Fluid-mixture type algorithm
for
compressible multifluid flows
in
generalized curvilinear grids

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Overview

- Mathematical model for homogeneous multifluid flow
  - Compressible Euler eqs. in generalized coordinates
  - Grid-movement conditions for moving grid system
  - Mixture equations of state
  - Transport eqs. for multifluid problems of concerns
- Finite volume numerical method
  - Godunov-type f-wave formulation of LeVeque *et al.*
- Numerical examples
  - Underwater explosions, shock-bubble, $\cdots$
- Future direction

Motivations

- Some basic facts
  - Lagrangian method can resolve material or slip lines sharply if there is not too much grid tangling
  - Generalized curvilinear grid is often superior to Cartesian grid when they are employed in numerical methods for complex fixed or moving geometries

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- Some examples done by Cartesian-based method
  - Falling liquid drop problem
  - Shock-bubble interaction
  - Flying projectile & ocean surface
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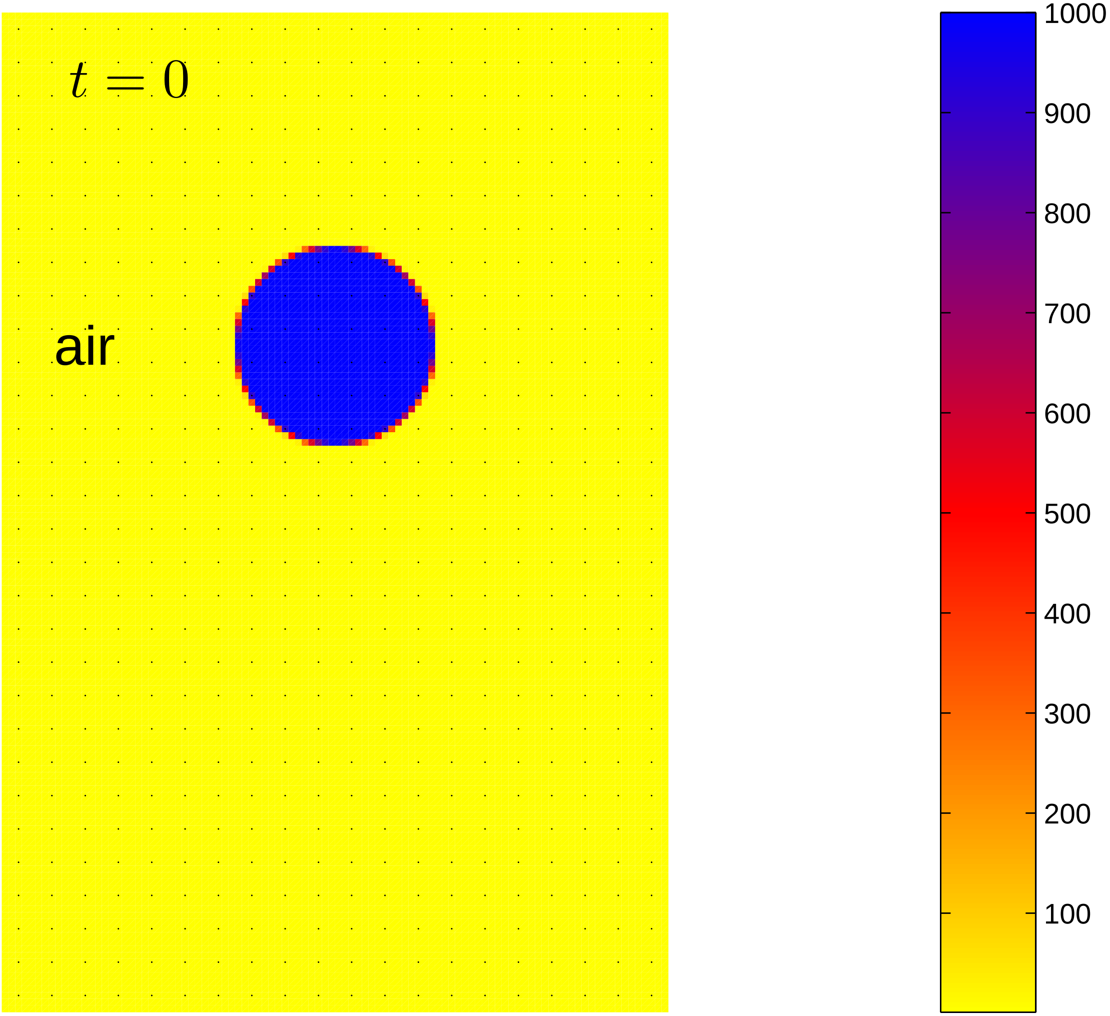
- Some examples done by Cartesian-based method
  - Falling liquid drop problem
  - Shock-bubble interaction
  - Flying projectile & ocean surface
  - Falling rigid object in water tank
- Search for more robust method (work present here is preliminary)

Falling Liquid Drop Problem

- Interface capturing with gravity



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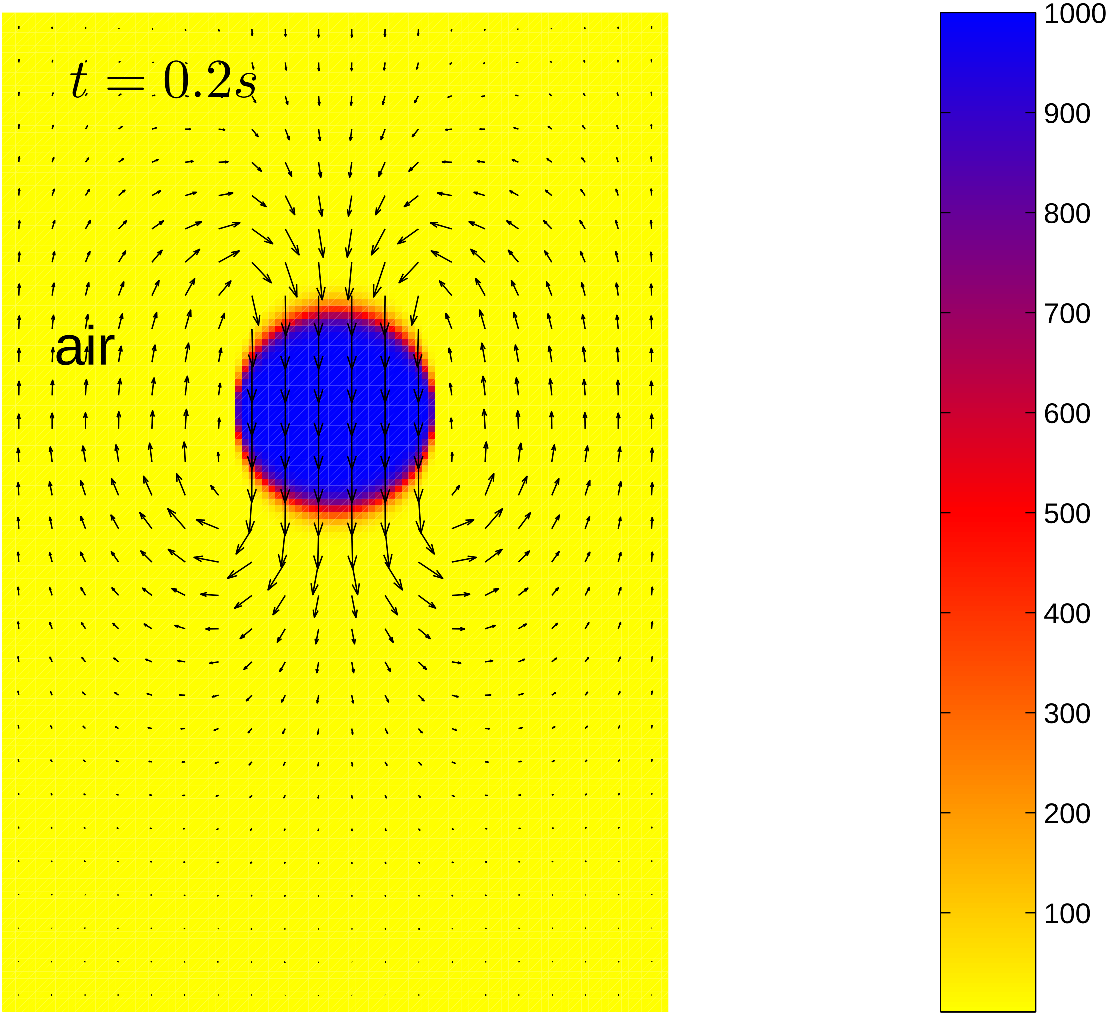
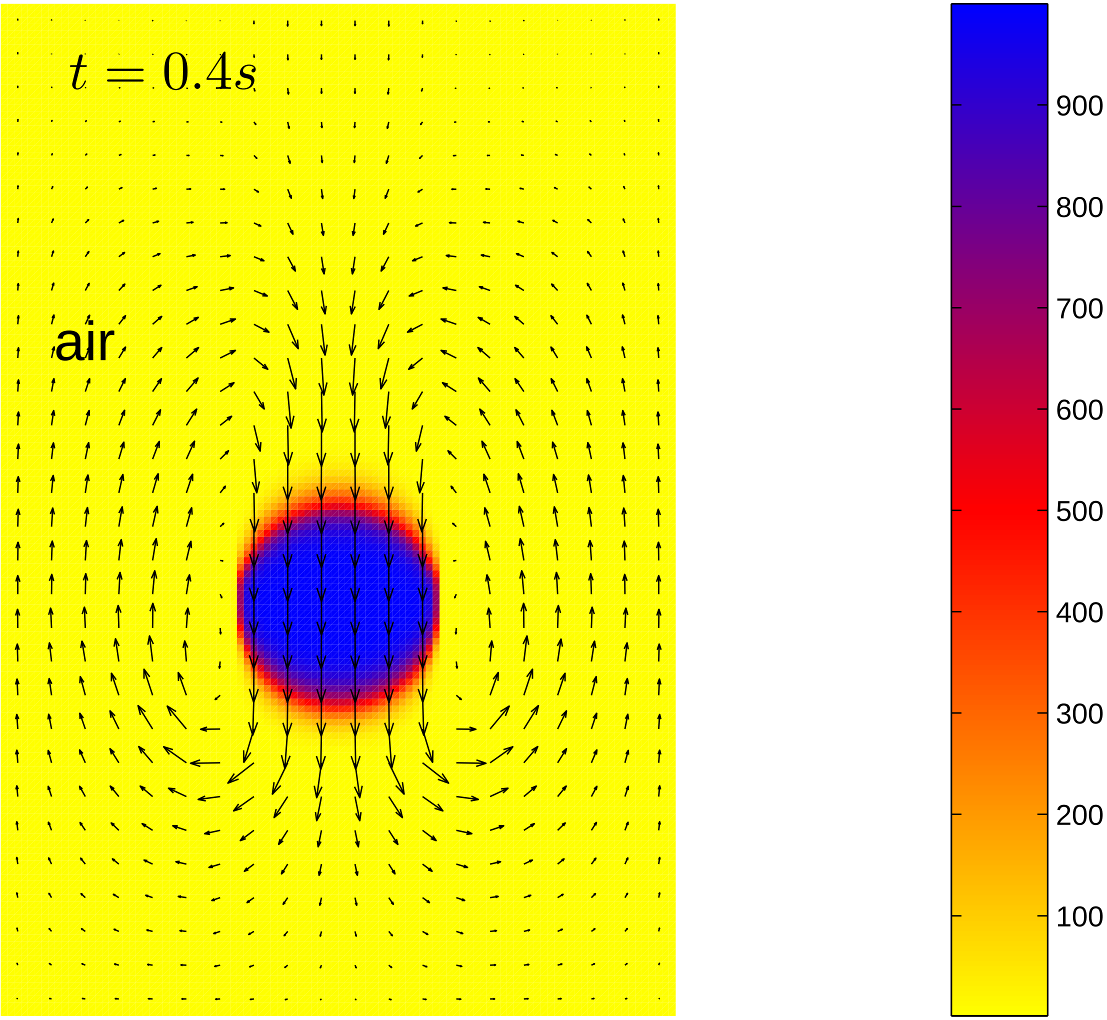


Falling Liquid Drop Problem

- Interface diffused badily



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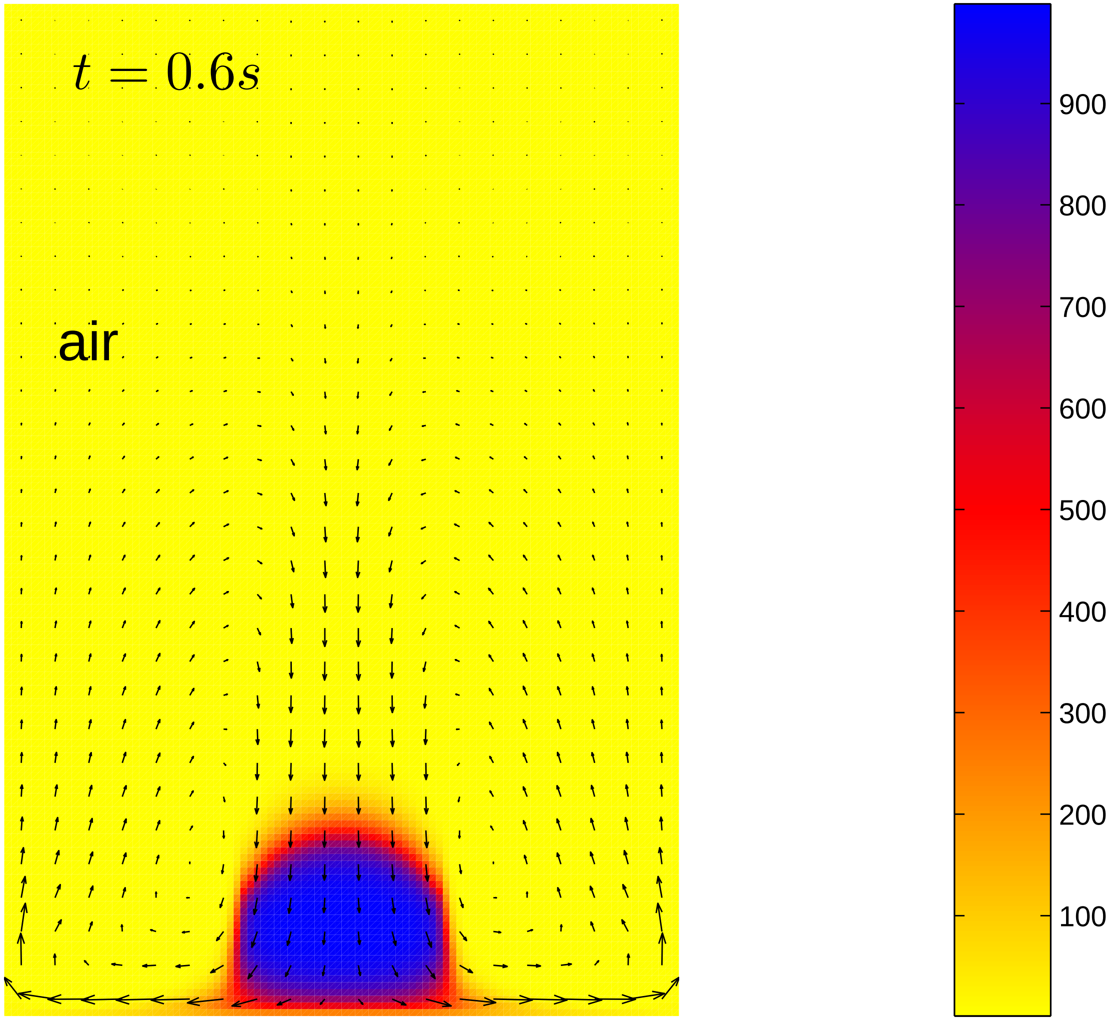

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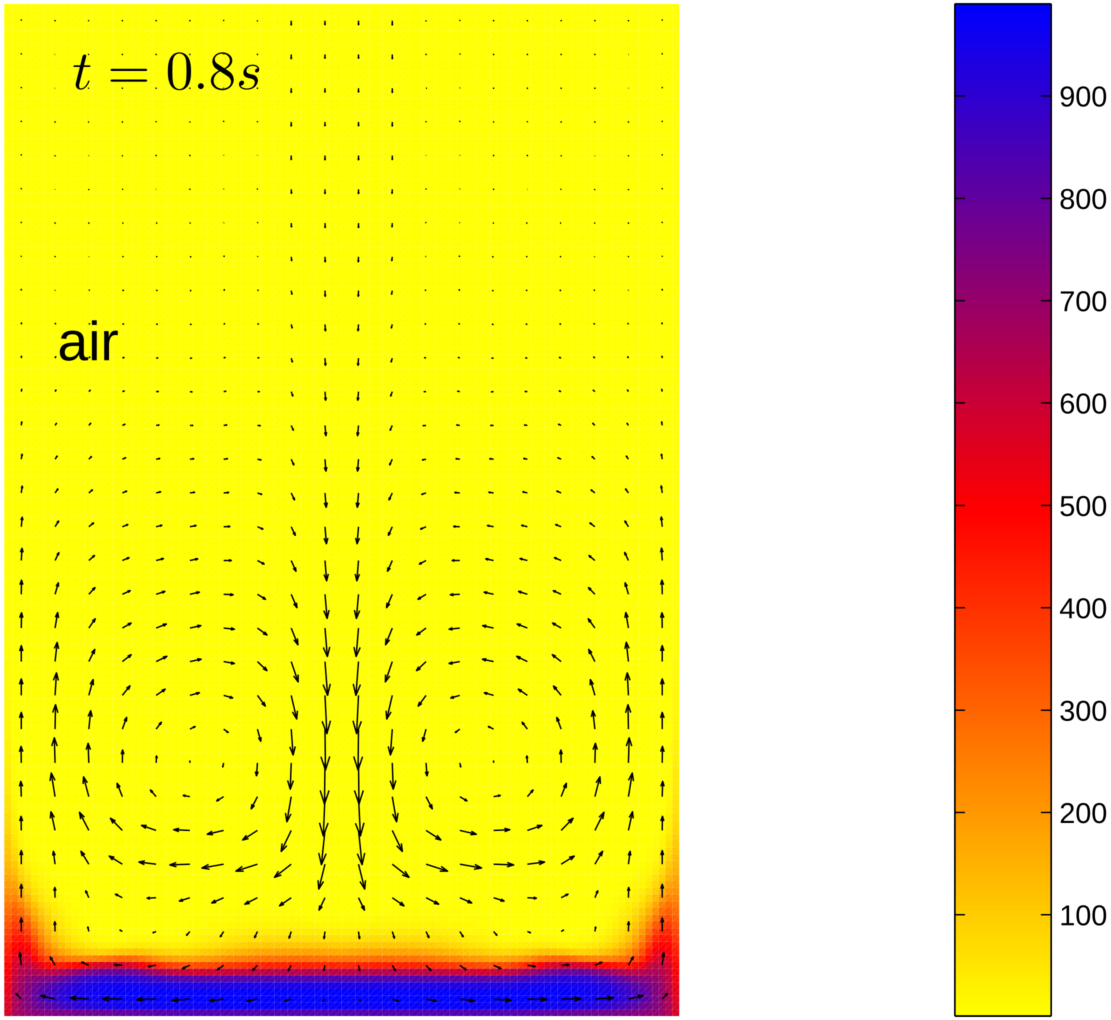

