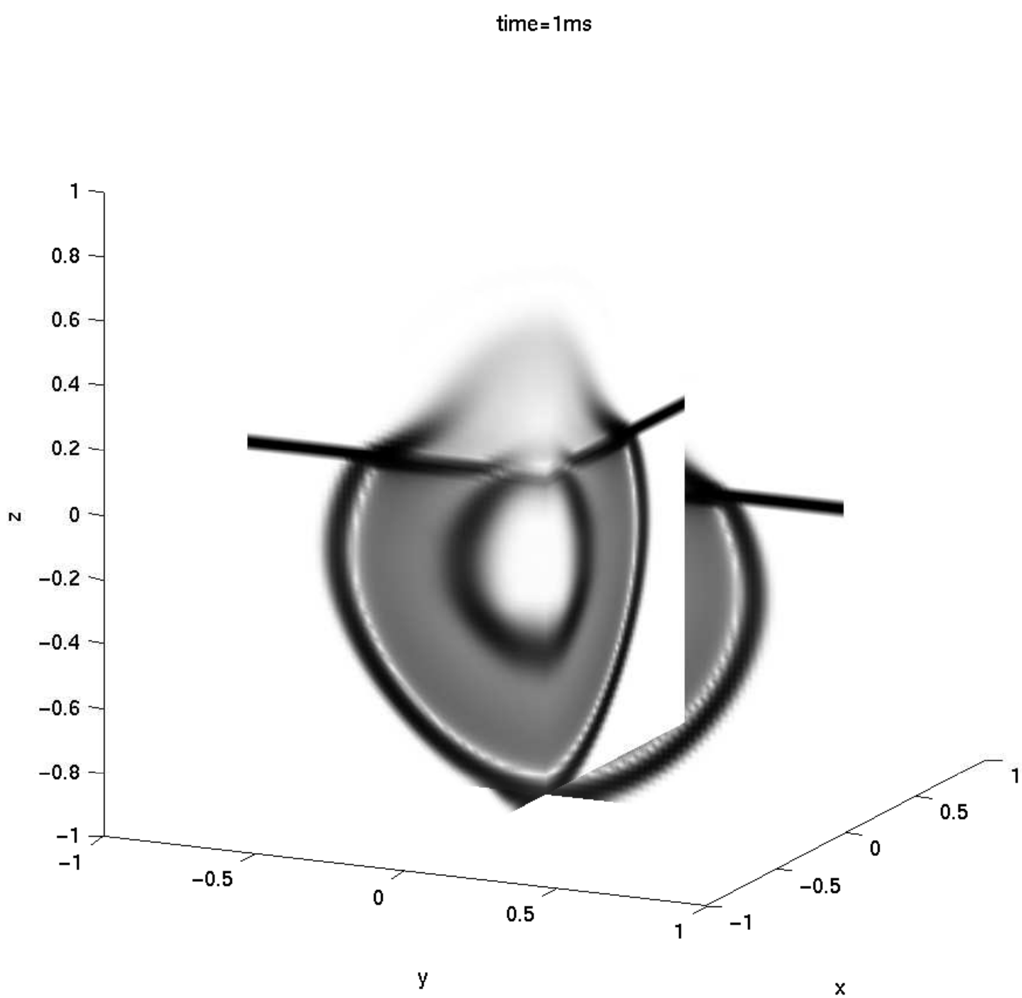

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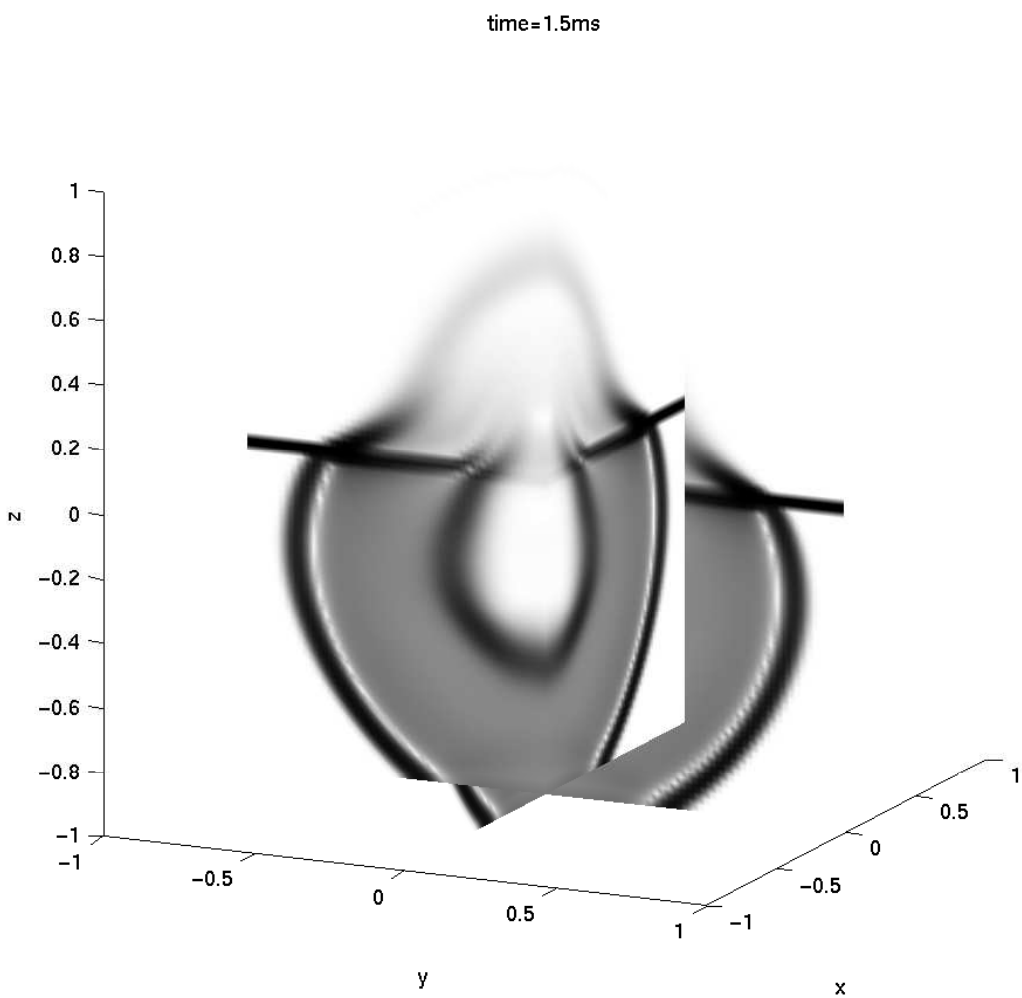
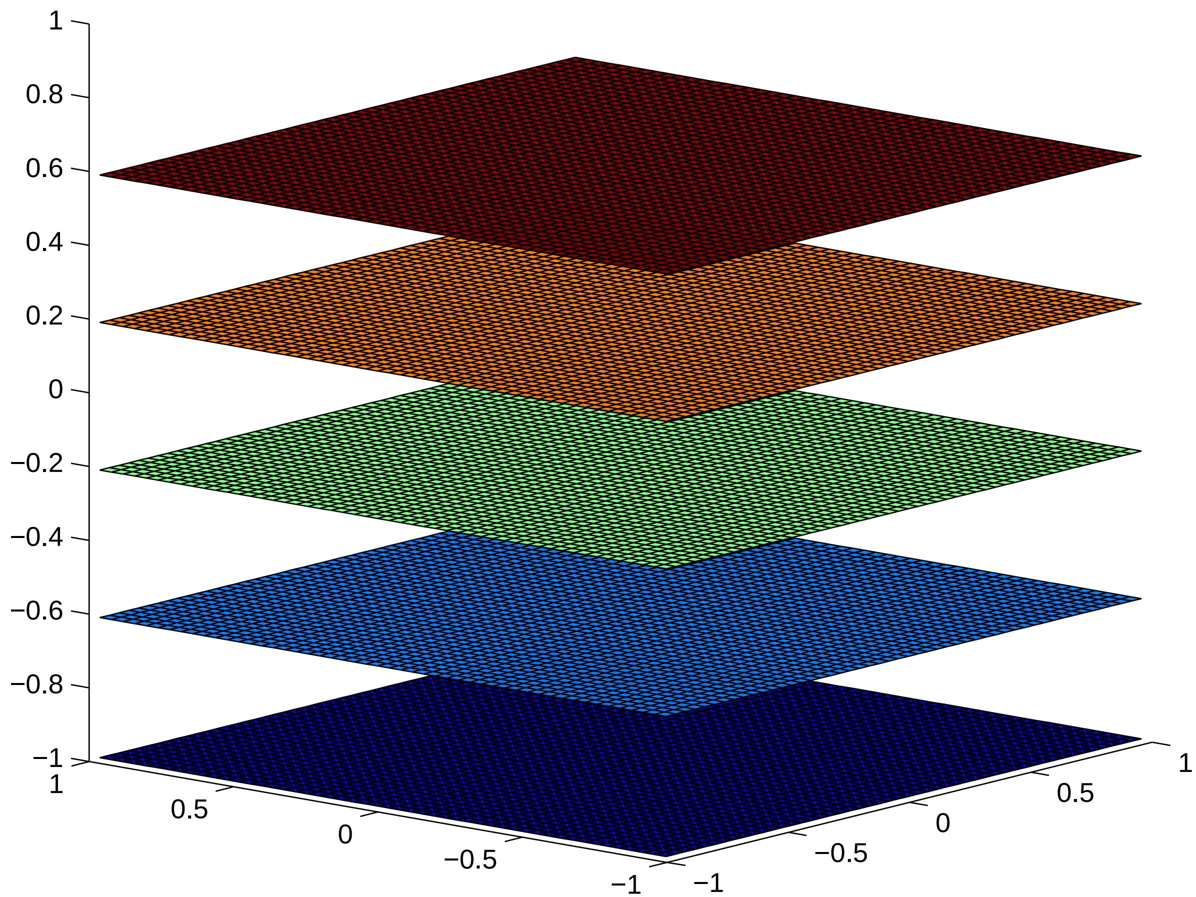