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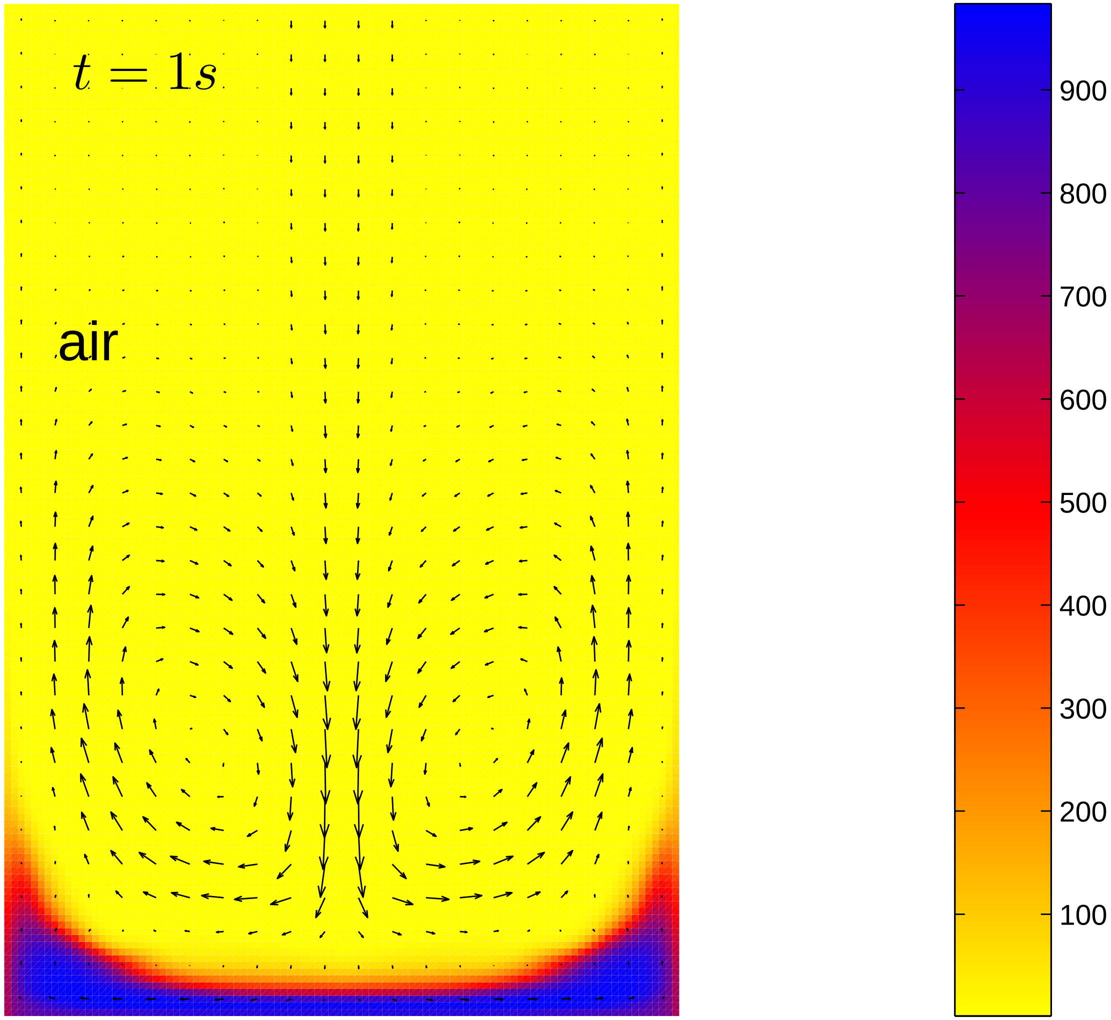
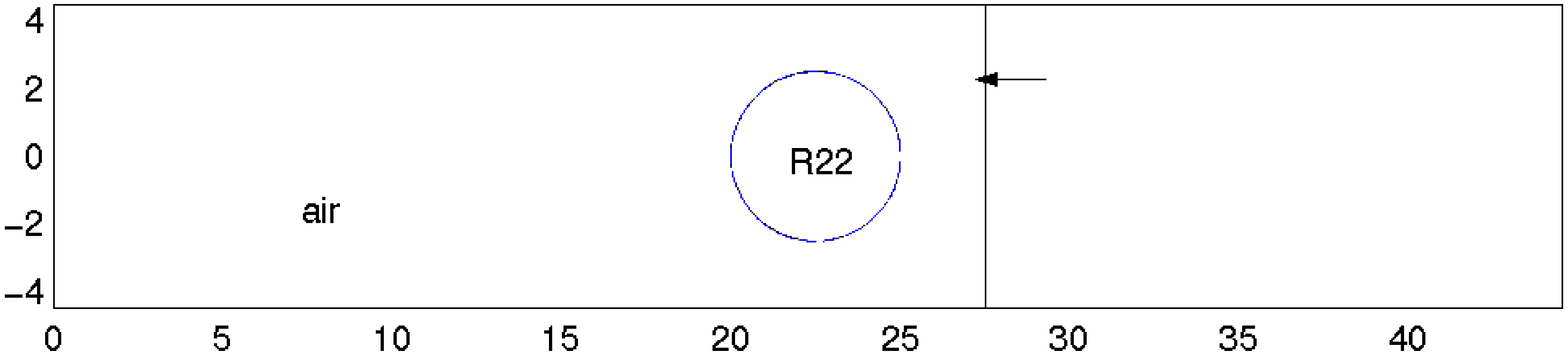


Shock-Bubble Interaction

- Volume tracking for material interface



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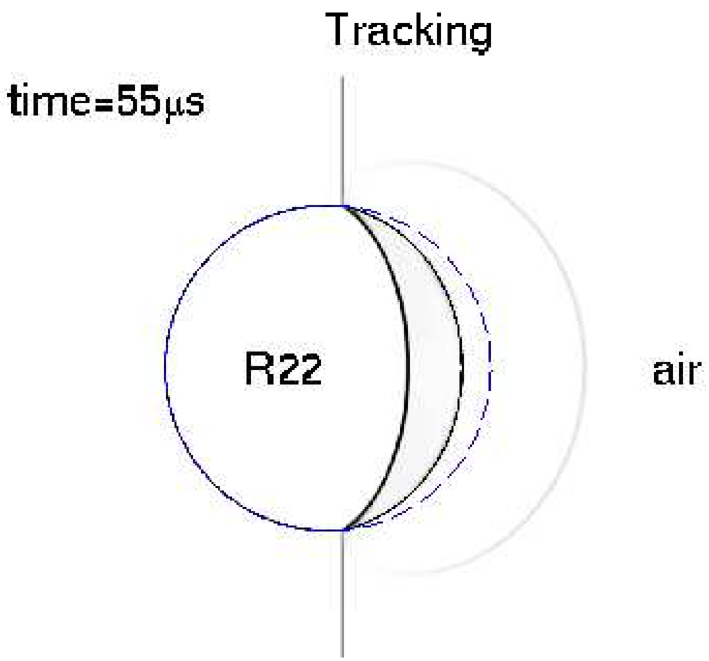

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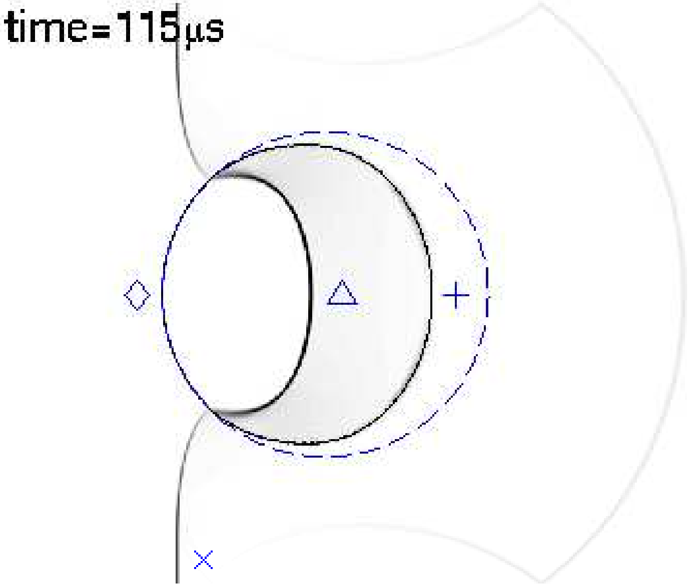

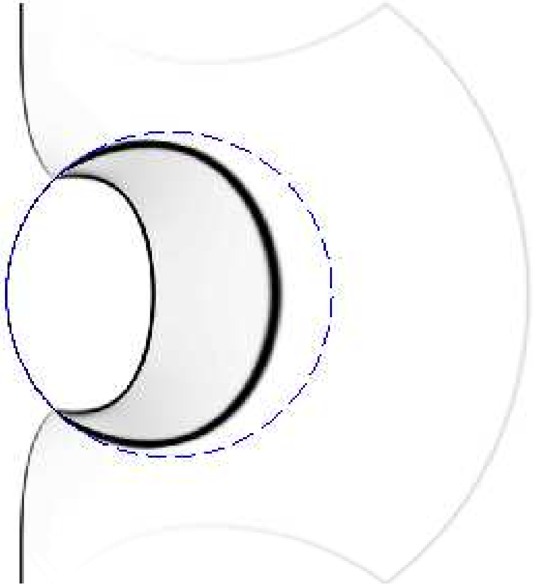

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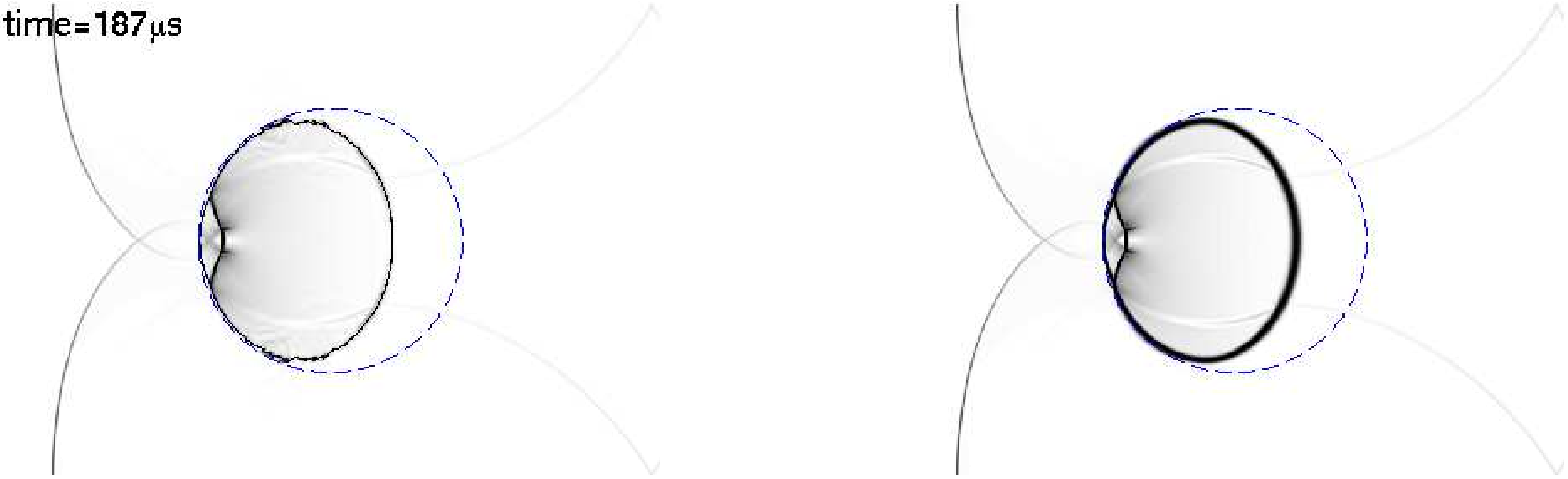

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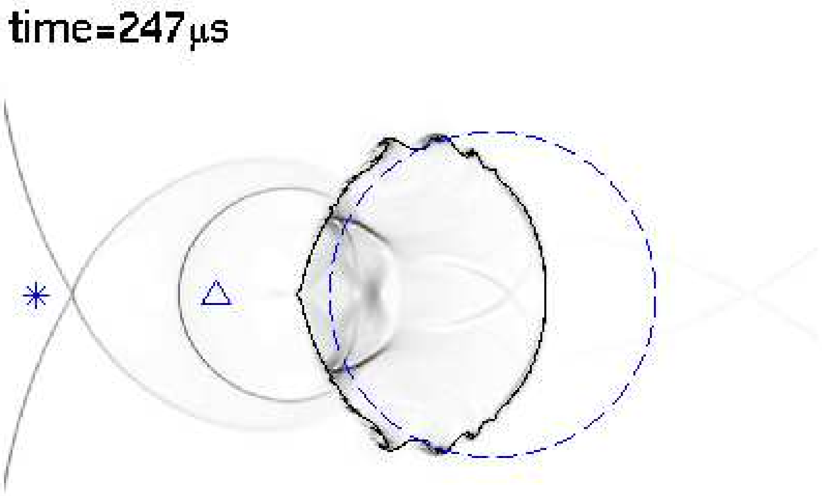

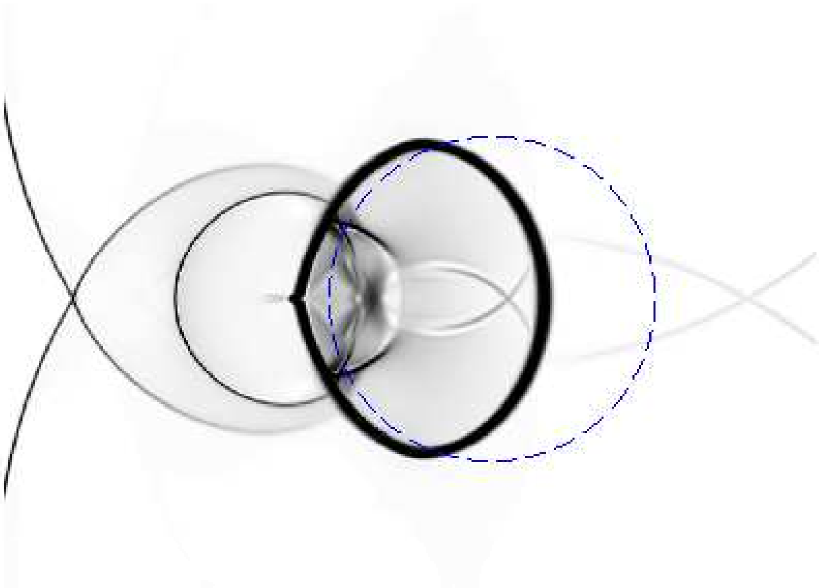

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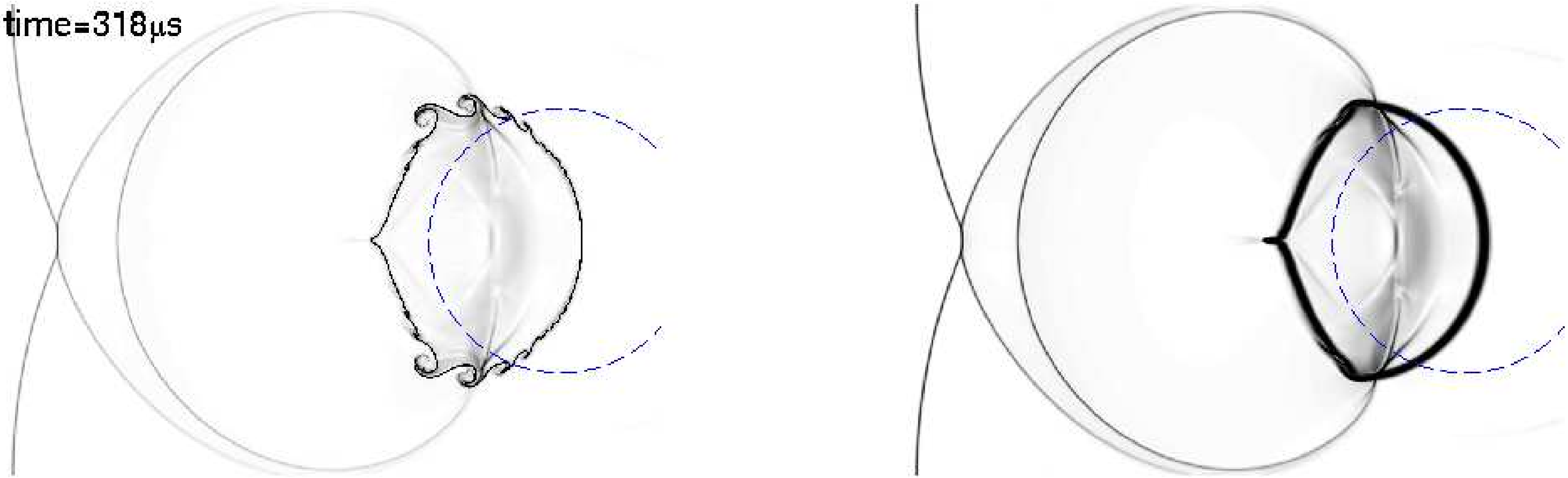

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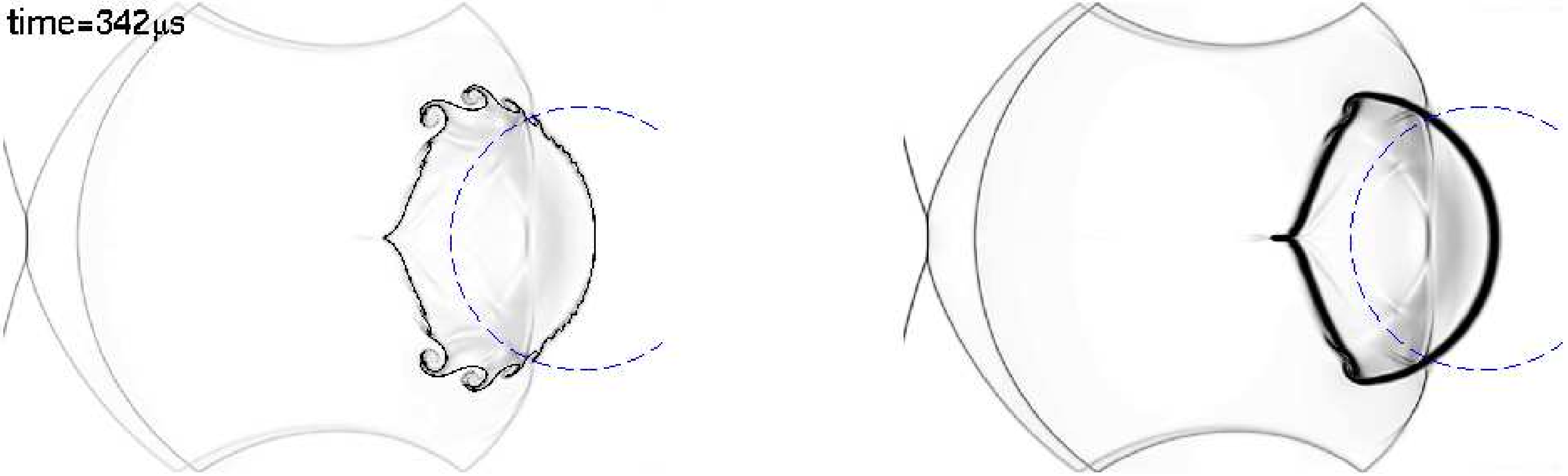