3D Underwater Explosions (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$

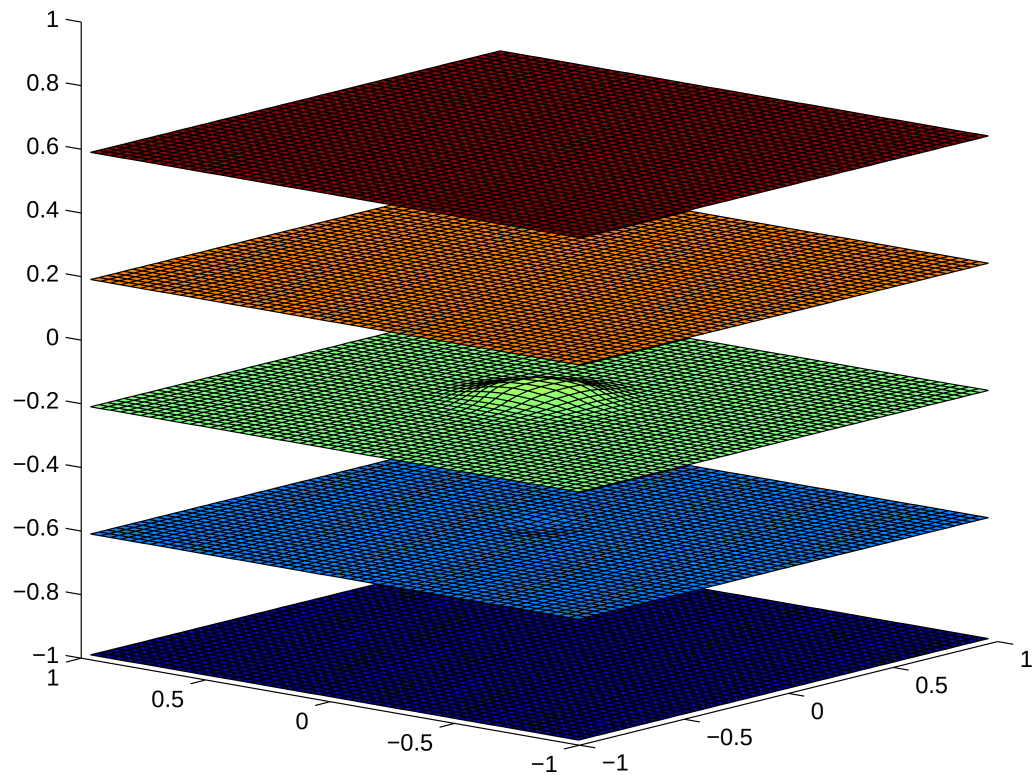
time $= 0$



3D Underwater Explosions (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$

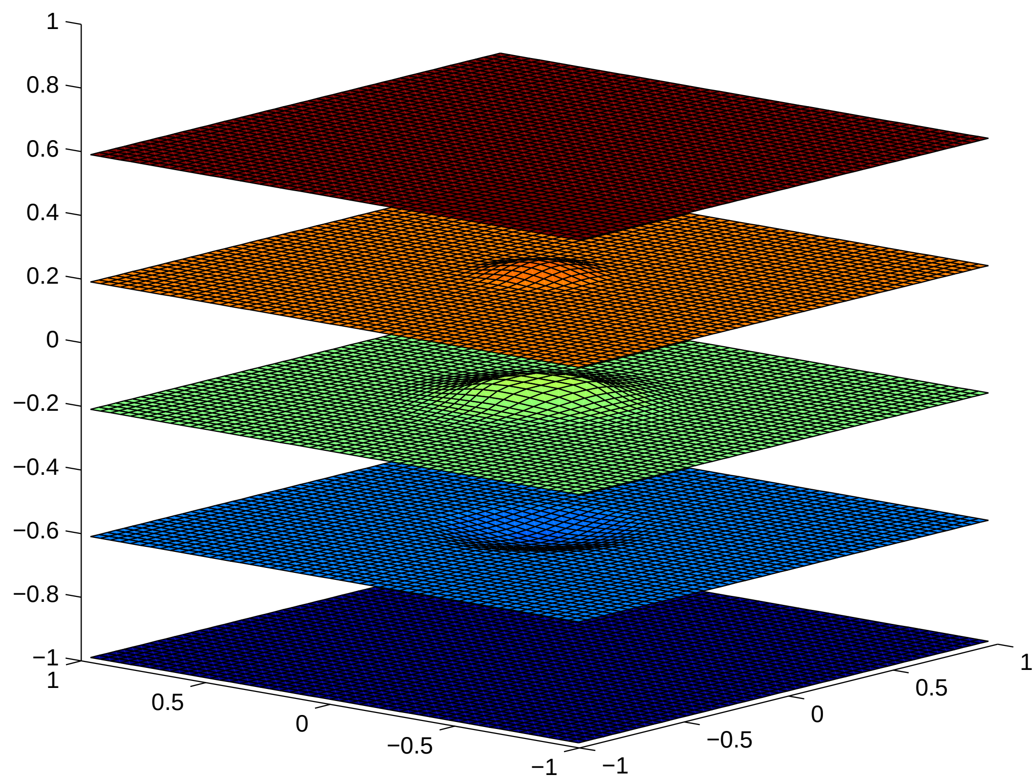
time $= 0.25$ms



3D Underwater Explosions (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$

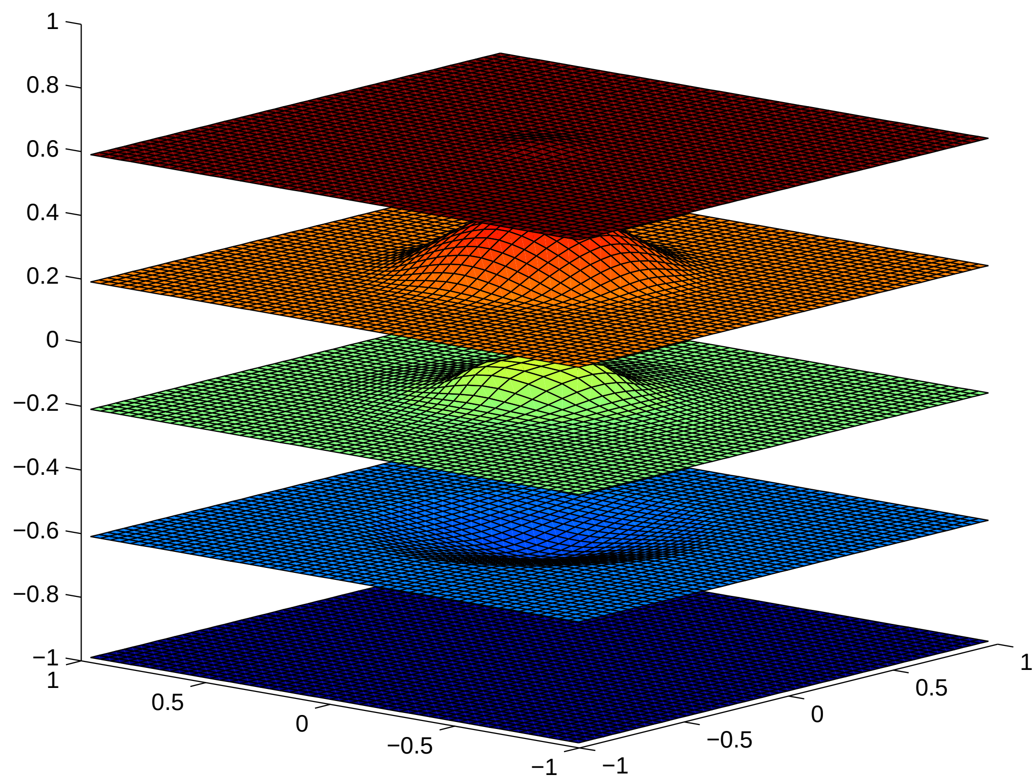
time $= 0.5$ms



3D Underwater Explosions (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$

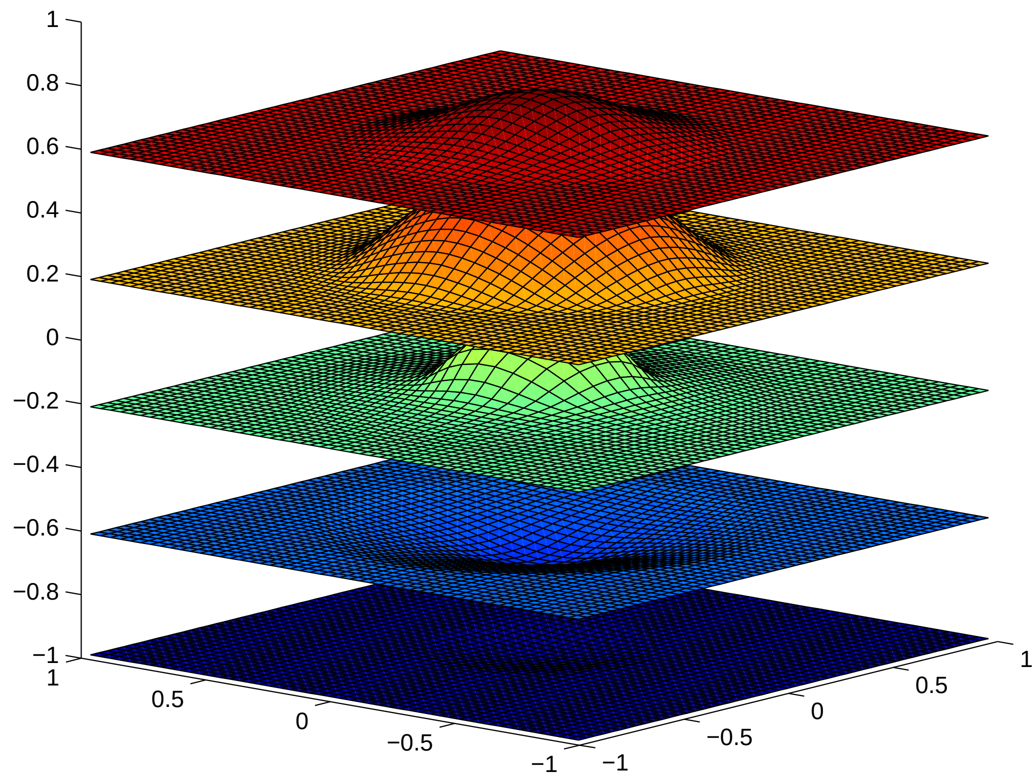
time $= 1.0$ms



3D Underwater Explosions (Cont.)