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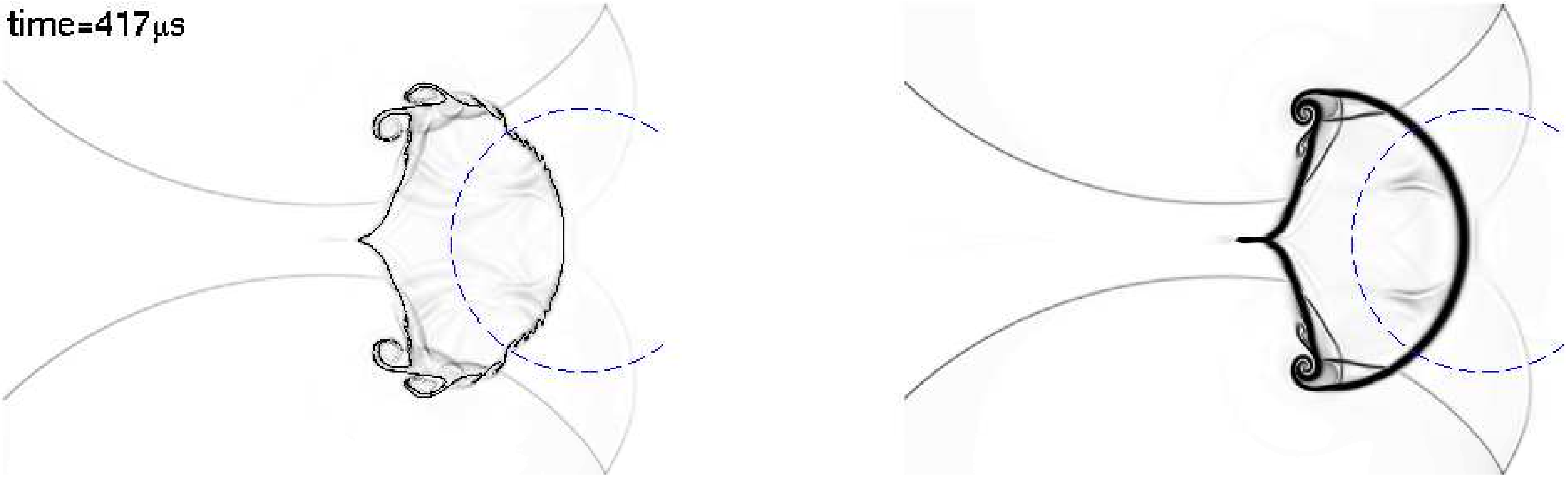
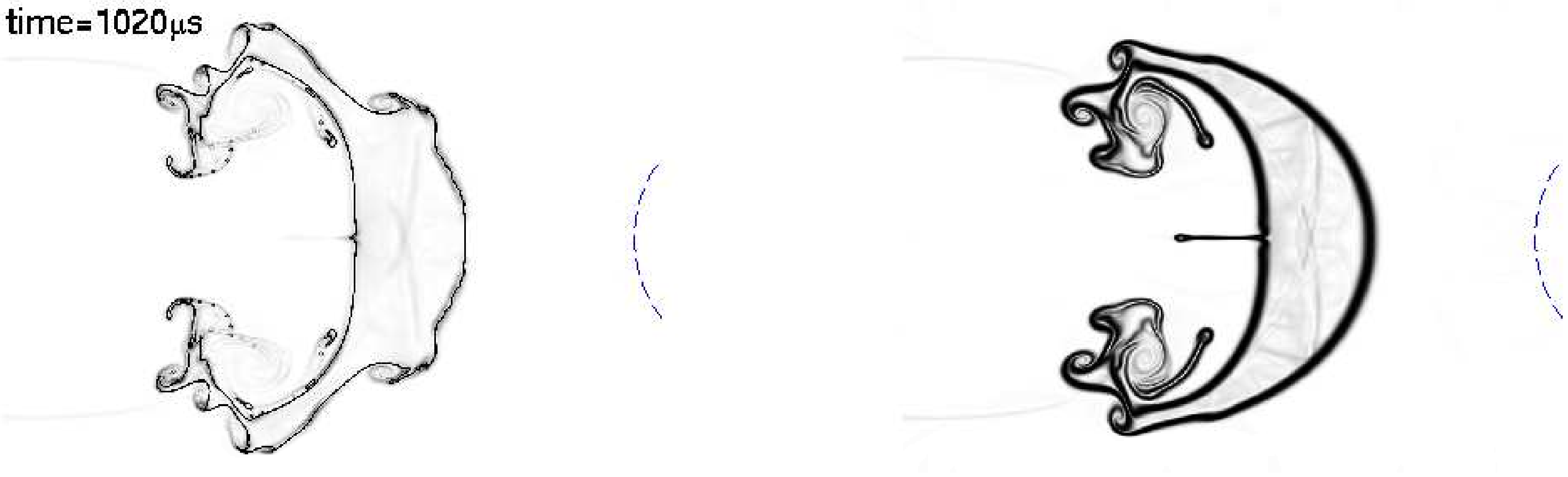


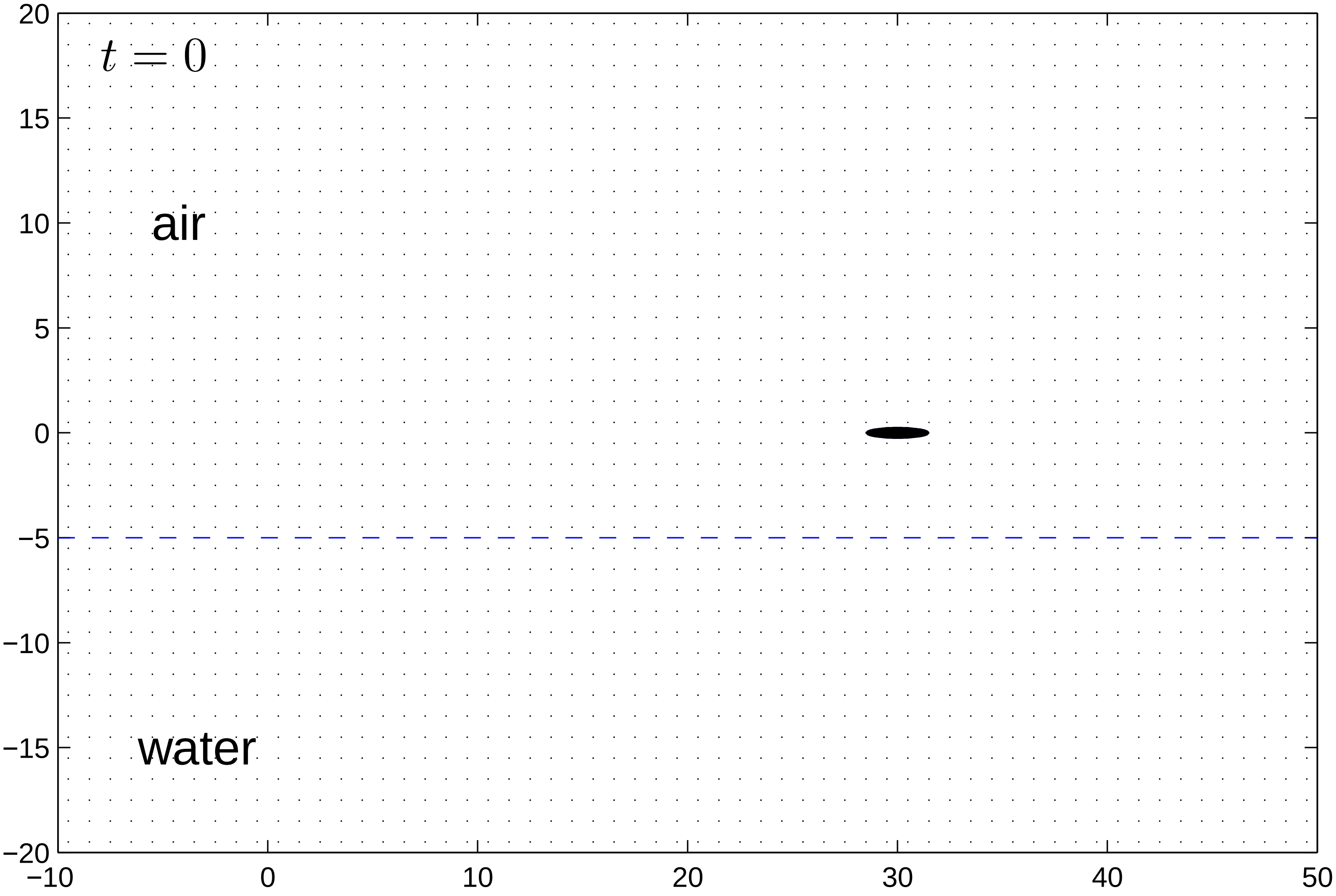
Shock-Bubble Interaction

- Small moving irregular cells: stability & accuracy



Flying Projectile & Ocean Surface

- Moving boundary tracking & interface capturing



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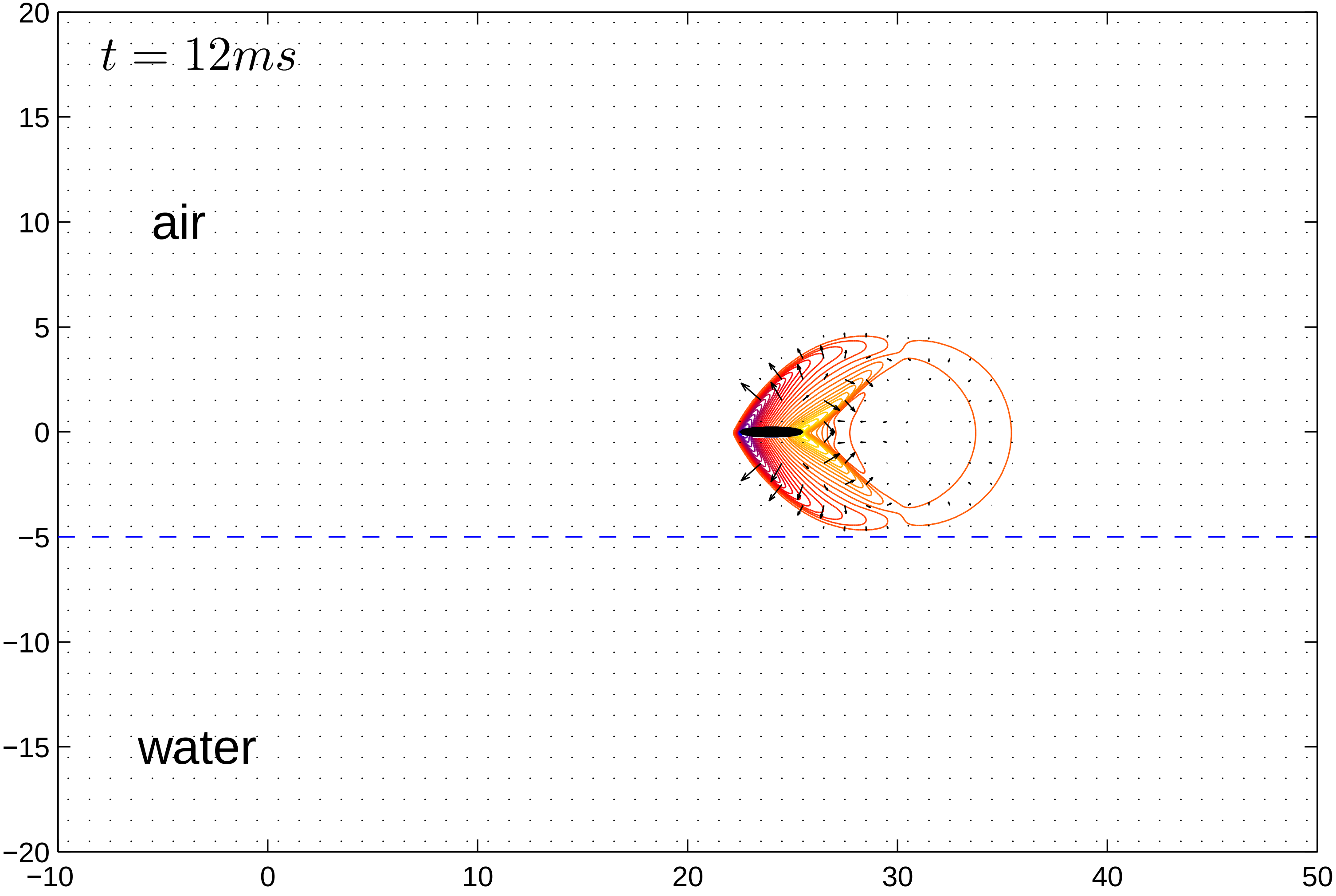

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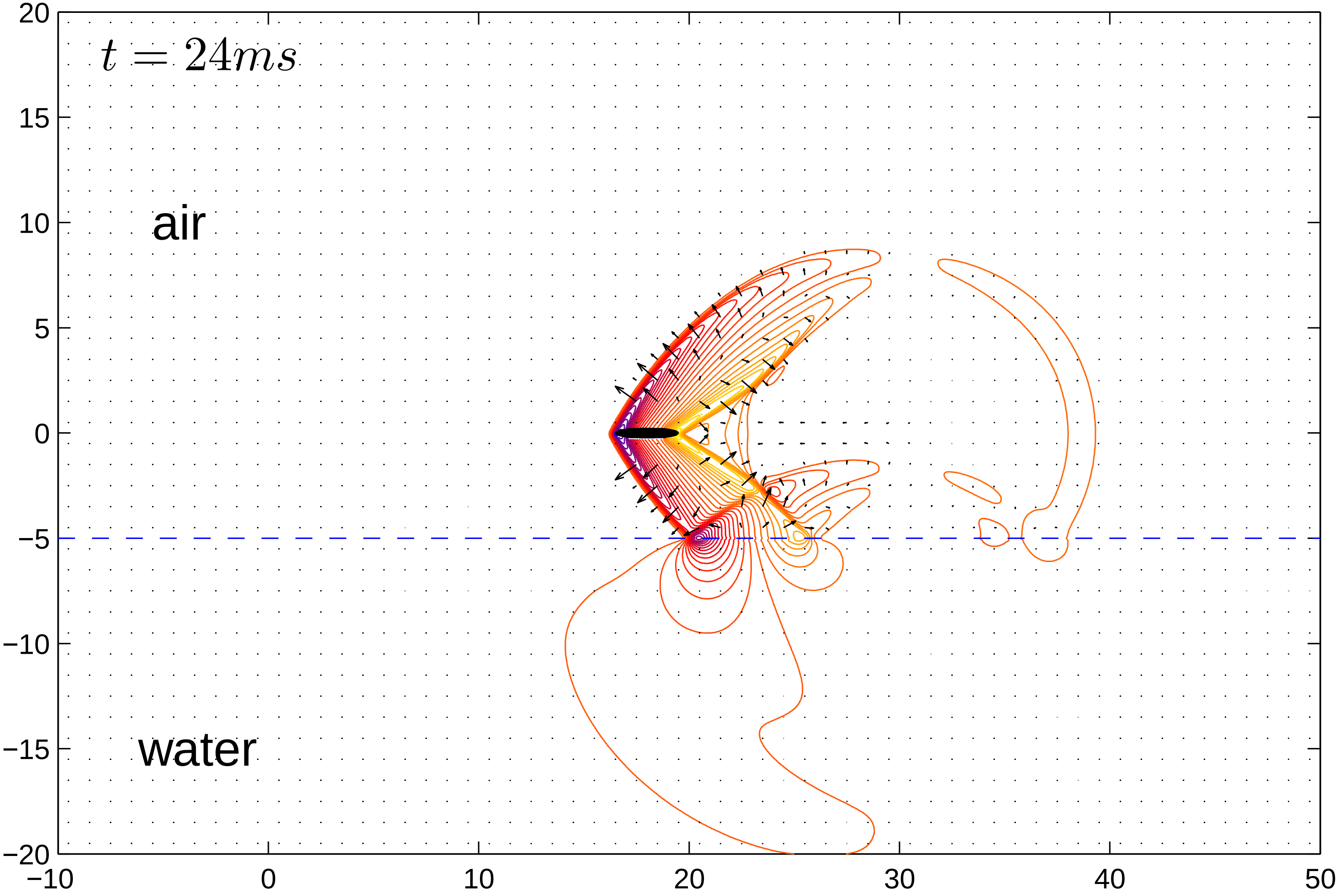

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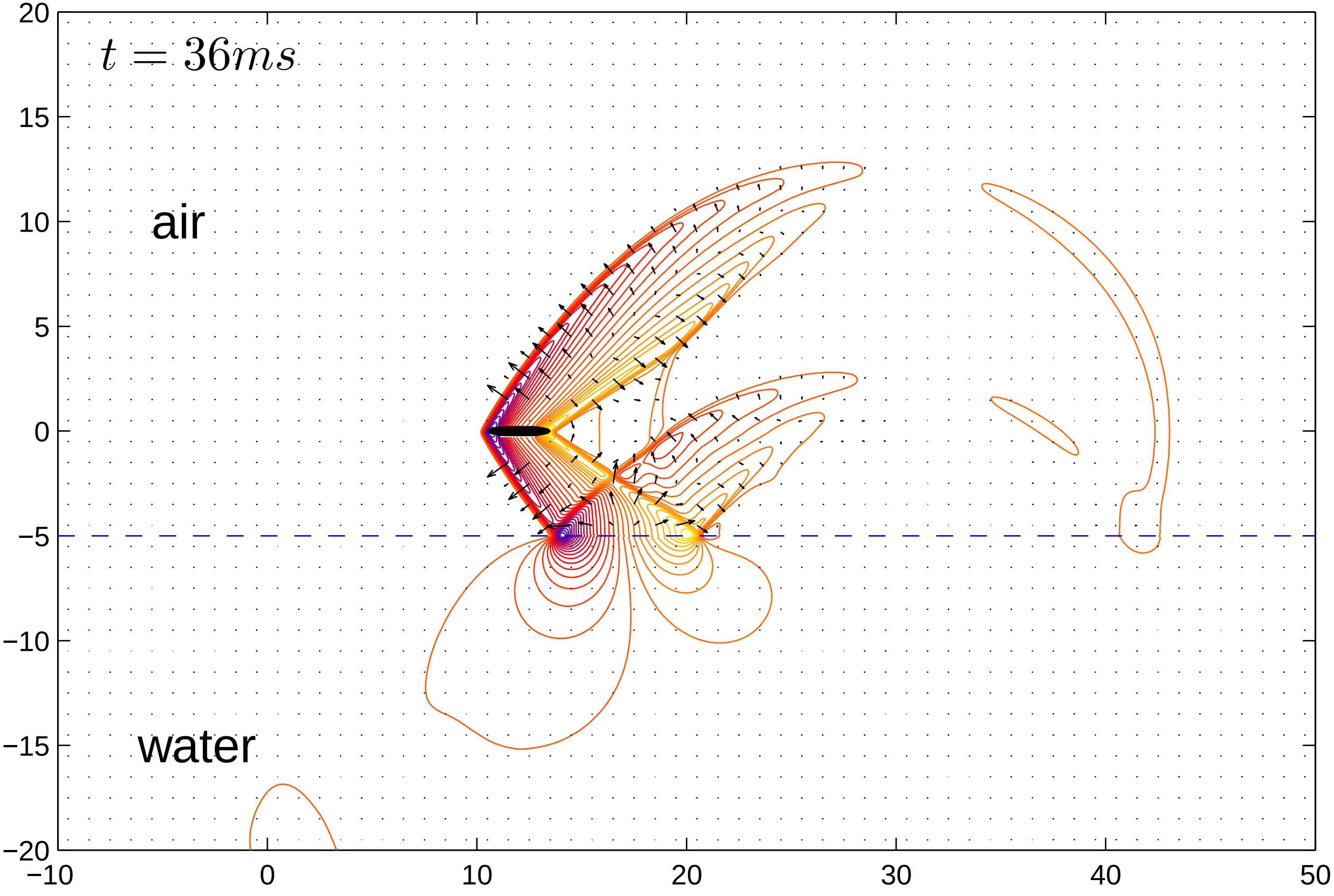

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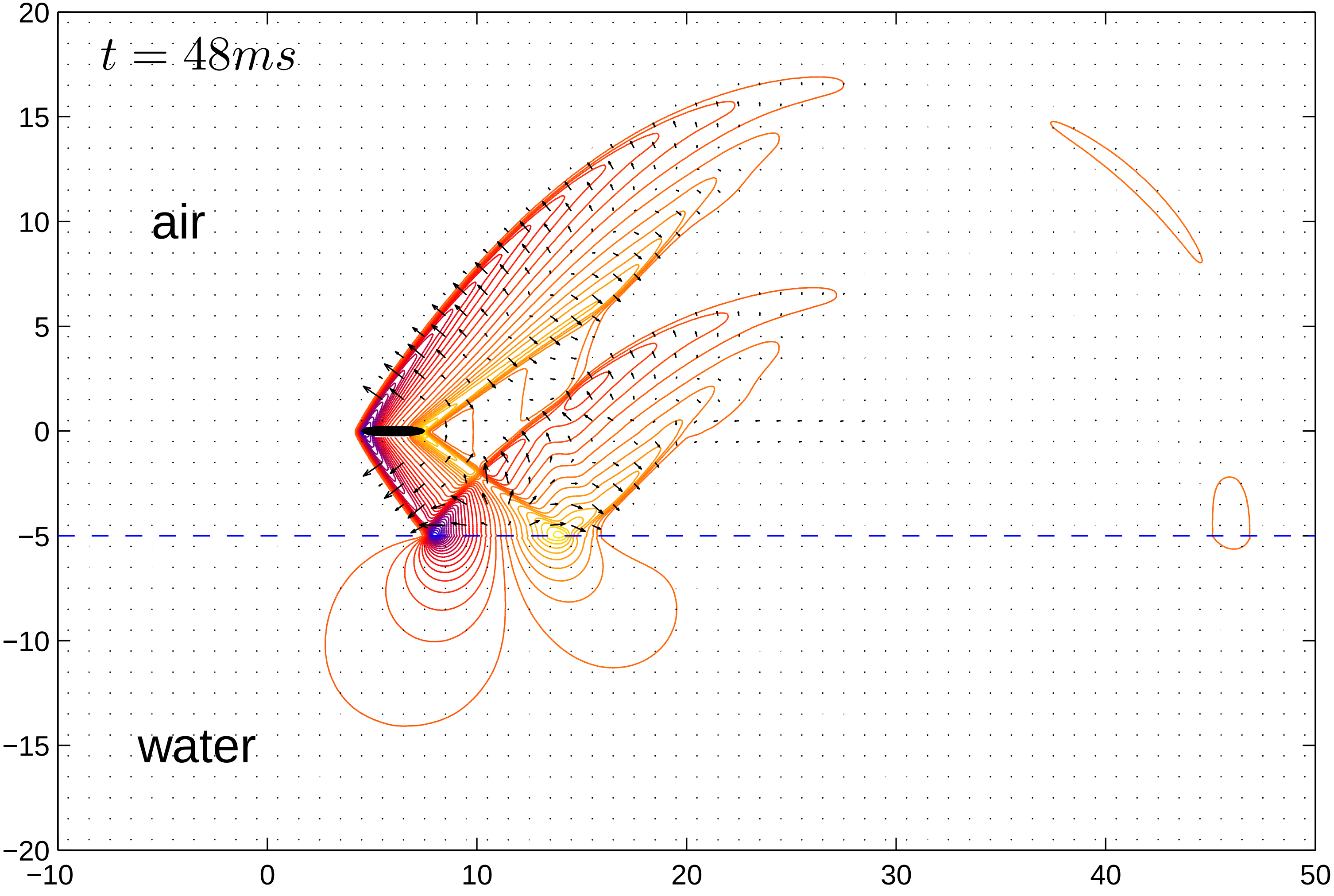

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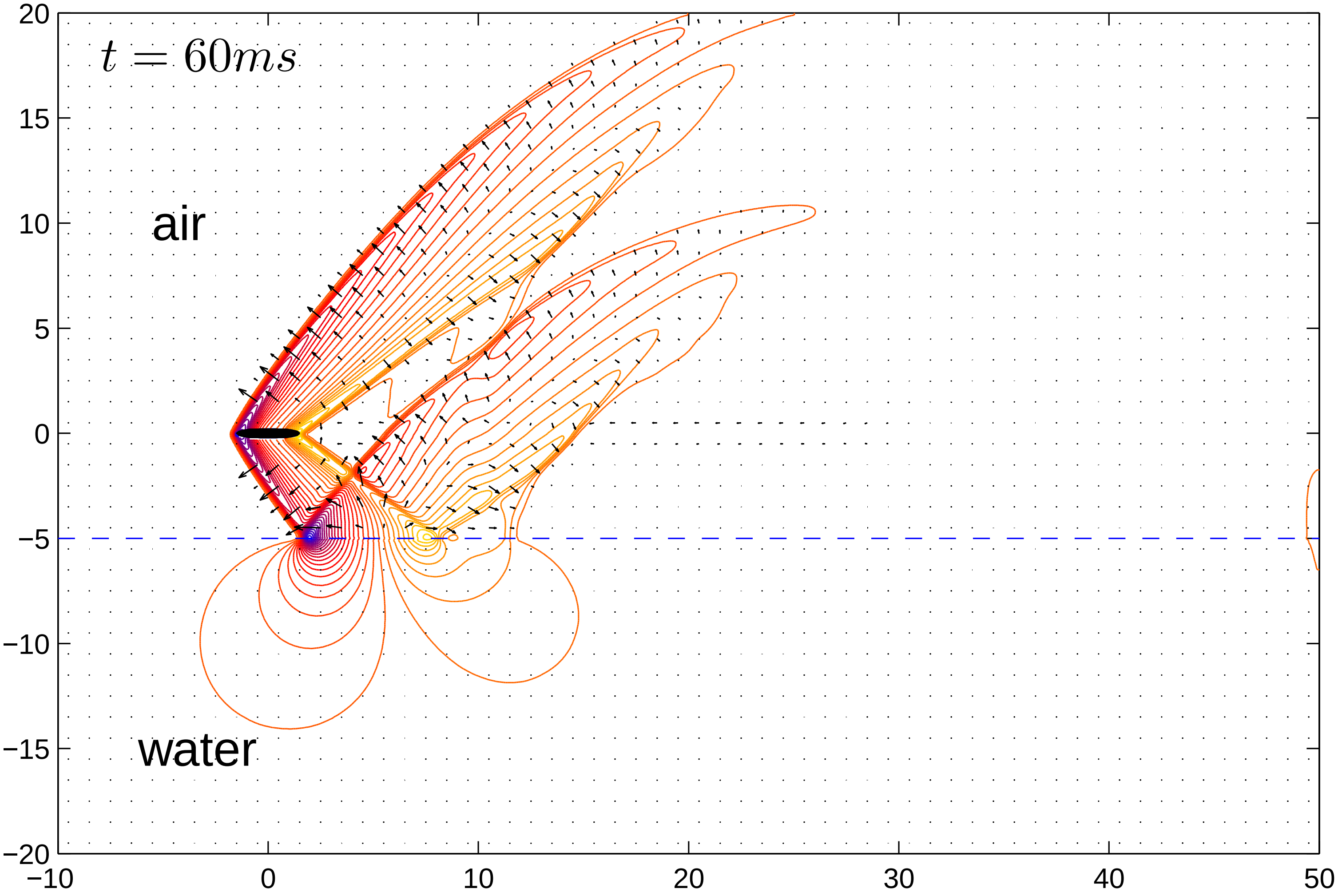
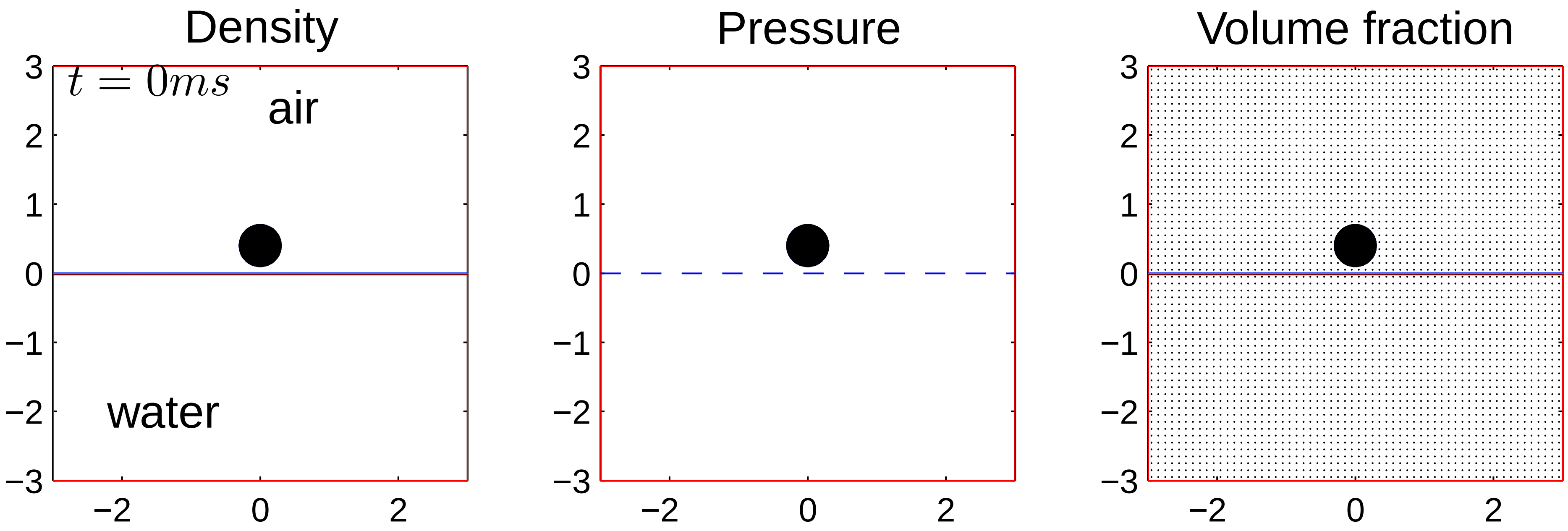


Falling Rigid Object in Water Tank

- Moving boundary tracking & interface capturing



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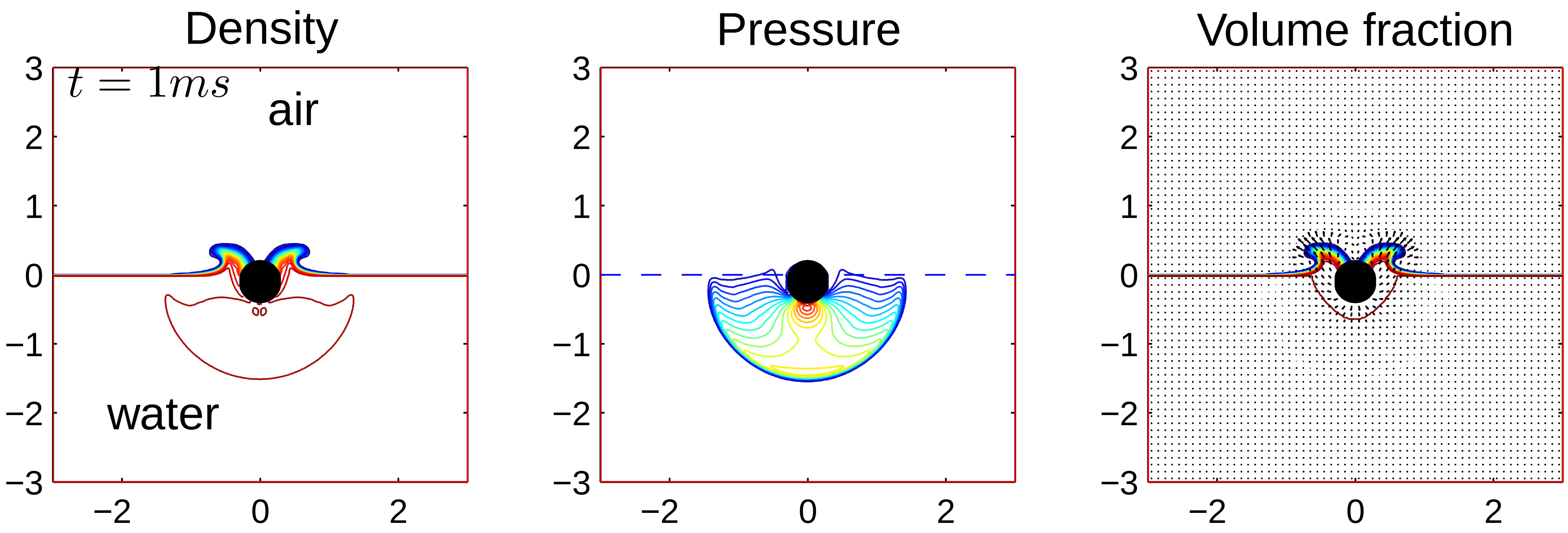

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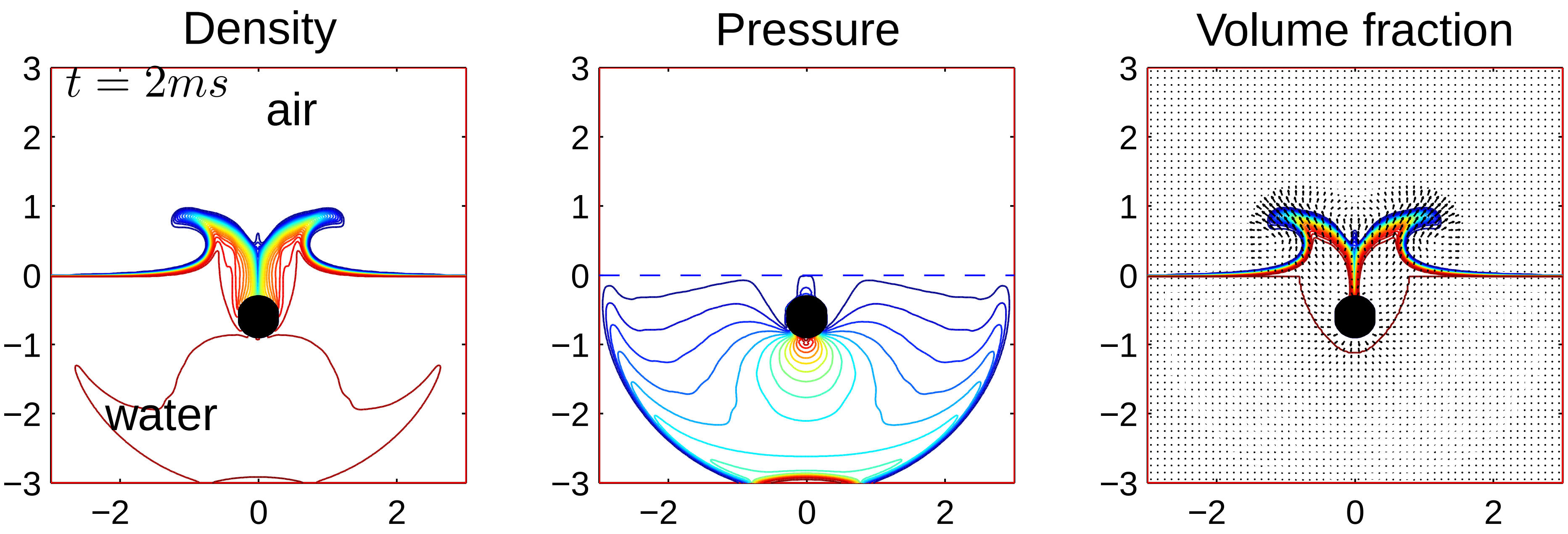

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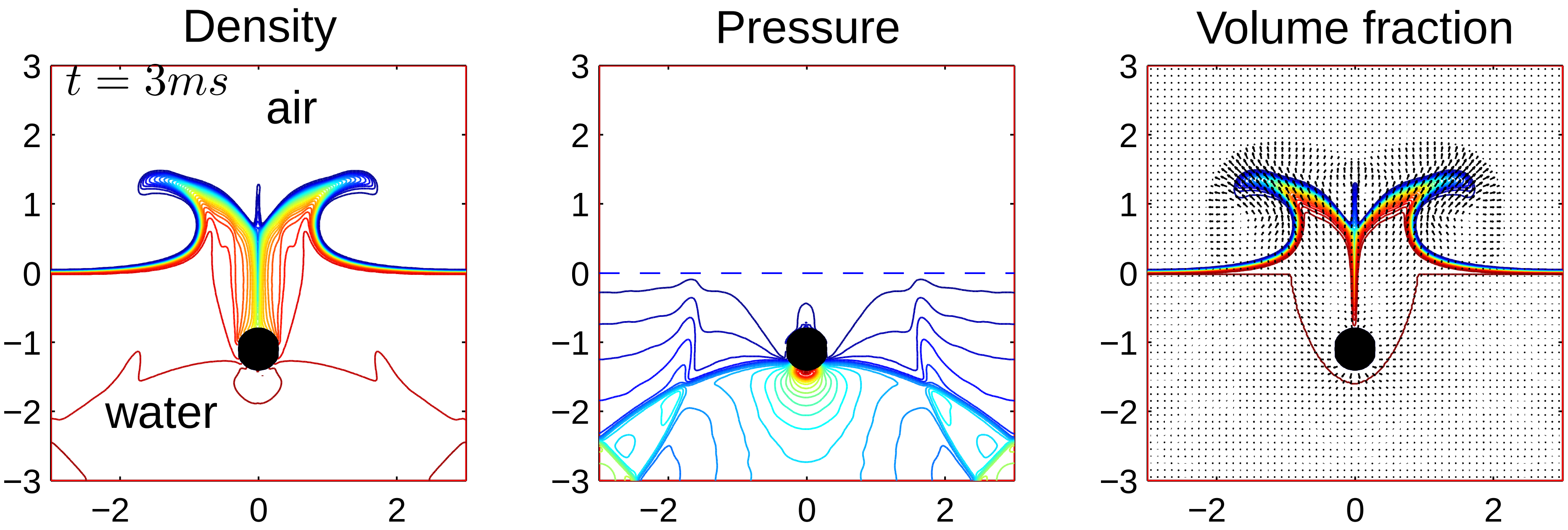

Euler Eqs. in Generalized Coord.

With gravity effect included, for example, 2D compressible Euler eqs. in Cartesian coordinates take

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + \frac{\partial g(q)}{\partial y} = \psi(q)$$

where

$$q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \quad f(q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ Eu + pu \end{bmatrix}, \quad g(q) = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ Ev + pv \end{bmatrix}, \quad \psi = \begin{bmatrix} 0 \\ 0 \\ \rho g \\ \rho g v \end{bmatrix}$$

ρ : density, (u, v) : vector of particle velocity

p : pressure, $E = \rho[e + (u^2 + v^2)/2]$: total energy

$e(\rho, p)$: internal energy, ψ : gravitational source term

Euler in General. Coord. (Cont.)