- Grid system (coarsen by factor 2) with $h_0 = 0.6$

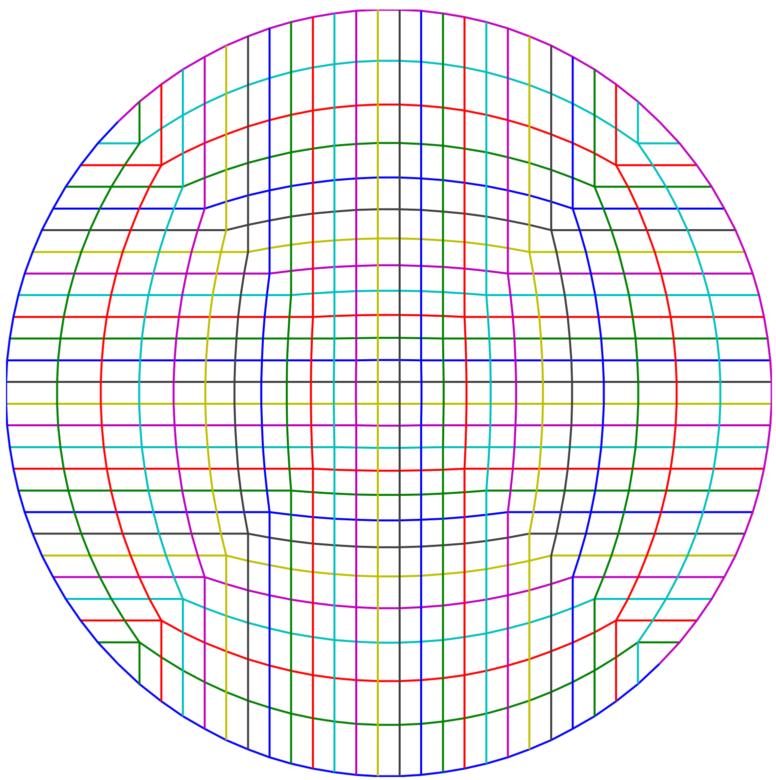
time $= 1.5$ms



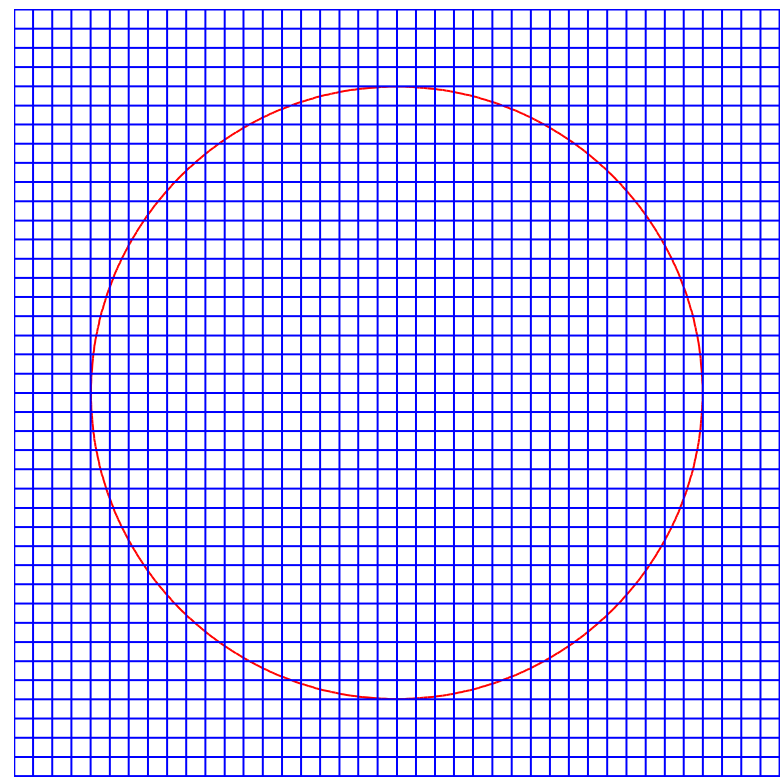
Blast Wave Computation