- Introduce transformation $(t,\ x,\ y) \leftrightarrow (\tau,\ \xi,\ \eta)$ via

$$\begin{pmatrix} dt \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x_\tau & x_\xi & x_\eta \\ y_\tau & y_\xi & y_\eta \end{pmatrix} \begin{pmatrix} d\tau \\ d\xi \\ d\eta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} d\tau \\ d\xi \\ d\eta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \xi_t & \xi_x & \xi_y \\ \eta_t & \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \end{pmatrix}$$

- Basic grid-metric relations:

$$\begin{pmatrix} 1 & 0 & 0 \\ \xi_t & \xi_x & \xi_y \\ \eta_t & \eta_x & \eta_y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x_\tau & x_\xi & x_\eta \\ y_\tau & y_\xi & y_\eta \end{pmatrix}^{-1} = \frac{1}{J} \begin{bmatrix} x_\xi y_\eta - x_\eta y_\xi & 0 & 0 \\ -x_\tau y_\eta + y_\tau x_\eta & y_\eta & -y_\xi \\ x_\tau y_\xi - y_\tau x_\xi & -x_\eta & x_\xi \end{bmatrix}$$

  - $J = x_\xi y_\eta - x_\eta y_\xi$: grid Jacobian

Euler in General. Coord. (Cont.)

With these notations, Euler eqs. in generalized coord. are

$$\frac{\partial \tilde{q}}{\partial \tau} + \frac{\partial \tilde{f}}{\partial \xi} + \frac{\partial \tilde{g}}{\partial \eta} = \tilde{\psi}$$

where

$$\tilde{q} = J \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix}, \tilde{f} = J \begin{bmatrix} \rho U \\ \rho u U + \xi_x p \\ \rho v U + \xi_y p \\ EU + pU - \xi_t p \end{bmatrix}, \tilde{g} = J \begin{bmatrix} \rho V \\ \rho u V + \eta_x p \\ \rho v V + \eta_y p \\ EV + pV - \eta_t p \end{bmatrix}, \tilde{\psi} = J \begin{bmatrix} 0 \\ 0 \\ \rho g \\ \rho g v \end{bmatrix}$$

with contravariant velocities U & V defined by

$$U = \xi_t + \xi_x u + \xi_y v \quad \& \quad V = \eta_t + \eta_x u + \eta_y v$$

Grid Movement Conditions

Continuity on mixed derivatives of grid coordinates gives geometrical conservation laws

$$\frac{\partial}{\partial \tau}\begin{pmatrix} x_\xi \\ y_\xi \\ x_\eta \\ y_\eta \end{pmatrix} + \frac{\partial}{\partial \xi}\begin{pmatrix} -x_\tau \\ -y_\tau \\ 0 \\ 0 \end{pmatrix} + \frac{\partial}{\partial \eta}\begin{pmatrix} 0 \\ 0 \\ -x_\tau \\ -y_\tau \end{pmatrix} = 0$$

with (x_τ, y_τ) to be specified as, for example,

- Eulerian case: $(x_\tau, y_\tau) = \vec{0}$
- Lagrangian case: $(x_\tau, y_\tau) = (u, v)$
- Lagrangian-like case: $(x_\tau, y_\tau) = h_0(u, v)$ or $(h_0 u, k_0 v)$
    - $h_0 \in [0, 1]$ & $k_0 \in [0, 1]$

Grid Movement Conditions (Cont.)

- General 1-parameter case: $(x_\tau, y_\tau) = h(u, v)$
  - Mesh-area preserving condition

$$\begin{aligned} \frac{\partial J}{\partial \tau} &= \frac{\partial}{\partial \tau}\left(x_\xi y_\eta - x_\eta y_\xi\right) \\ &= x_{\xi\tau}\, y_\eta + x_\xi\, y_{\eta\tau} - x_{\eta\tau}\, y_\xi - x_\eta\, y_{\xi\tau} \\ &= \cdots \\ &= \mathcal{A} h_\xi + \mathcal{B} h_\eta + \mathcal{C} h = 0 \qquad (\text{1st order PDE for } h \in [0,1]) \end{aligned}$$

with

$$\mathcal{A} = u y_\eta - v x_\eta, \quad \mathcal{B} = v x_\xi - u y_\xi$$

$$\mathcal{C} = u_\xi y_\eta + v_\eta x_\xi - u_\eta y_\xi - v_\xi x_\eta$$

  - Initial & boundary conditions for h-equation ?

Grid Movement Conditions (Cont.)

- General 1-parameter case: $(x_\tau, y_\tau) = h(u,v)$
  - Grid-angle preserving condition (Hui *et al.* JCP 1999)

$$\frac{\partial}{\partial \tau} \cos^{-1}\left(\frac{\nabla\xi}{|\nabla\xi|}\cdot\frac{\nabla\eta}{|\nabla\eta|}\right) = \frac{\partial}{\partial \tau} \cos^{-1}\left(\frac{-y_\eta x_\eta - y_\xi x_\xi}{\sqrt{y_\xi^2 + y_\eta^2}\sqrt{x_\xi^2 + x_\eta^2}}\right)$$

$$= \cdots$$

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with

$$\mathcal{A} = \sqrt{x_\eta^2 + y_\eta^2}\,(v x_\xi - u y_\xi)\,, \quad \mathcal{B} = \sqrt{x_\xi^2 + y_\xi^2}\,(u y_\eta - v x_\eta)$$

$$\mathcal{C} = \sqrt{x_\xi^2 + y_\xi^2}\,(u_\eta y_\eta - v_\eta x_\eta) - \sqrt{x_\eta^2 + y_\eta^2}\,(u_\xi y_\xi - v_\xi x_\xi)$$

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Grid Movement Conditions (Cont.)

- 2-parameter case of Hui *et al.* (2005): $(x_\tau, y_\tau) = (U_g, V_g)$
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    1. Grid-angle preserving
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Roadmap of current work:

$$\boxed{(x_\tau, y_\tau) = h_0(u,v)} \longrightarrow \boxed{(x_\tau, y_\tau) = h(u,v)} \longrightarrow \cdots$$

Single-Fluid Model

With $(x_\tau, y_\tau) = h_0(u, v)$, our model system for single-phase flow reads

$$\frac{\partial}{\partial \tau}\begin{pmatrix} J\rho \\ J\rho u \\ J\rho v \\ JE \\ x_\xi \\ y_\xi \\ x_\eta \\ y_\eta \end{pmatrix} + \frac{\partial}{\partial \xi}\begin{pmatrix} J\rho U \\ J\rho u U + y_\eta p \\ J\rho v U - x_\eta p \\ JEU + (y_\eta u - x_\eta v)p \\ -h_0 u \\ -h_0 v \\ 0 \\ 0 \end{pmatrix} + \frac{\partial}{\partial \eta}\begin{pmatrix} J\rho V \\ J\rho u V - y_\xi p \\ J\rho v V + x_\xi p \\ JEV + (x_\xi v - y_\xi u)p \\ 0 \\ 0 \\ -h_0 u \\ -h_0 v \end{pmatrix} = \tilde{\psi}$$

where $U = (1 - h_0)(y_\eta u - x_\eta v)$ & $V = (1 - h_0)(x_\xi v - y_\xi u)$

Single-Fluid Model: Remarks

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- Canonical form
  - In Cartesian coordinates

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + \frac{\partial g(q)}{\partial y} = \psi(q)$$

  - In generalized coordinates

$$\frac{\partial q}{\partial \tau} + \frac{\partial f(q,\Xi)}{\partial \xi} + \frac{\partial g(q,\Xi)}{\partial \eta} = \psi(q),$$

Ξ: grid metrics

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- Canonical form
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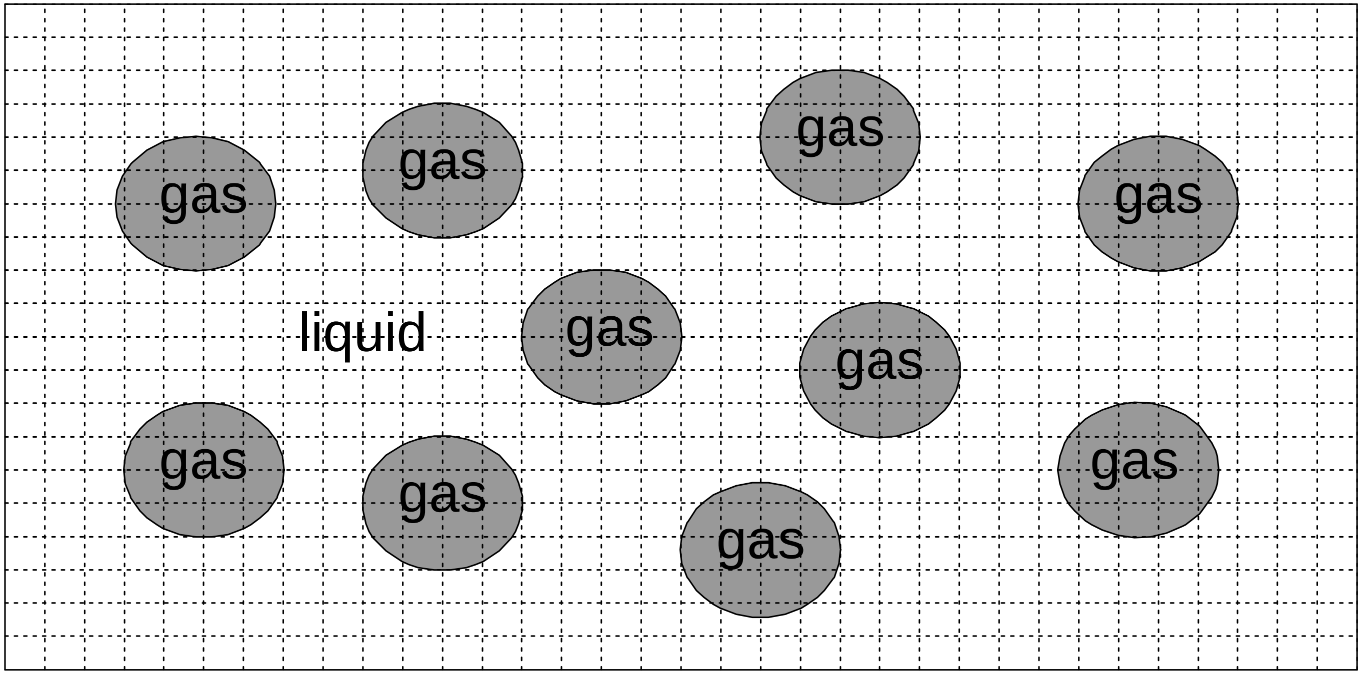
$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} + \frac{\partial g(q)}{\partial y} = \psi(q)$$

  - In generalized coordinates : spatially varying fluxes

$$\frac{\partial q}{\partial \tau} + \frac{\partial f(q,\Xi)}{\partial \xi} + \frac{\partial g(q,\Xi)}{\partial \eta} = \psi(q), \qquad \Xi\text{: grid metrics}$$

Extension to Multifluid

- Assume homogeneous (1-pressure & 1-velocity) flow; *i.e.*, across interfaces $p_\iota = p$ & $\vec{u}_\iota = \vec{u}$, ∀ fluid phase ι



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- Sample examples
  - Barotropic 2-phase flow
  - Hybrid barotropic & non-barotropic 2-phase flow

Barotropic 2-Phase Flow

- Equations of state
  - Fluid component 1 & 2: Tait EOS

$$p(\rho) = (p_{0\iota} + \mathcal{B}_\iota)\left(\frac{\rho}{\rho_{0\iota}}\right)^{\gamma_\iota} - \mathcal{B}_\iota, \quad \iota = 1, 2$$

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  - Mixture pressure law (Shyue, JCP 2004)

$$p(\rho, \rho e) = \begin{cases} (p_{0\iota} + \mathcal{B}_\iota)\left(\frac{\rho}{\rho_{0\iota}}\right)^{\gamma_\iota} - \mathcal{B}_\iota & \text{if} \quad \alpha = 0 \text{ or } 1 \\ (\gamma - 1)\left(\rho e + \frac{\rho\mathcal{B}}{\rho_0}\right) - \gamma\mathcal{B} & \text{if} \quad \alpha \in (0, 1) \end{cases}$$

Here α denotes volume fraction of one chosen fluid component

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variant form of

$$p(\rho, S) = \mathcal{A}(S)\,(p_0 + \mathcal{B})\left(\frac{\rho}{\rho_0}\right)^{\gamma} - \mathcal{B}$$

$\mathcal{A}(S) = e^{[(S - S_0)/C_V]}$, S, C_V: specific entropy & heat at constant volume

Barotropic 2-Phase Flow (Cont.)

- Transport equations for material quantities γ, $\mathcal{B}$, & ρ_0
  - γ-based equations

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\gamma-1}\right) + U\frac{\partial}{\partial \xi}\left(\frac{1}{\gamma-1}\right) + V\frac{\partial}{\partial \eta}\left(\frac{1}{\gamma-1}\right) = 0$$

$$\frac{\partial}{\partial \tau}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right) + U\frac{\partial}{\partial \xi}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right) + V\frac{\partial}{\partial \eta}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right) = 0$$

$$\frac{\partial}{\partial \tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right) + \frac{\partial}{\partial \xi}\left(J\frac{\mathcal{B}}{\rho_0}\rho U\right) + \frac{\partial}{\partial \eta}\left(J\frac{\mathcal{B}}{\rho_0}\rho V\right) = 0$$

Barotropic 2-Phase Flow (Cont.)

- Transport equations for material quantities γ, $\mathcal{B}$, & ρ_0
  - γ-based equations

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{1}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{1}{\gamma-1}\right)=0$$

$$\frac{\partial}{\partial \tau}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)=0$$

$$\frac{\partial}{\partial \tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right)+\frac{\partial}{\partial \xi}\left(J\frac{\mathcal{B}}{\rho_0}\rho U\right)+\frac{\partial}{\partial \eta}\left(J\frac{\mathcal{B}}{\rho_0}\rho V\right)=0$$

  - Above equations are derived from energy equation & make use of homogeneous equilibrium flow assumption together with mass conservation law

Barotropic 2-Phase Flow (Cont.)

- Transport equations for material quantities γ, $\mathcal{B}$, & ρ_0
  - γ-based equations

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{1}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{1}{\gamma-1}\right)=0$$

$$\frac{\partial}{\partial \tau}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)=0$$

$$\cancel{\frac{\partial}{\partial \tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right)+\frac{\partial}{\partial \xi}\left(J\frac{\mathcal{B}}{\rho_0}\rho U\right)+\frac{\partial}{\partial \eta}\left(J\frac{\mathcal{B}}{\rho_0}\rho V\right)=0}$$

    - If we ignore $J\mathcal{B}\rho/\rho_0$ term, they are essentially equations proposed by Saurel & Abgrall (SISC 1999), but are written in generalized coord.

Barotropic 2-Phase Flow (Cont.)

- Transport equations for material quantities γ, $\mathcal{B}$, & ρ_0
  - γ-based equations

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{1}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{1}{\gamma-1}\right)=0$$

$$\frac{\partial}{\partial \tau}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)=0$$

$$\frac{\partial}{\partial \tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right)+\frac{\partial}{\partial \xi}\left(J\frac{\mathcal{B}}{\rho_0}\rho U\right)+\frac{\partial}{\partial \eta}\left(J\frac{\mathcal{B}}{\rho_0}\rho V\right)=0$$

  - α-based equations

$$\frac{\partial \alpha}{\partial \tau}+U\frac{\partial \alpha}{\partial \xi}+V\frac{\partial \alpha}{\partial \eta}=0, \qquad \text{with} \quad z=\sum_{\iota=1}^{2}\alpha_\iota z_\iota, \quad z=\frac{1}{\gamma-1}\,\&\,\frac{\gamma\mathcal{B}}{\gamma-1}$$

$$\frac{\partial}{\partial \tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right)+\frac{\partial}{\partial \xi}\left(J\frac{\mathcal{B}}{\rho_0}\rho U\right)+\frac{\partial}{\partial \eta}\left(J\frac{\mathcal{B}}{\rho_0}\rho V\right)=0$$

Barotropic 2-Phase Flow (Cont.)

- Transport equations for material quantities γ, $\mathcal{B}$, & ρ_0
  - γ-based equations

$$\frac{\partial}{\partial\tau}\left(\frac{1}{\gamma-1}\right)+U\frac{\partial}{\partial\xi}\left(\frac{1}{\gamma-1}\right)+V\frac{\partial}{\partial\eta}\left(\frac{1}{\gamma-1}\right)=0$$

$$\frac{\partial}{\partial\tau}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)+U\frac{\partial}{\partial\xi}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)+V\frac{\partial}{\partial\eta}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)=0$$

$$\frac{\partial}{\partial\tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right)+\frac{\partial}{\partial\xi}\left(J\frac{\mathcal{B}}{\rho_0}\rho U\right)+\frac{\partial}{\partial\eta}\left(J\frac{\mathcal{B}}{\rho_0}\rho V\right)=0$$

  - α-based equations (Allaire *et al.* , JCP 2002)

$$\frac{\partial\alpha}{\partial\tau}+U\frac{\partial\alpha}{\partial\xi}+V\frac{\partial\alpha}{\partial\eta}=0 \qquad \text{with} \qquad z=\sum_{\iota=1}^{2}\alpha_\iota z_\iota, \qquad z=\frac{1}{\gamma-1}\,\&\,\frac{\gamma\mathcal{B}}{\gamma-1}$$

$$\frac{\partial}{\partial\tau}(J\rho_1\alpha)+\frac{\partial}{\partial\xi}(J\rho_1\alpha U)+\frac{\partial}{\partial\eta}(J\rho_1\alpha V)=0 \qquad \text{with} \qquad z=\frac{\mathcal{B}}{\rho_0}\rho$$

Barotropic 2-Phase Flow (Cont.)

- Transport equations for material quantities γ, $\mathcal{B}$, & ρ_0
  - γ-based equations

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{1}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{1}{\gamma-1}\right)=0$$

$$\frac{\partial}{\partial \tau}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)+U\frac{\partial}{\partial \xi}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)+V\frac{\partial}{\partial \eta}\left(\frac{\gamma\mathcal{B}}{\gamma-1}\right)=0$$

$$\frac{\partial}{\partial \tau}\left(J\frac{\mathcal{B}}{\rho_0}\rho\right)+\frac{\partial}{\partial \xi}\left(J\frac{\mathcal{B}}{\rho_0}\rho U\right)+\frac{\partial}{\partial \eta}\left(J\frac{\mathcal{B}}{\rho_0}\rho V\right)=0$$

  - α-based equations (Kapila *et al.* , Phys. Fluid 2001)

$$\frac{\partial \alpha}{\partial \tau}+U\frac{\partial \alpha}{\partial \xi}+V\frac{\partial \alpha}{\partial \eta}=\alpha_1\alpha_2\left(\frac{\rho_1 c_1^2-\rho_2 c_2^2}{\sum_{k=1}^{2}\alpha_k\rho_k c_k^2}\right)\nabla\cdot\vec{u}$$

$\cdots$ will not be discussed here

Barotropic & Non-Barotropic Flow

- Equations of state
  - Fluid component 1: Tait EOS

$$p(\rho) = (p_0 + \mathcal{B}) \left(\frac{\rho}{\rho_0} \right)^{\gamma} - \mathcal{B}$$

  - Fluid component 2: Noble-Abel EOS

$$p(\rho, \rho e) = \left(\frac{\gamma - 1}{1 - b\rho} \right) \rho e \qquad (0 \leq b \leq 1/\rho)$$

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$$p(\rho, \rho e) = \left(\frac{\gamma - 1}{1 - b\rho}\right) \rho e \qquad (0 \le b \le 1/\rho)$$

  - Mixture pressure law (Shyue, Shock Waves 2006)

$$p(\rho, \rho e) = \begin{cases} (p_0 + \mathcal{B}) \left(\dfrac{\rho}{\rho_0}\right)^{\gamma} - \mathcal{B} & \text{if} \quad \alpha = 1 \qquad (\text{fluid } 1) \\ \left(\dfrac{\gamma - 1}{1 - b\rho}\right) (\rho e - \mathcal{B}) - \mathcal{B} & \text{if} \quad \alpha \ne 1 \end{cases}$$

Baro. & Non-Baro. Flow (Cont.)

- Equations of state
  - Fluid component 1: Tait EOS

$$p(V) = \mathcal{A}(S_0)\,(p_0 + \mathcal{B})\left(\frac{V_0}{V}\right)^{\gamma} - \mathcal{B}, \qquad V = 1/\rho$$

  - Fluid component 2: Noble-Abel EOS

$$p(V,S) = \mathcal{A}(S)p_0\left(\frac{V_0 - b}{V - b}\right)^{\gamma}$$

  - Mixture pressure law

$$p(V,S) = \mathcal{A}(S)\,(p_0 + \mathcal{B})\left(\frac{V_0 - b}{V - b}\right)^{\gamma} - \mathcal{B}$$

Baro. & Non-Baro. Flow (Cont.)

- Equations of state
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$$p(V, S) = \mathcal{A}(S) p_0 \left(\frac{V_0 - b}{V - b} \right)^{\gamma}$$

  - Mixture pressure law

$$p(V, S) = \mathcal{A}(S)\,(p_0 + \mathcal{B}) \left(\frac{V_0 - b}{V - b} \right)^{\gamma} - \mathcal{B}$$

variant form of $$p(\rho, \rho e) = \left(\frac{\gamma - 1}{1 - b\rho} \right) (\rho e - \mathcal{B}) - \mathcal{B}$$

Baro. & Non-Baro. Flow (Cont.)

- Transport equations for material quantities γ, b, & $\mathcal{B}$
  - α-based equations

$$\frac{\partial \alpha}{\partial \tau} + U \frac{\partial \alpha}{\partial \xi} + V \frac{\partial \alpha}{\partial \eta} = 0$$

$$\frac{\partial}{\partial \tau}\left(J\rho_1\alpha\right) + \frac{\partial}{\partial \xi}\left(J\rho_1\alpha U\right) + \frac{\partial}{\partial \eta}\left(J\rho_1\alpha V\right) = 0$$

with $z = \sum_{\iota=1}^{2} \alpha_\iota z_\iota$, $z = \frac{1}{\gamma-1}$, $\frac{b\rho}{\gamma-1}$, & $\frac{\gamma-b\rho}{\gamma-1}\mathcal{B}$

Baro. & Non-Baro. Flow (Cont.)

- Transport equations for material quantities γ, b, & $\mathcal{B}$
  - α-based equations

$$\frac{\partial \alpha}{\partial \tau} + U \frac{\partial \alpha}{\partial \xi} + V \frac{\partial \alpha}{\partial \eta} = 0$$

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with $\boxed{z = \sum_{\iota=1}^{2} \alpha_\iota z_\iota, \quad z = \frac{1}{\gamma-1}, \quad \frac{b\rho}{\gamma-1}, \quad \& \ \frac{\gamma-b\rho}{\gamma-1}\mathcal{B}}$

Note: $$\frac{1-b\rho}{\gamma-1}\, p + \frac{\gamma-b\rho}{\gamma-1}\mathcal{B} = \rho e = \sum_{\iota=1}^{2} \alpha_\iota \rho_\iota e_\iota$$

$$= \sum_{\iota=1}^{2} \alpha_\iota \left(\frac{1-b_\iota \rho_\iota}{\gamma_\iota - 1}\, p_\iota + \frac{\gamma_\iota - b_\iota \rho_\iota}{\gamma_\iota - 1} \mathcal{B}_\iota \right)$$

Baro. & Non-Baro. Flow (Cont.)

- Transport equations for material quantities γ, b, & $\mathcal{B}$
  - α-based equations

$$\frac{\partial \alpha}{\partial \tau} + U \frac{\partial \alpha}{\partial \xi} + V \frac{\partial \alpha}{\partial \eta} = 0$$

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Multifluid Model

With $(x_\tau, y_\tau) = h_0(u, v)$ & sample EOS described above, our α-based model for multifluid flow is

$$\frac{\partial}{\partial \tau}\begin{pmatrix} J\rho \\ J\rho u \\ J\rho v \\ JE \\ x_\xi \\ y_\xi \\ x_\eta \\ y_\eta \\ J\rho_1\alpha \end{pmatrix} + \frac{\partial}{\partial \xi}\begin{pmatrix} J\rho U \\ J\rho u U + y_\eta p \\ J\rho v U - x_\eta p \\ JEU + (y_\eta u - x_\eta v)p \\ -h_0 u \\ -h_0 v \\ 0 \\ 0 \\ J\rho_1 \alpha U \end{pmatrix} + \frac{\partial}{\partial \eta}\begin{pmatrix} J\rho V \\ J\rho u V - y_\xi p \\ J\rho v V + x_\xi p \\ JEV + (x_\xi v - y_\xi u)p \\ 0 \\ 0 \\ -h_0 u \\ -h_0 v \\ J\rho_1 \alpha V \end{pmatrix} = \tilde{\psi}$$

$$\frac{\partial \alpha}{\partial \tau} + U\frac{\partial \alpha}{\partial \xi} + V\frac{\partial \alpha}{\partial \eta} = 0$$

Multifluid Model (Cont.)

For convenience, our multifluid model is written into

$$\frac{\partial q}{\partial \tau} + f\left(\frac{\partial}{\partial \xi}, q, \Xi\right) + g\left(\frac{\partial}{\partial \eta}, q, \Xi\right) = \tilde{\psi}$$

with

$$\begin{aligned}
q &= [J\rho, J\rho u, J\rho v, JE, x_\xi, y_\xi, x_\eta, y_\eta, J\rho_1\alpha, \alpha]^T \\
f &= \left[\frac{\partial}{\partial \xi}(J\rho U), \frac{\partial}{\partial \xi}(J\rho u U + y_\eta p), \frac{\partial}{\partial \xi}(J\rho v U - x_\eta p), \frac{\partial}{\partial \xi}(JEU + (y_\eta u - x_\eta v)p),\right. \\
&\qquad \left.\frac{\partial}{\partial \xi}(-h_0 u), \frac{\partial}{\partial \xi}(-h_0 v), 0, 0, \frac{\partial}{\partial \xi}(J\rho_1 \alpha U), U\frac{\partial \alpha}{\partial \xi}\right]^T \\
g &= \left[\frac{\partial}{\partial \eta}(J\rho V), \frac{\partial}{\partial \eta}(J\rho u V - y_\xi p), \frac{\partial}{\partial \eta}(J\rho v V + x_\xi p), \frac{\partial}{\partial \eta}(JEV + (x_\xi v - y_\xi u)p),\right. \\
&\qquad \left.0, 0, \frac{\partial}{\partial \eta}(-h_0 u), \frac{\partial}{\partial \eta}(-h_0 v), \frac{\partial}{\partial \eta}(J\rho_1 \alpha V), V\frac{\partial \alpha}{\partial \eta}\right]^T
\end{aligned}$$

Multifluid model: Remarks

- As before, under thermodyn. stability condition, our multifluid model in generalized coordinates is hyperbolic when $h_0 \neq 1$, & is weakly hyperbolic when $h_0 = 1$
- Our model system is written in quasi-conservative form with spatially varying fluxes in generalized coordinates
- Our grid system is a time-varying grid
- Extension of the model to general non-barotropic multifluid flow can be made in an analogous manner

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Numerical approximation ?

Numerical Approximation

- Equations to be solved are

$$\frac{\partial q}{\partial \tau} + f\left(\frac{\partial}{\partial \xi}, q, \Xi\right) + g\left(\frac{\partial}{\partial \eta}, q, \Xi\right) = \tilde{\psi}$$

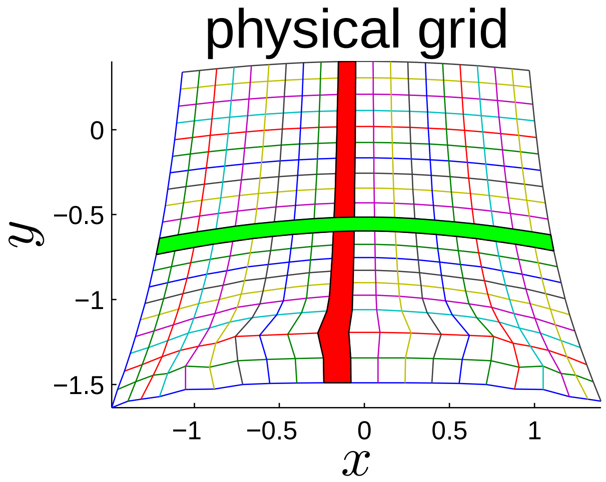
- A simple dimensional-splitting approach based on f-wave formulation of LeVeque *et al.* is used
  - Solve one-dimensional generalized Riemann problem (defined below) at each cell interfaces
  - Use resulting jumps of fluxes (decomposed into each wave family) of Riemann solution to update cell averages
  - Introduce limited jumps of fluxes to achieve high resolution

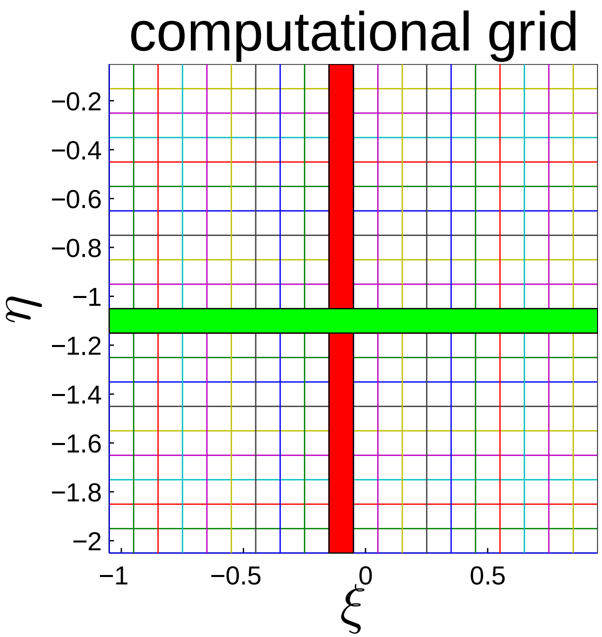
Numerical Approximation (Cont.)

Employ finite volume formulation of numerical solution

$$Q_{ij}^{n} \approx \frac{1}{\Delta\xi\Delta\eta}\int_{C_{ij}} q(\xi,\eta,\tau_n)\,dA$$

that gives approximate value of cell average of solution q over cell $C_{ij} = [\xi_i, \xi_{i+1}] \times [\eta_j, \eta_{j+1}]$ at time τ_n


physical grid


computational grid

Generalized Riemann Problem

Generalized Riemann problem of our multifluid model at cell interface $\xi_{i-1/2}$ consists of the equation

$$\frac{\partial q}{\partial \tau} + F_{i-\frac{1}{2},j}\left(\partial_\xi, q, \Xi\right) = 0$$

together with flux function

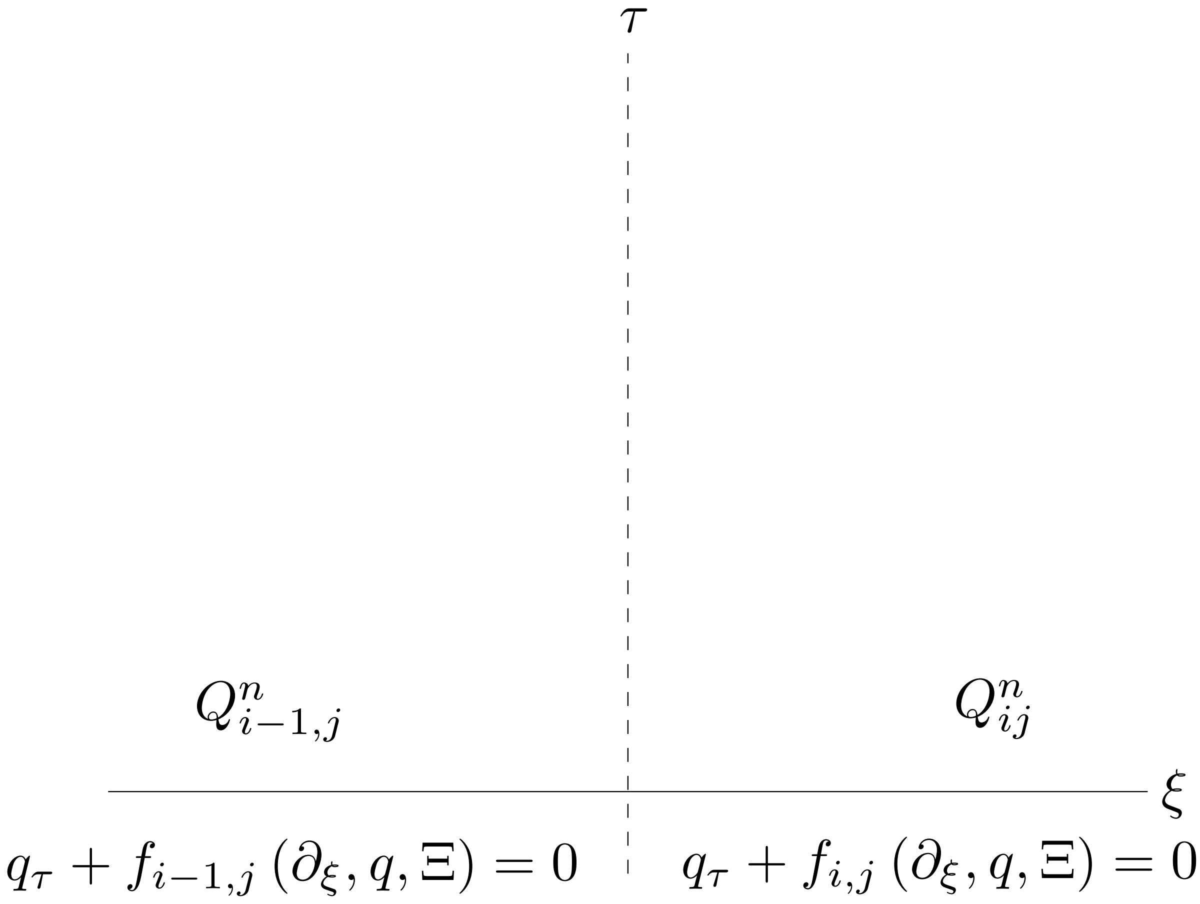
$$F_{i-\frac{1}{2},j} = \begin{cases} f_{i-1,j}\left(\partial_\xi, q, \Xi\right) & \text{for} \quad \xi < \xi_{i-1/2} \\ f_{ij}\left(\partial_\xi, q, \Xi\right) & \text{for} \quad \xi > \xi_{i-1/2} \end{cases}$$

and piecewise constant initial data

$$q(\xi, 0) = \begin{cases} Q^n_{i-1,j} & \text{for} \quad \xi < \xi_{i-1/2} \\ Q^n_{ij} & \text{for} \quad \xi > \xi_{i-1/2} \end{cases}$$

General. Riemann Problem (Cont.)

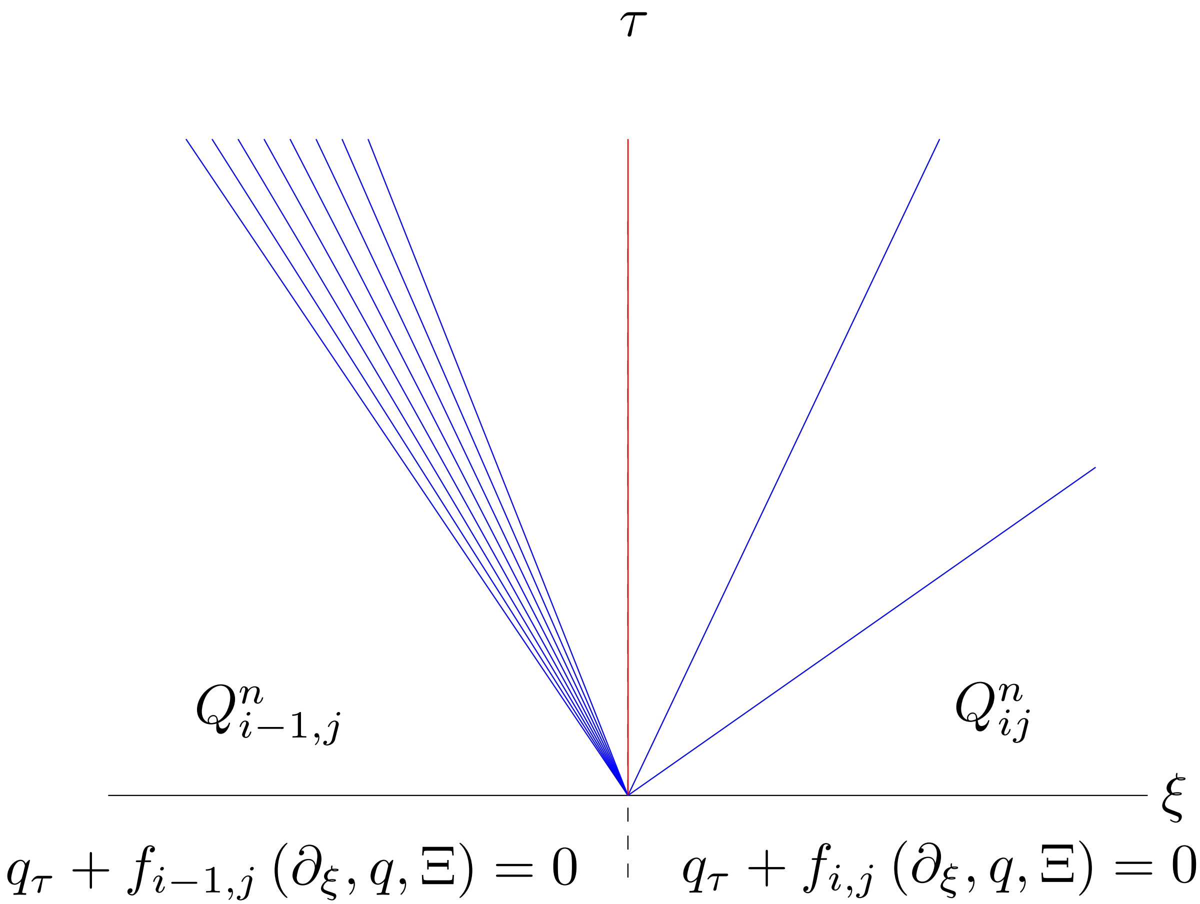
Generalized Riemann problem at time $\tau = 0$



$q_\tau + f_{i-1,j}\left(\partial_\xi, q, \Xi\right) = 0$ $\quad$ $q_\tau + f_{i,j}\left(\partial_\xi, q, \Xi\right) = 0$