- Two sample grid systems used in computation with $h_0 = 0$

mapped grid



Cartesian grid



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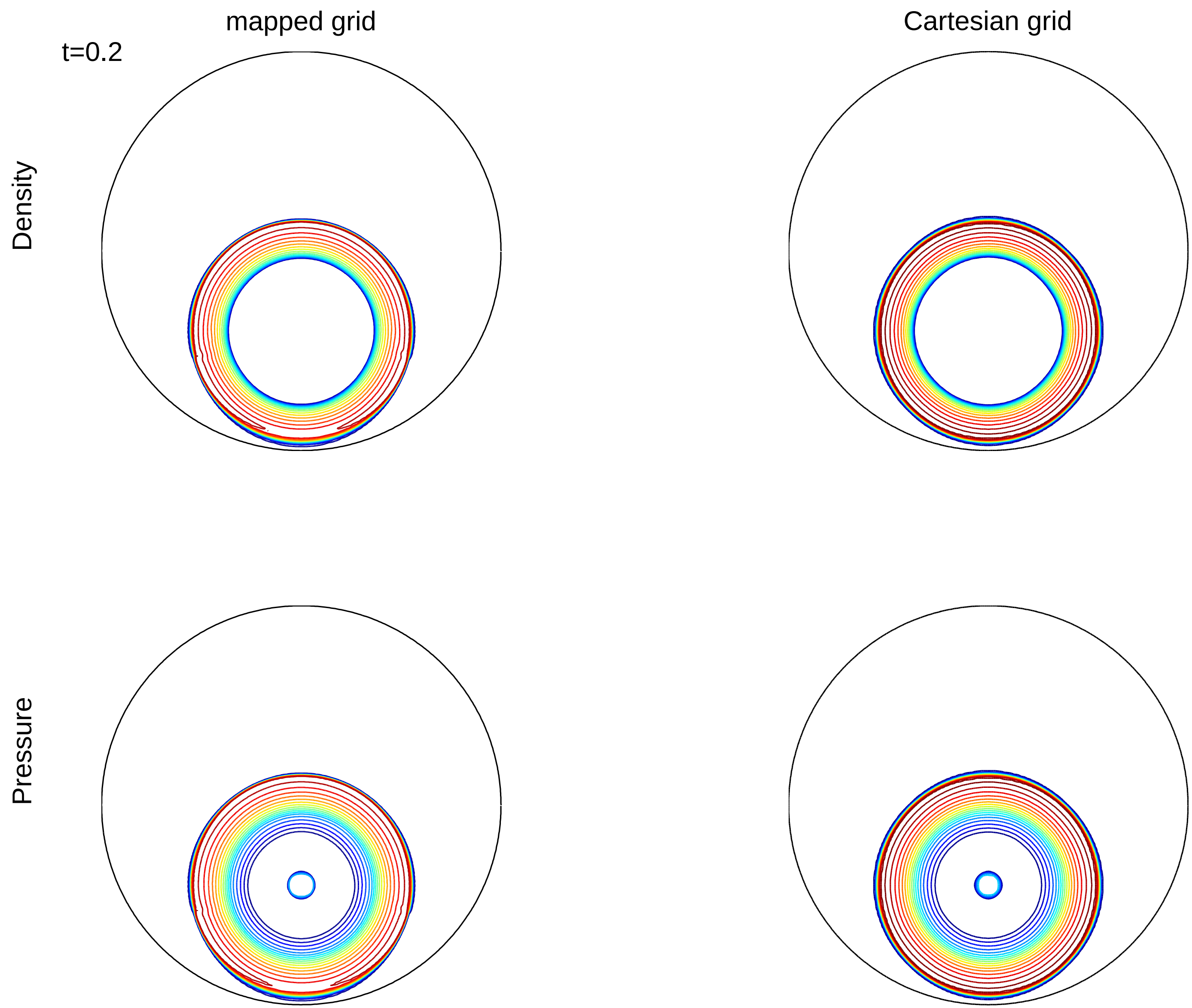
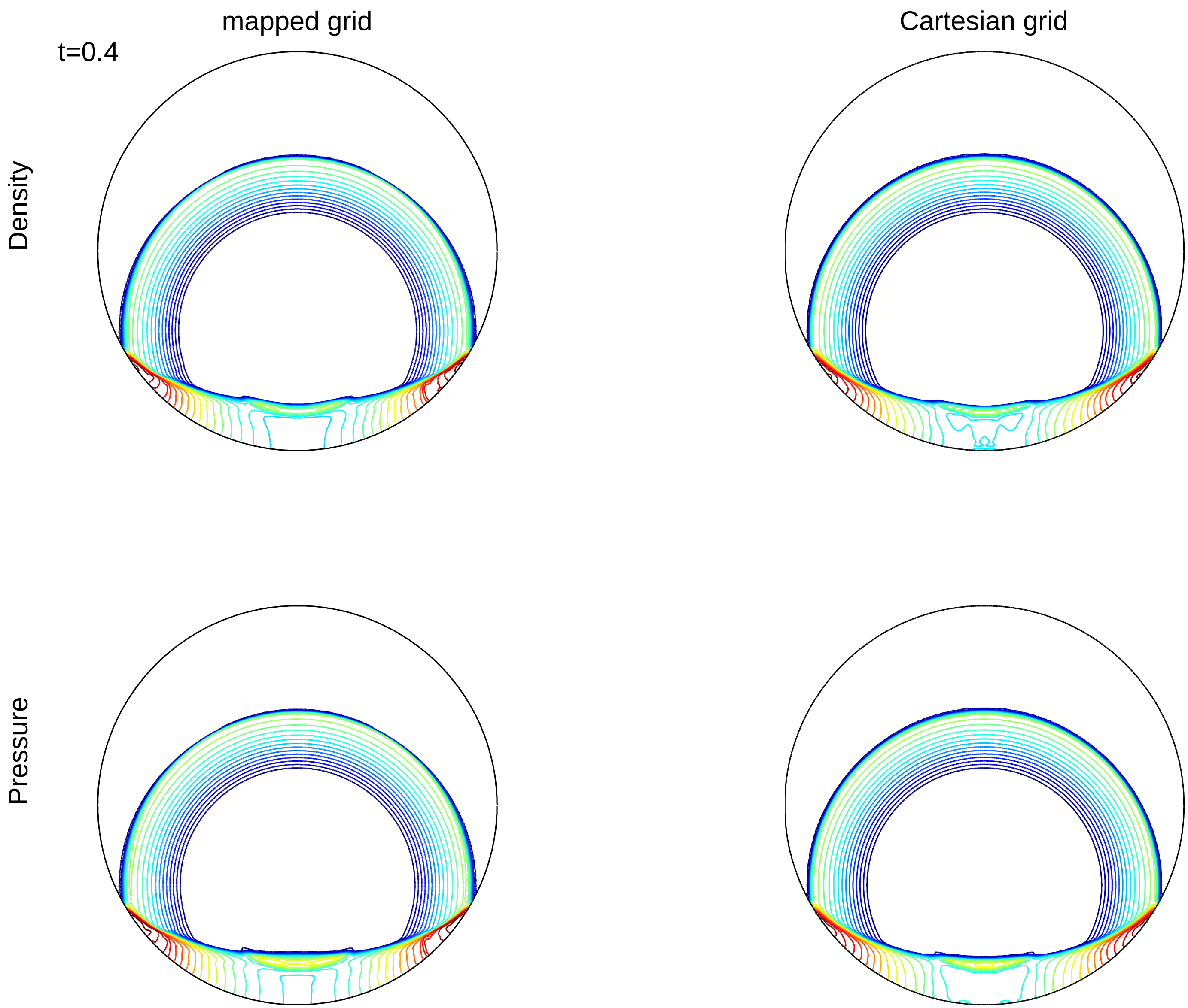


Blast Wave Computation



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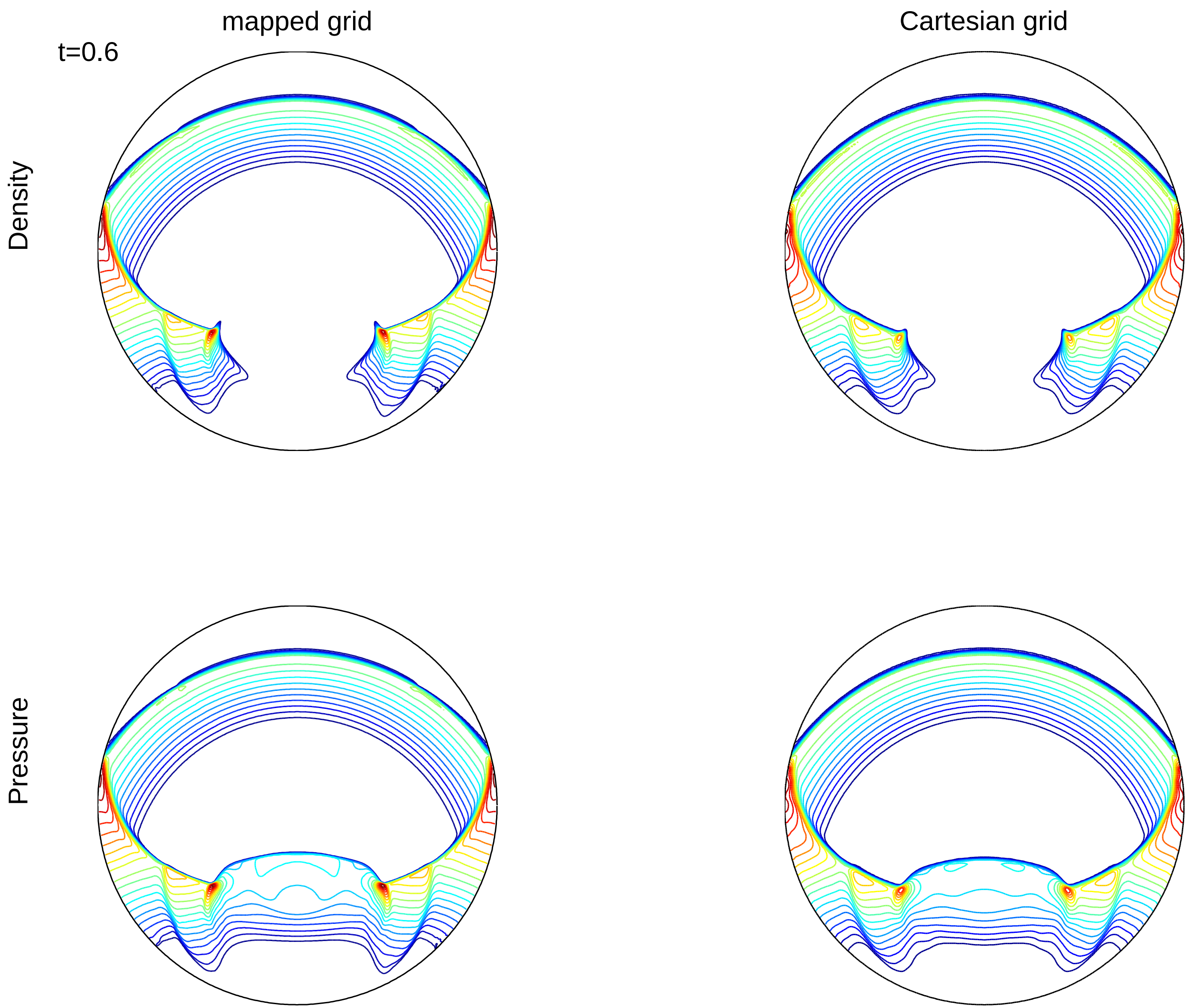

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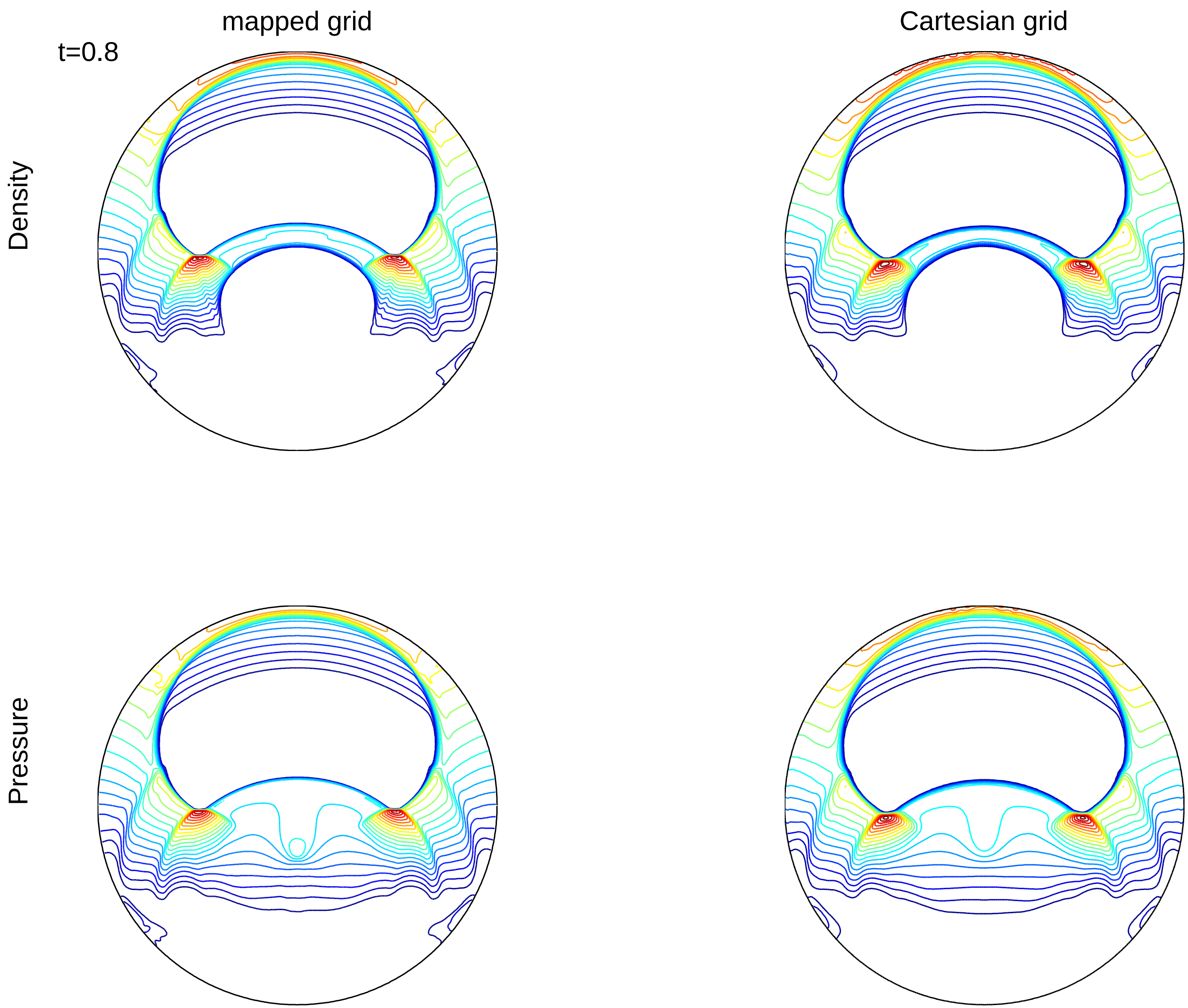