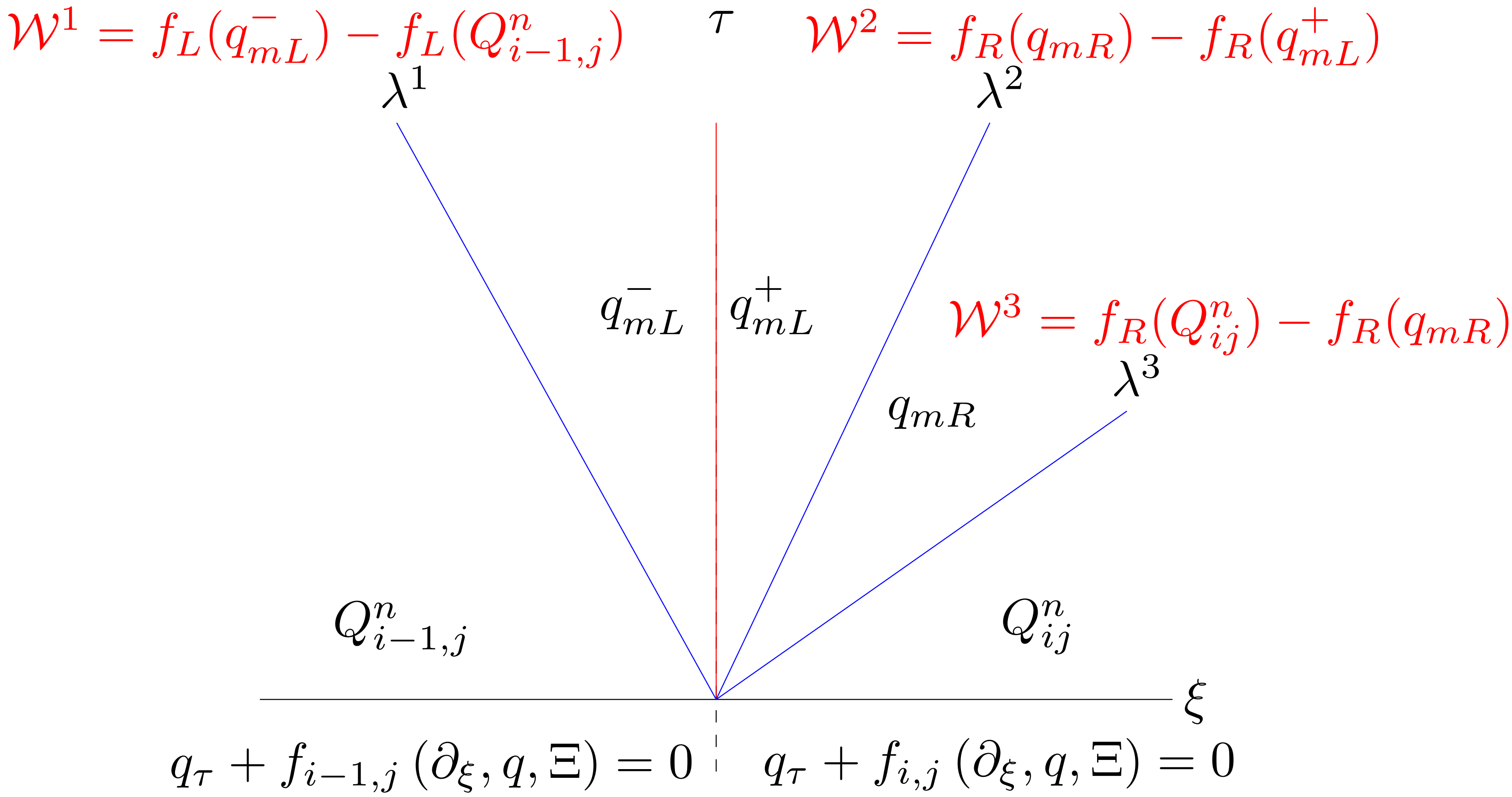
General. Riemann Problem (Cont.)

Exact generalized Riemann solution: basic structure



General. Riemann Problem (Cont.)

Shock-only approximate Riemann solution: basic structure



$\mathcal{W}^1 = f_L(q_{mL}^-) - f_L(Q_{i-1,j}^n)$

$\mathcal{W}^2 = f_R(q_{mR}) - f_R(q_{mL}^+)$

$\mathcal{W}^3 = f_R(Q_{ij}^n) - f_R(q_{mR})$

$q_\tau + f_{i-1,j}\left(\partial_\xi, q, \Xi\right) = 0$

$q_\tau + f_{i,j}\left(\partial_\xi, q, \Xi\right) = 0$

Numerical Approximation (Cont.)

Basic steps of a dimensional-splitting scheme

- ξ-sweeps: solve

$$\frac{\partial q}{\partial \tau} + f\left(\frac{\partial}{\partial \xi}, q, \Xi\right) = 0$$

  updating Q_{ij}^n to $Q_{i,j}^*$

- η-sweeps: solve

$$\frac{\partial q}{\partial \tau} + g\left(\frac{\partial}{\partial \eta}, q, \Xi\right) = 0$$

  updating Q_{ij}^* to $Q_{i,j}^{n+1}$

Numerical Approximation (Cont.)

That is to say,

- ξ-sweeps: we use

$$Q_{ij}^{*} = Q_{ij}^{n} - \frac{\Delta\tau}{\Delta\xi}\left(\mathcal{F}^{-}_{i+\frac{1}{2},j} - \mathcal{F}^{+}_{i-\frac{1}{2},j}\right) - \frac{\Delta\tau}{\Delta\xi}\left(\tilde{\mathcal{F}}_{i+\frac{1}{2},j} - \tilde{\mathcal{F}}_{i-\frac{1}{2},j}\right)$$

with $\tilde{\mathcal{F}}_{i-\frac{1}{2},j} = \frac{1}{2}\sum_{p=1}^{m_w} \text{sign}\left(\lambda^{p}_{i-\frac{1}{2},j}\right)\left(1 - \frac{\Delta\tau}{\Delta\xi}\left|\lambda^{p}_{i-\frac{1}{2},j}\right|\right)\tilde{\mathcal{W}}^{p}_{i-\frac{1}{2},j}$

- η-sweeps: we use

$$Q_{ij}^{n+1} = Q_{ij}^{*} - \frac{\Delta\tau}{\Delta\eta}\left(\mathcal{G}^{-}_{i,j+\frac{1}{2}} - \mathcal{G}^{+}_{i,j-\frac{1}{2}}\right) - \frac{\Delta\tau}{\Delta\eta}\left(\tilde{\mathcal{G}}_{i,j+\frac{1}{2}} - \tilde{\mathcal{G}}_{i,j-\frac{1}{2}}\right)$$

with $\tilde{\mathcal{G}}_{i,j-\frac{1}{2}} = \frac{1}{2}\sum_{p=1}^{m_w} \text{sign}\left(\lambda^{p}_{i,j-\frac{1}{2}}\right)\left(1 - \frac{\Delta\tau}{\Delta\eta}\left|\lambda^{p}_{i,j-\frac{1}{2}}\right|\right)\tilde{\mathcal{W}}^{p}_{i,j-\frac{1}{2}}$

Numerical Approximation (Cont.)

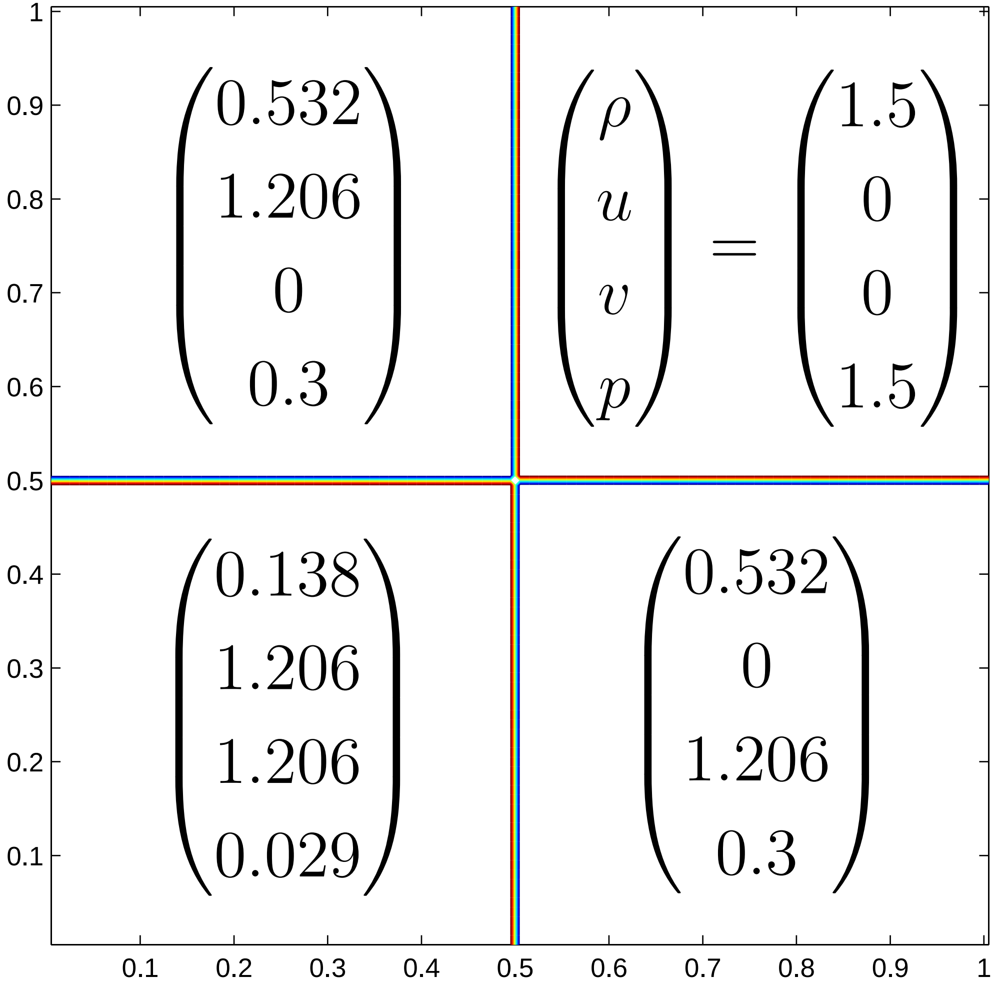
- Some care should be taken on the limited jump of fluxes $\tilde{\mathcal{W}}^p$, for $p = 2$ (contact wave), in particular to ensure correct pressure equilibrium across material interfaces
- First order or high resolution method for geometric conservation laws ? Their effect to the grid uniformity, $\ldots$

Numerical Examples

- 2D Riemann problem
- Underwater explosion
- Shock-bubble interaction
    - Helium bubble case
    - Refrigerant bubble case

2D Riemann Problem

Initial condition for 4-shock wave pattern

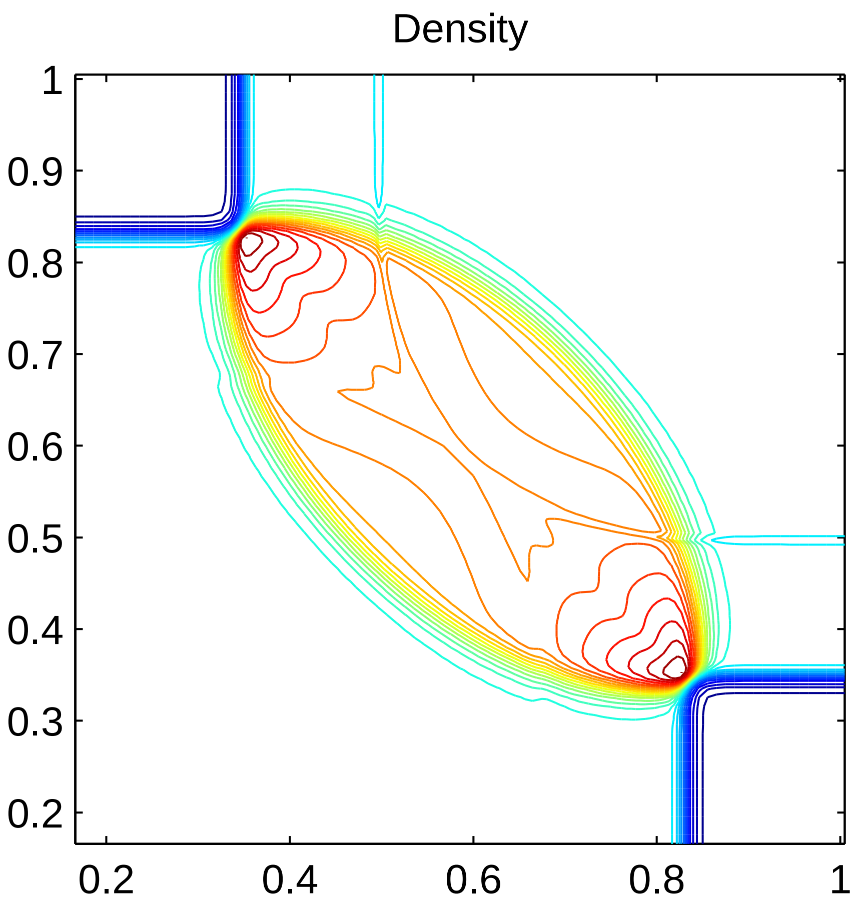


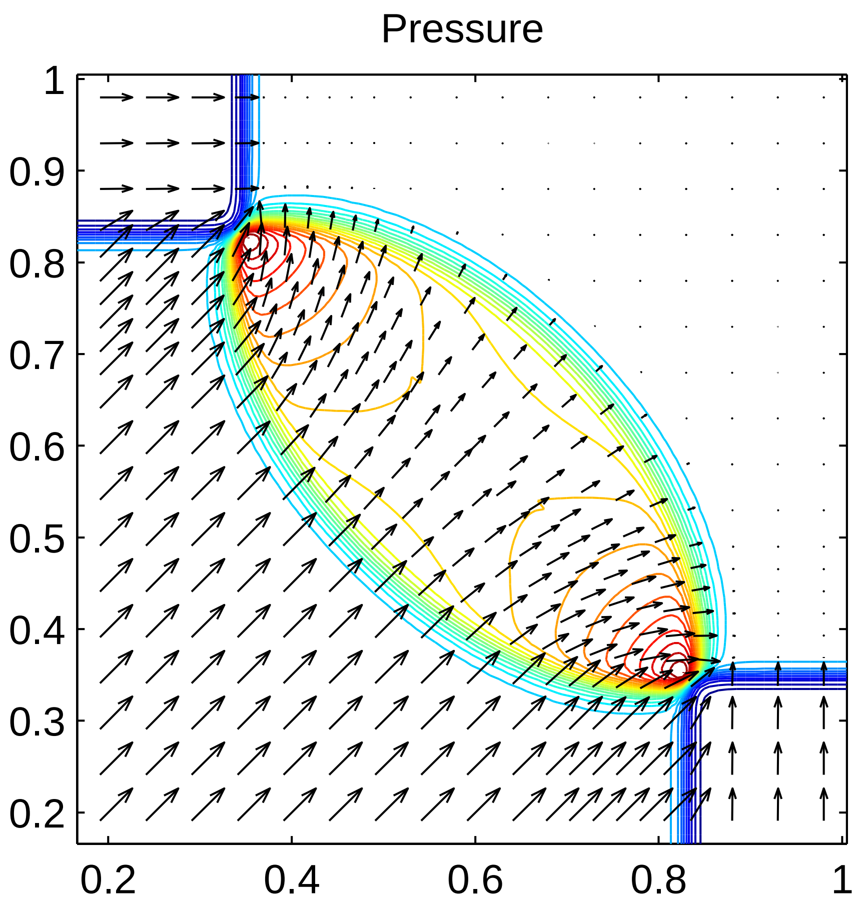
$$\begin{pmatrix}0.532\\1.206\\0\\0.3\end{pmatrix} \qquad \begin{pmatrix}\rho\\u\\v\\p\end{pmatrix} = \begin{pmatrix}1.5\\0\\0\\1.5\end{pmatrix}$$

$$\begin{pmatrix}0.138\\1.206\\1.206\\0.029\end{pmatrix} \qquad \begin{pmatrix}0.532\\0\\1.206\\0.3\end{pmatrix}$$

2D Riemann problem (Cont.)

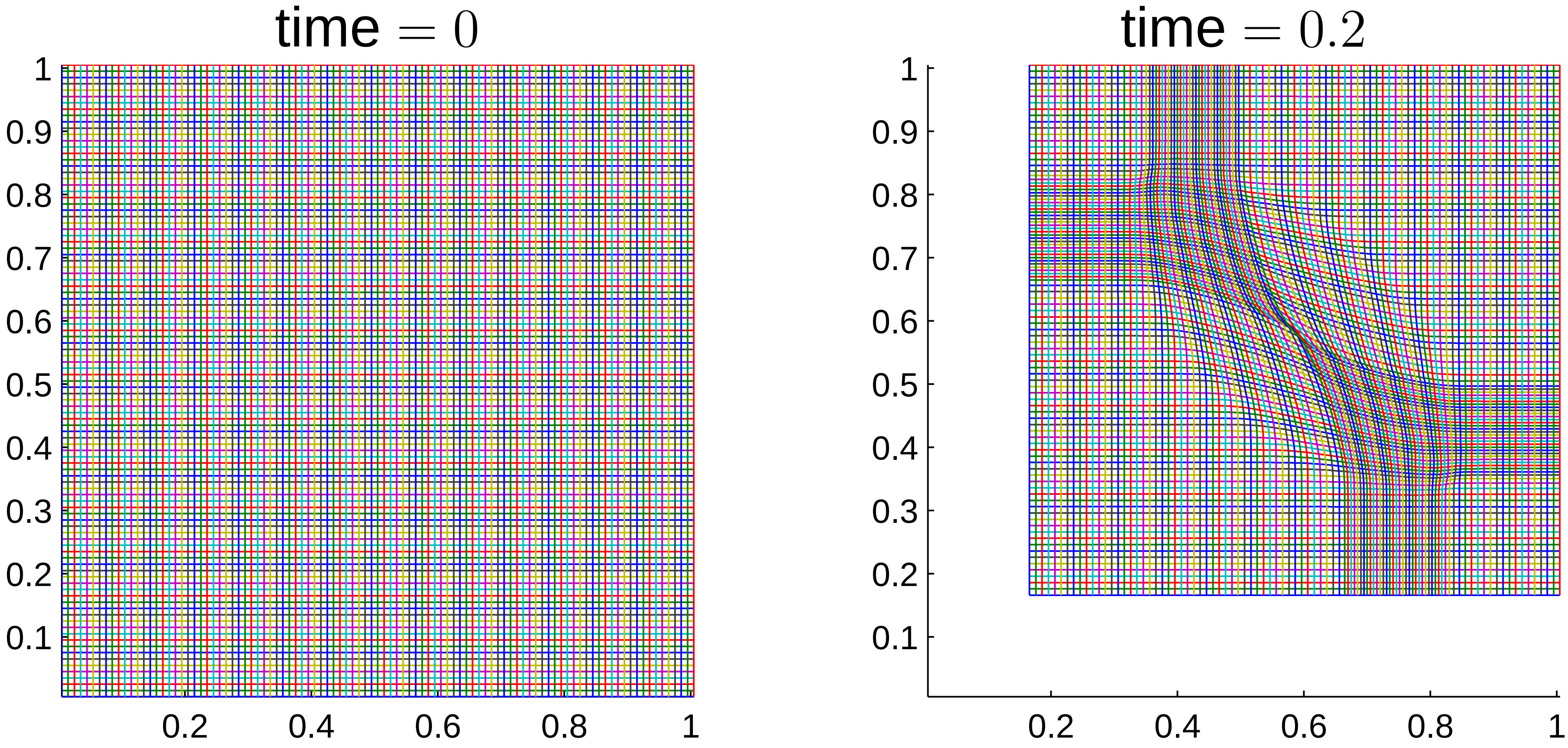
- Numerical contours for density and pressure