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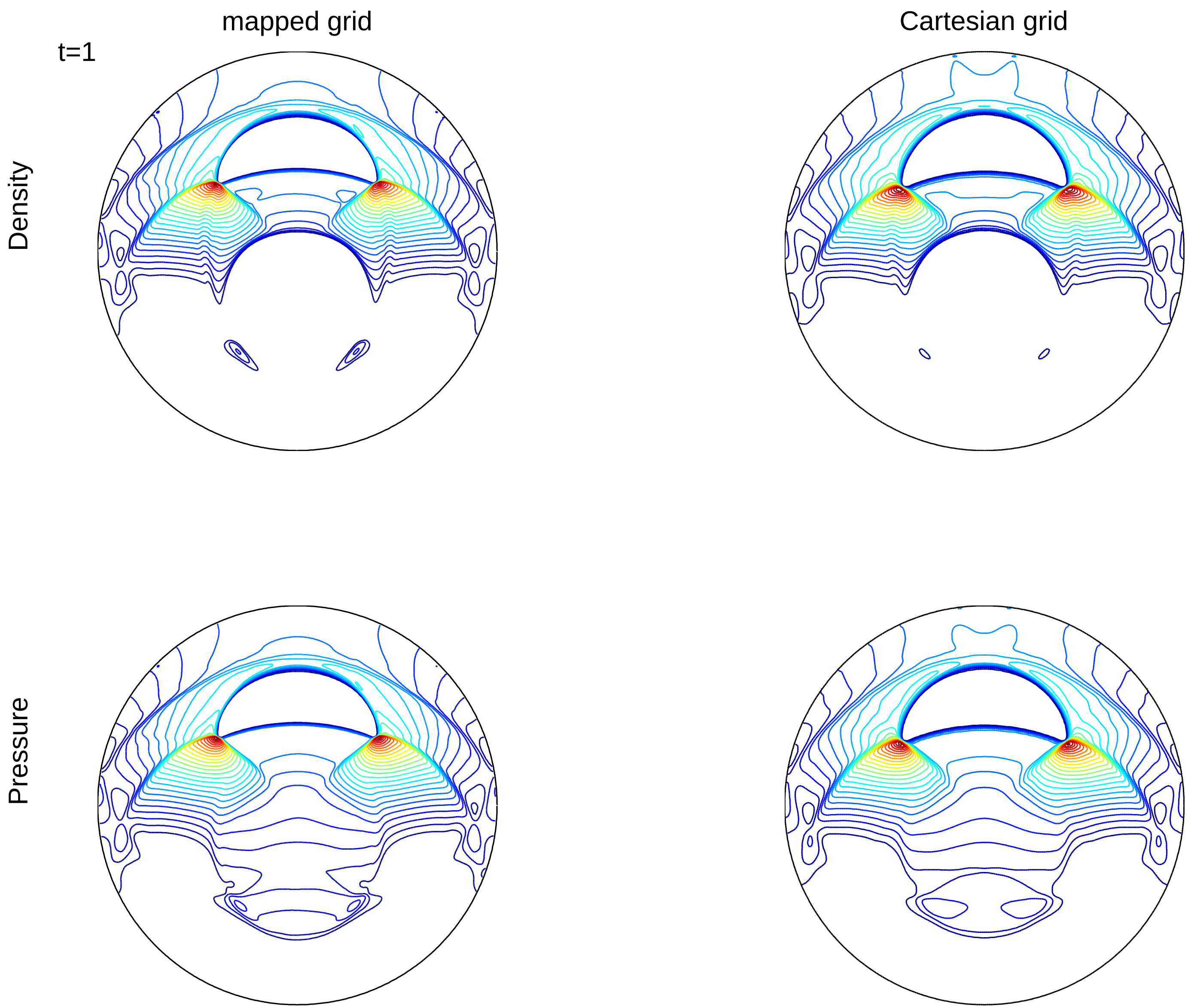

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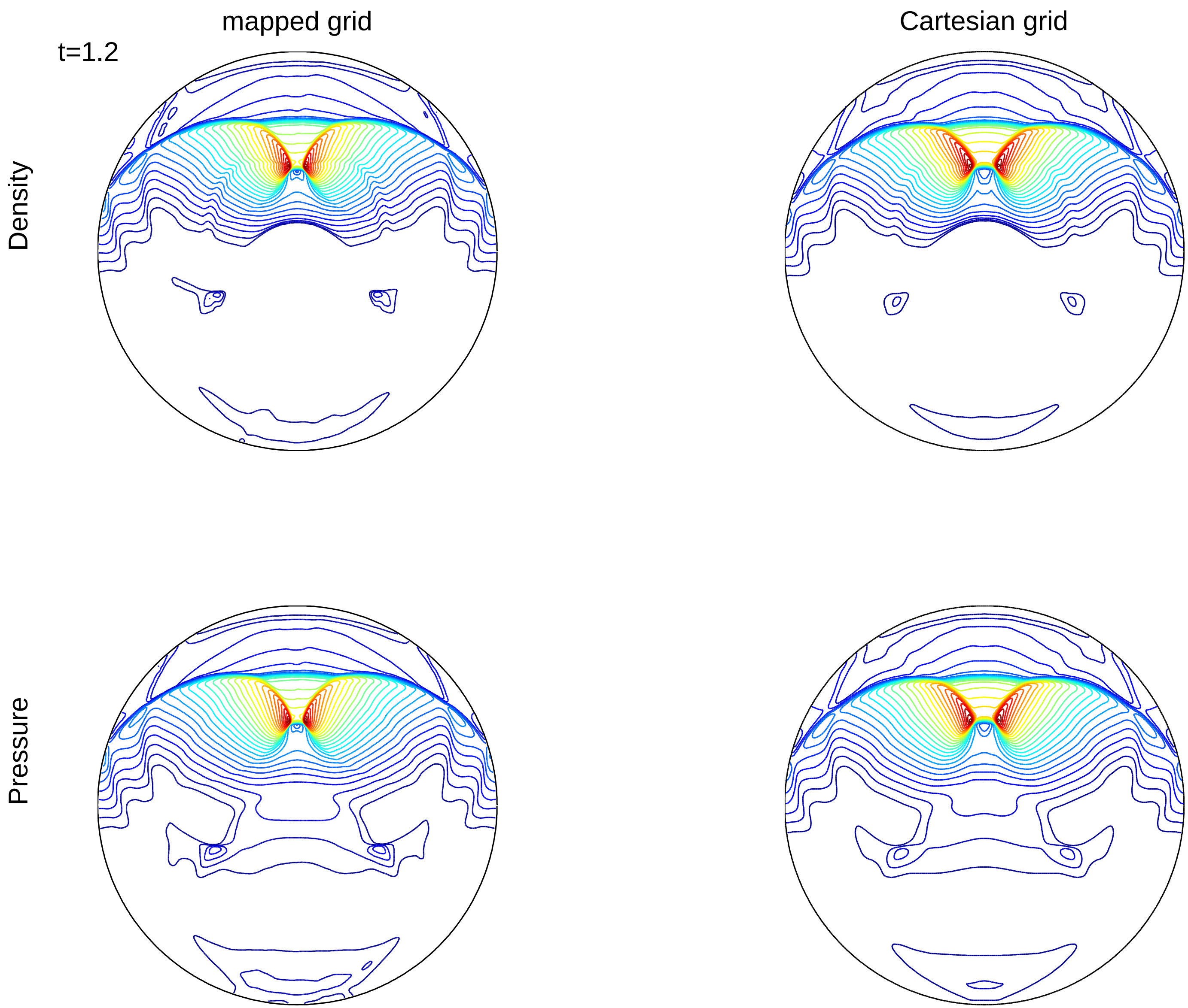
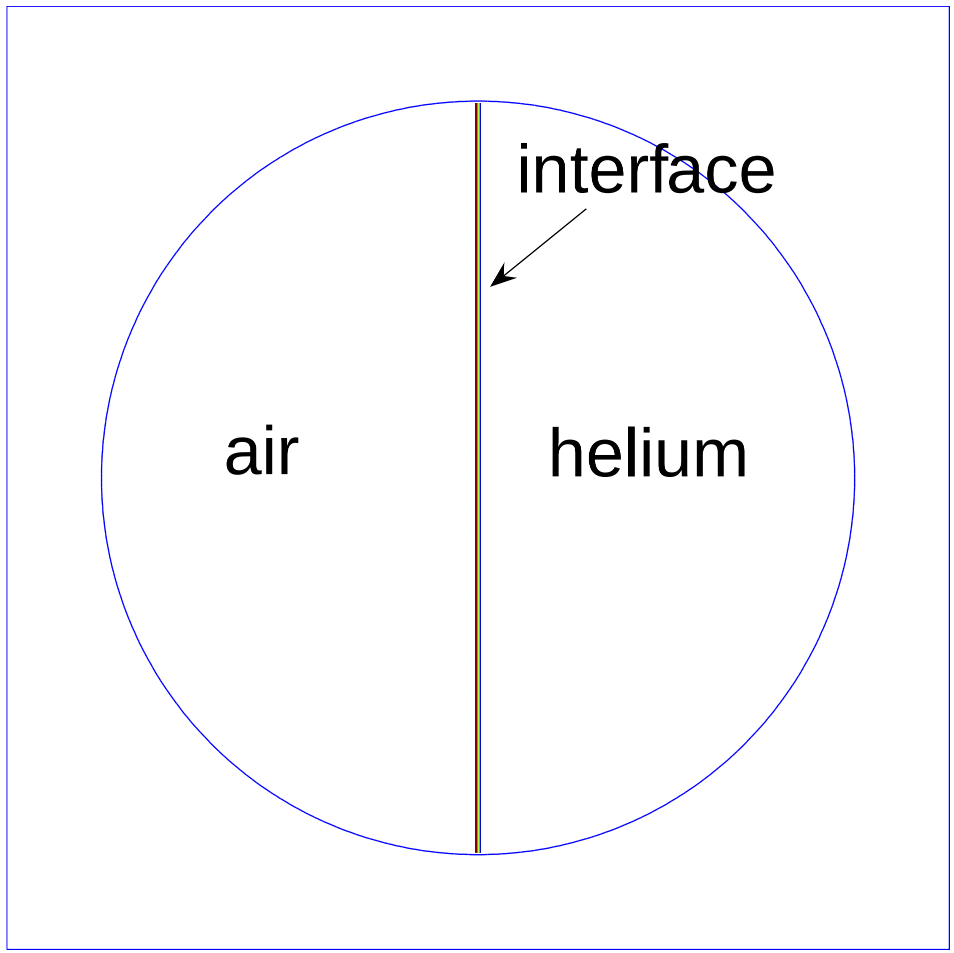


Moving Cylindrical Vessel

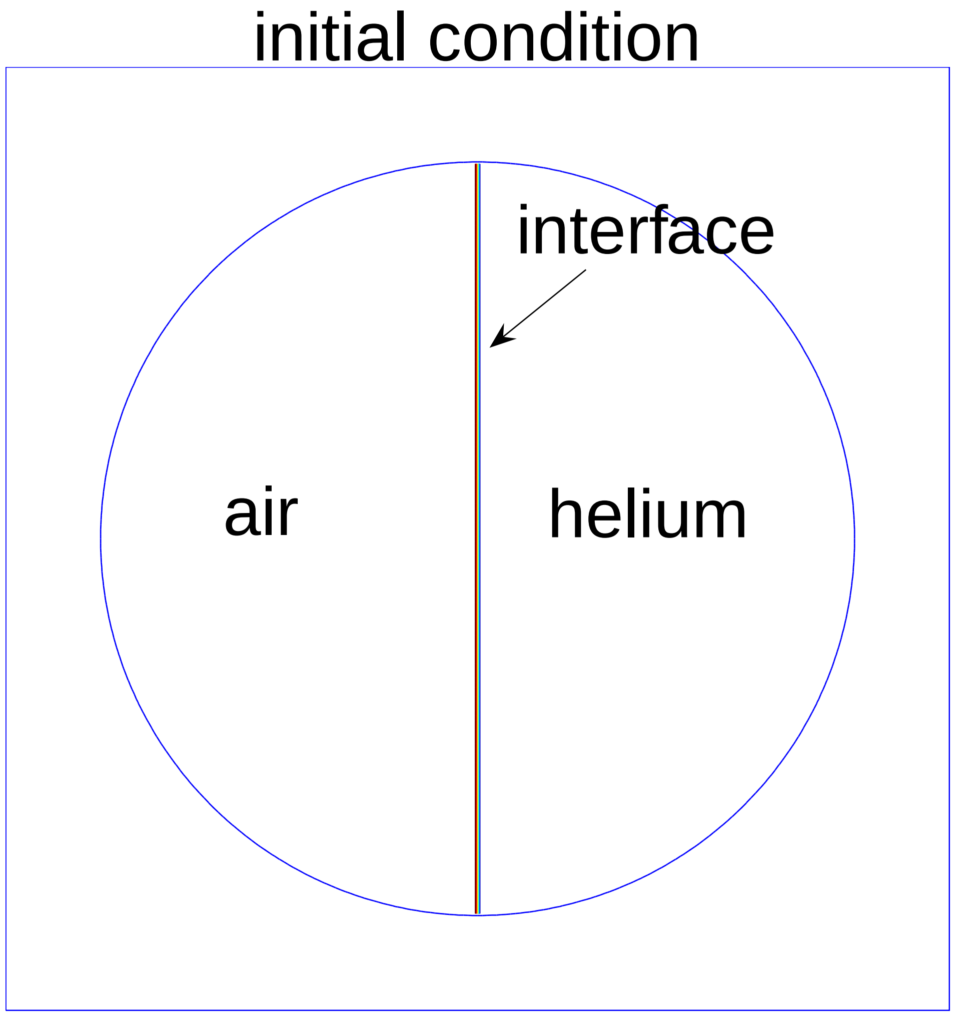
Mapped grid results with $h_0 = 0$ & $\vec{u}_b = (1,0)$

initial condition



Moving Cylindrical Vessel

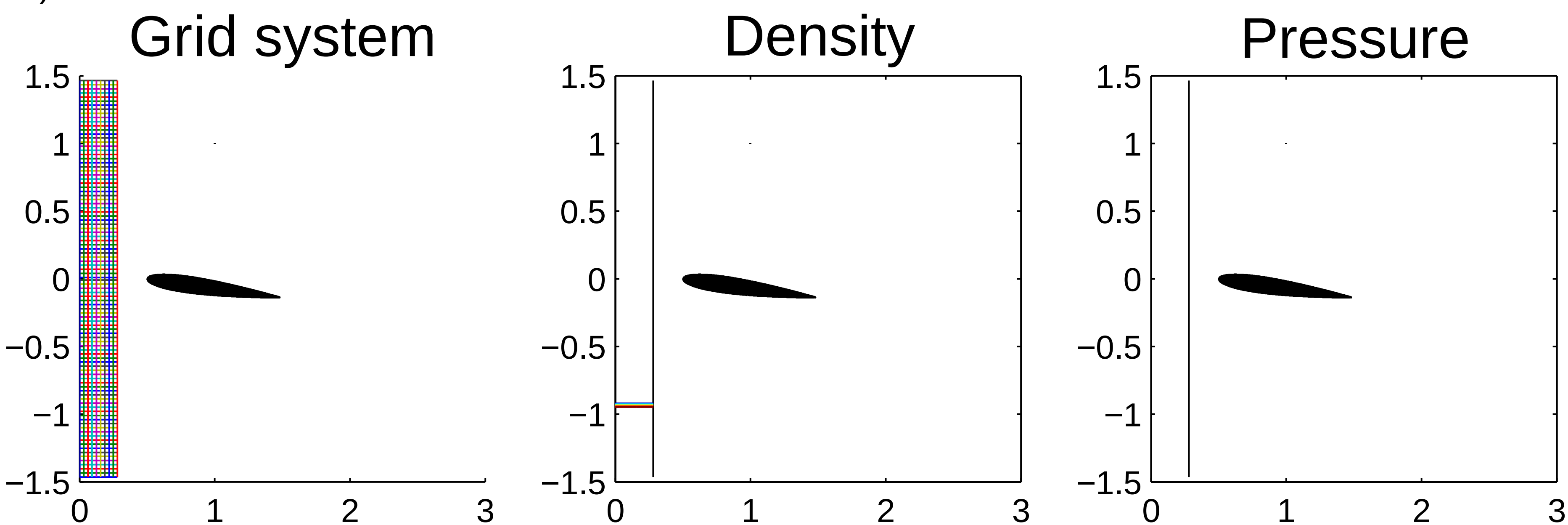
Cartesian grid results with embedded moving boundary



Automatic Time-Marching Grid

- Supersonic NACA0012 over heavier gas

a)



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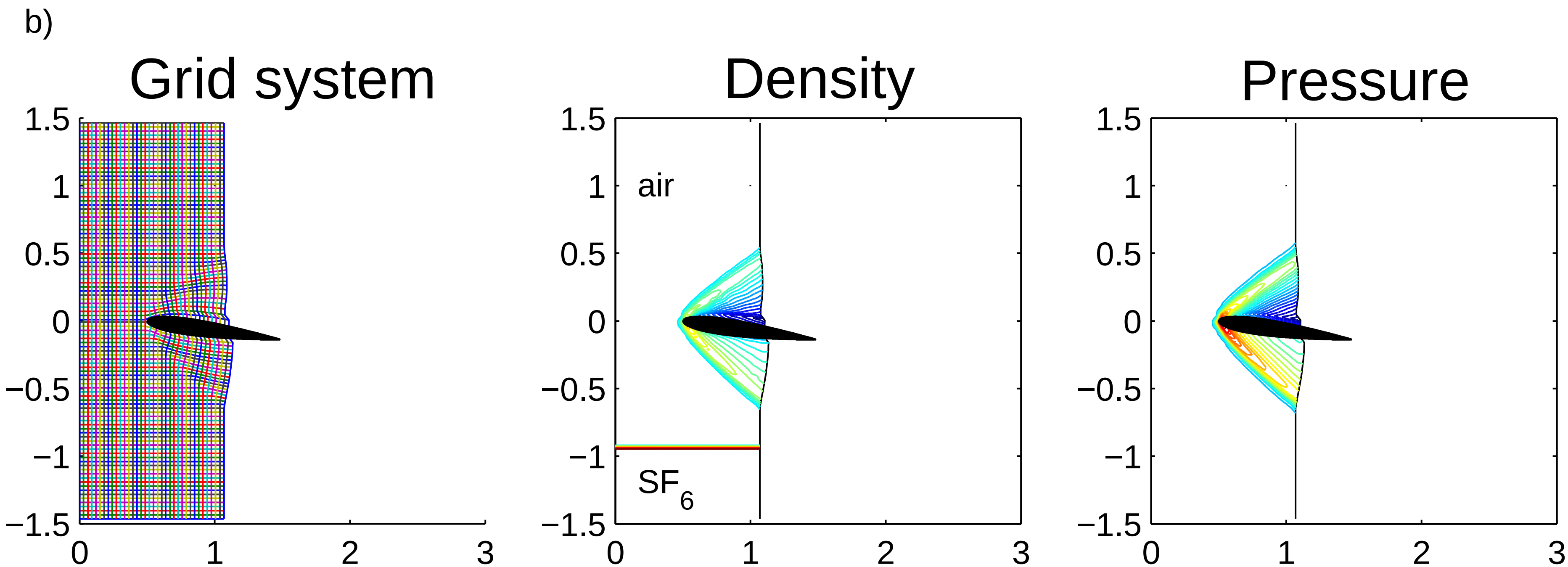

Automatic Time-Marching Grid

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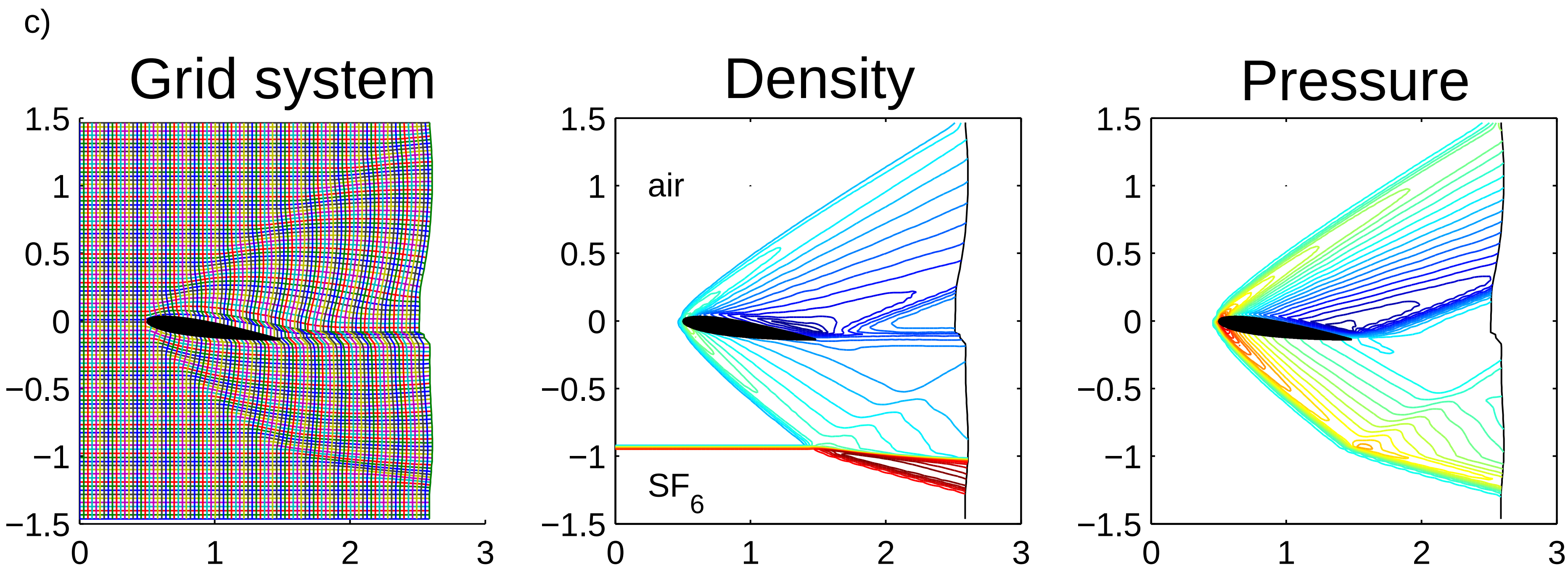

Conclusion

- Have described Lagrangian-like moving grid methods for hyperbolic balance laws
- Have shown results in 1, 2 & 3D to demonstrate feasibility of method for inviscid compressible flow problems

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  - Efficient & accurate grid movement strategy
  - Static & Moving 3D geometry problems
  - Weakly compressible flow
  - Viscous flow extension
  - $\cdots$

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Thank You