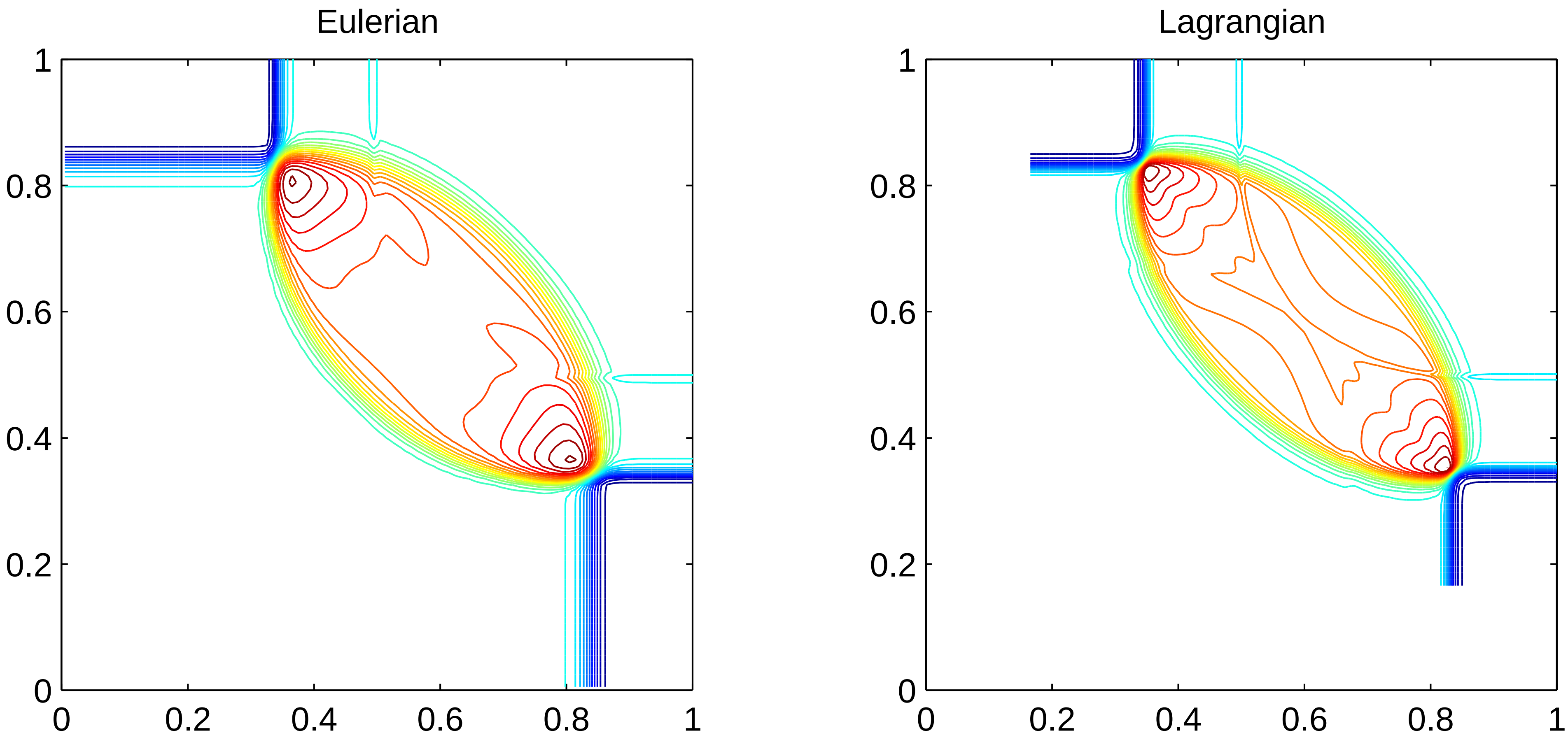
2D Riemann problem (Cont.)

- Grid system with $h_0 = 0.99$



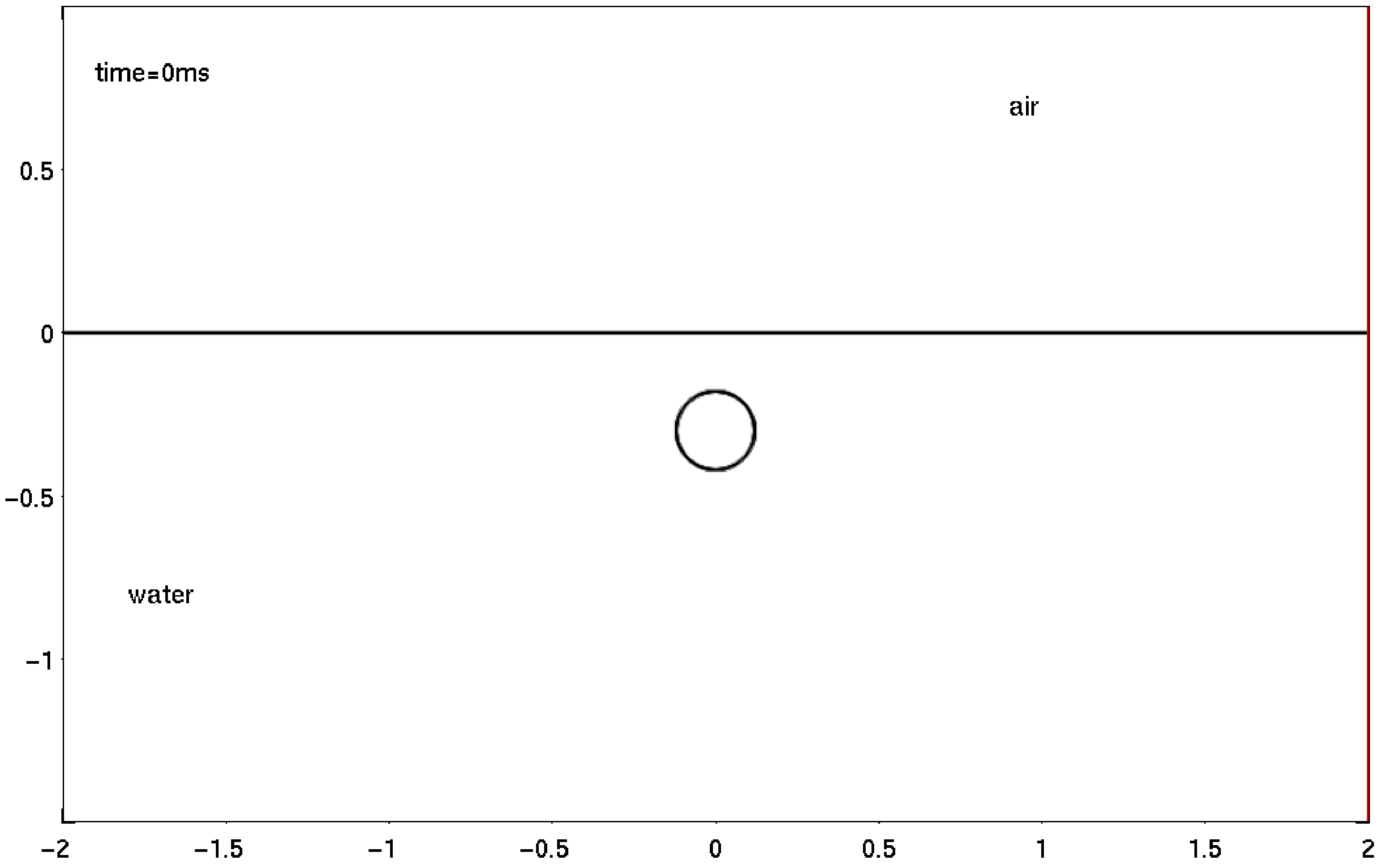
2D Riemann problem (Cont.)

- Euler vs. generalized coord.



Underwater Explosions

- Numerical schlieren images $h_0 = 0.9$, 800×500 grid



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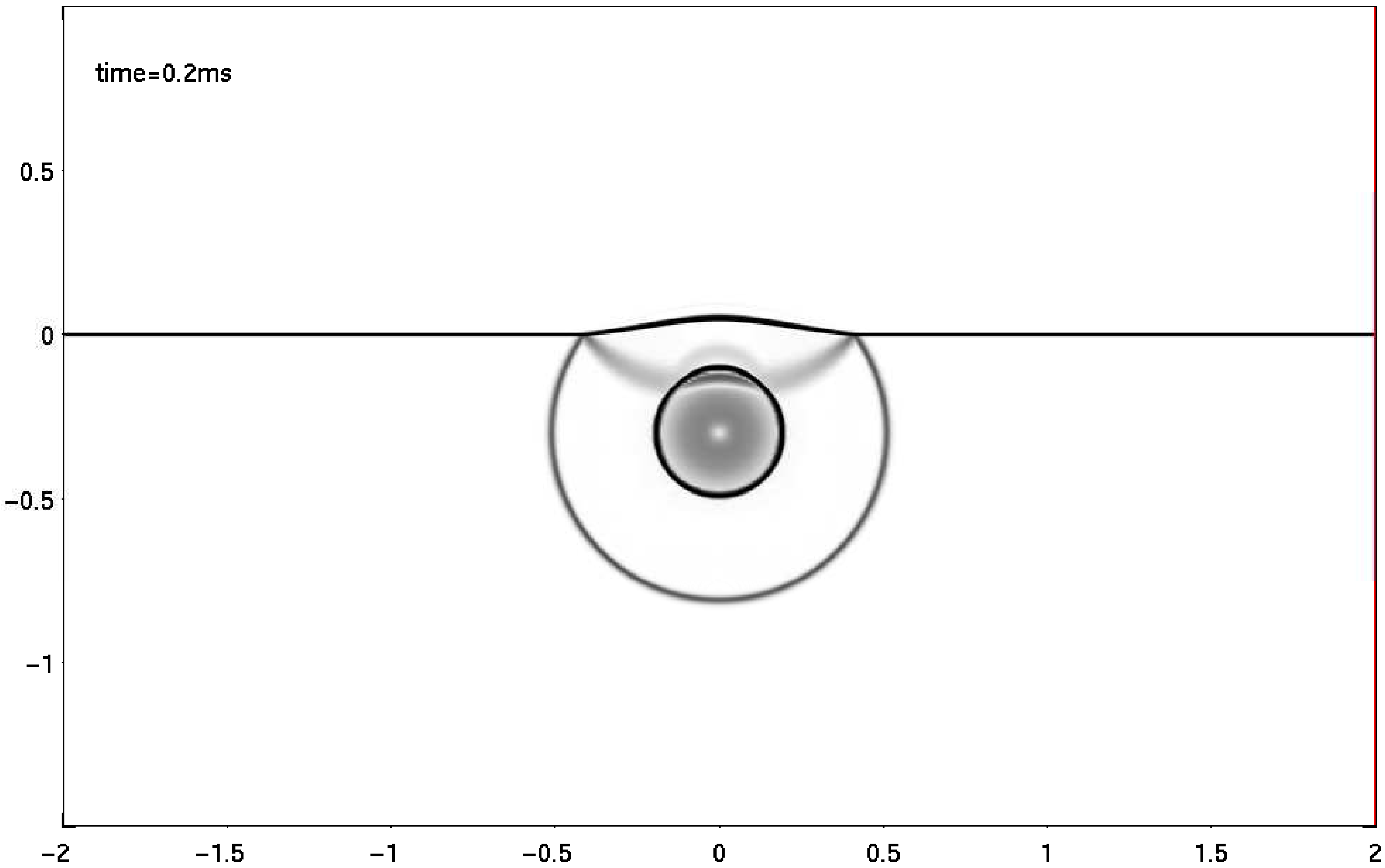
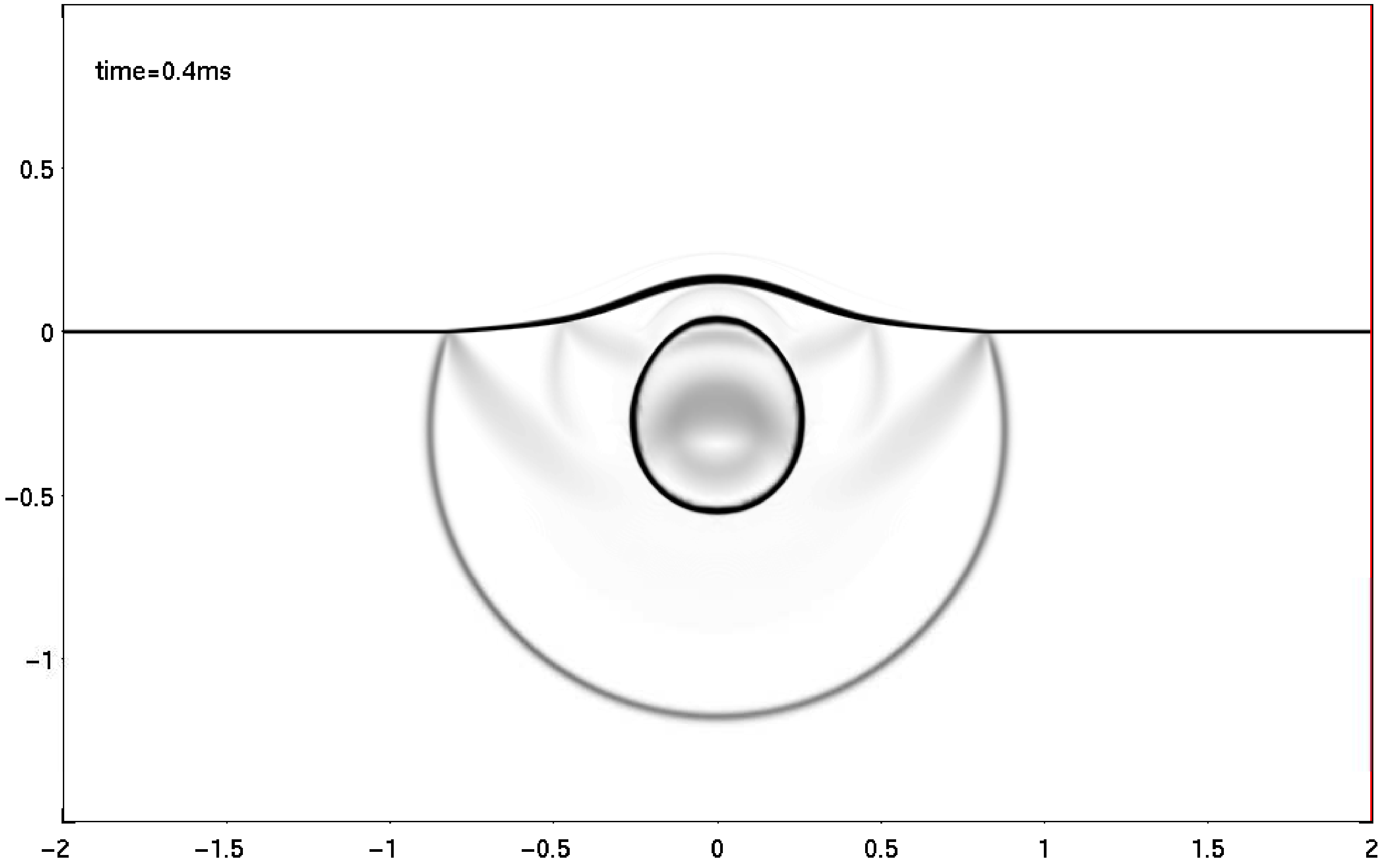


Underwater Explosions

- Numerical schlieren images $h_0 = 0.9$, 800×500 grid



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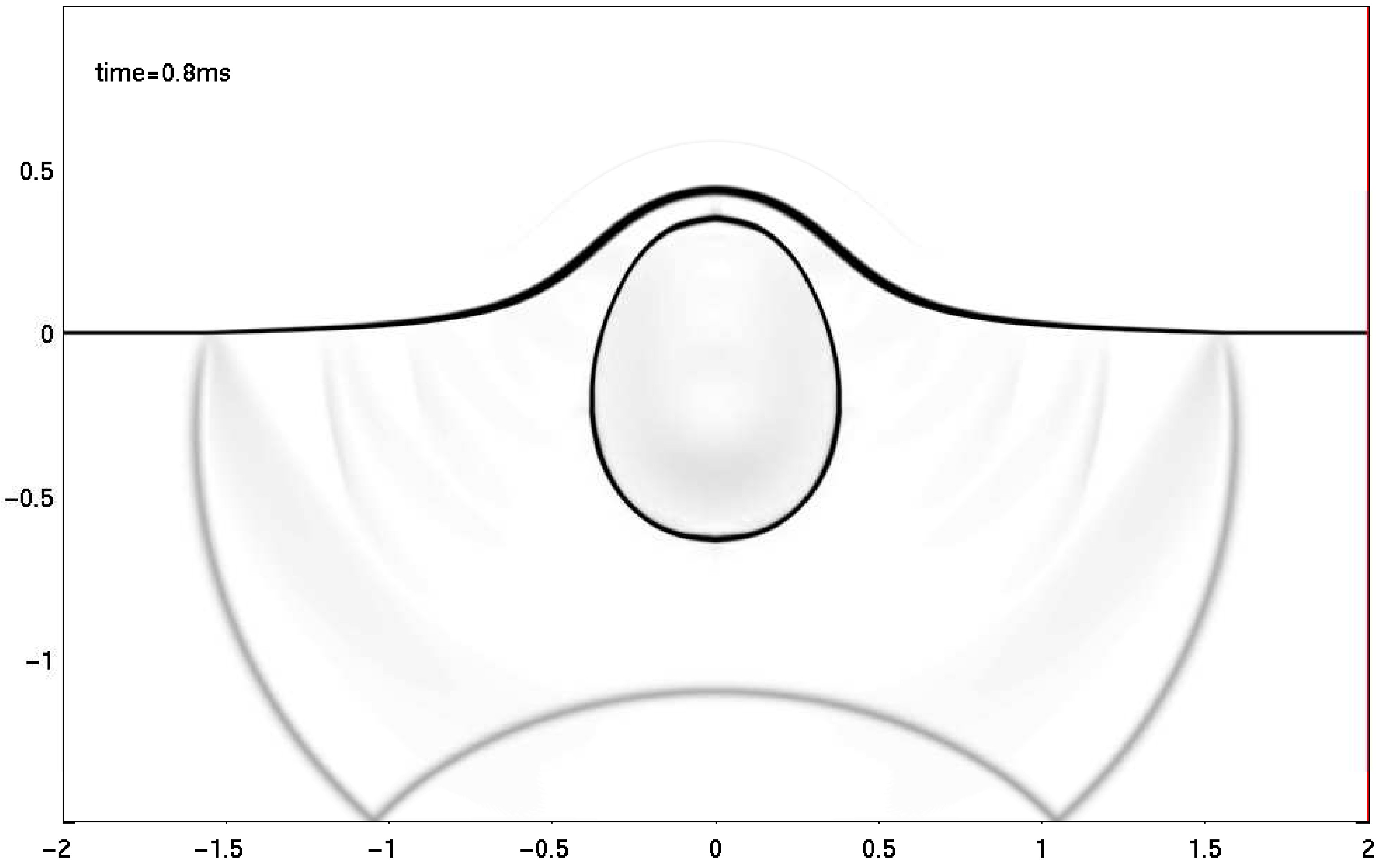

Underwater Explosions

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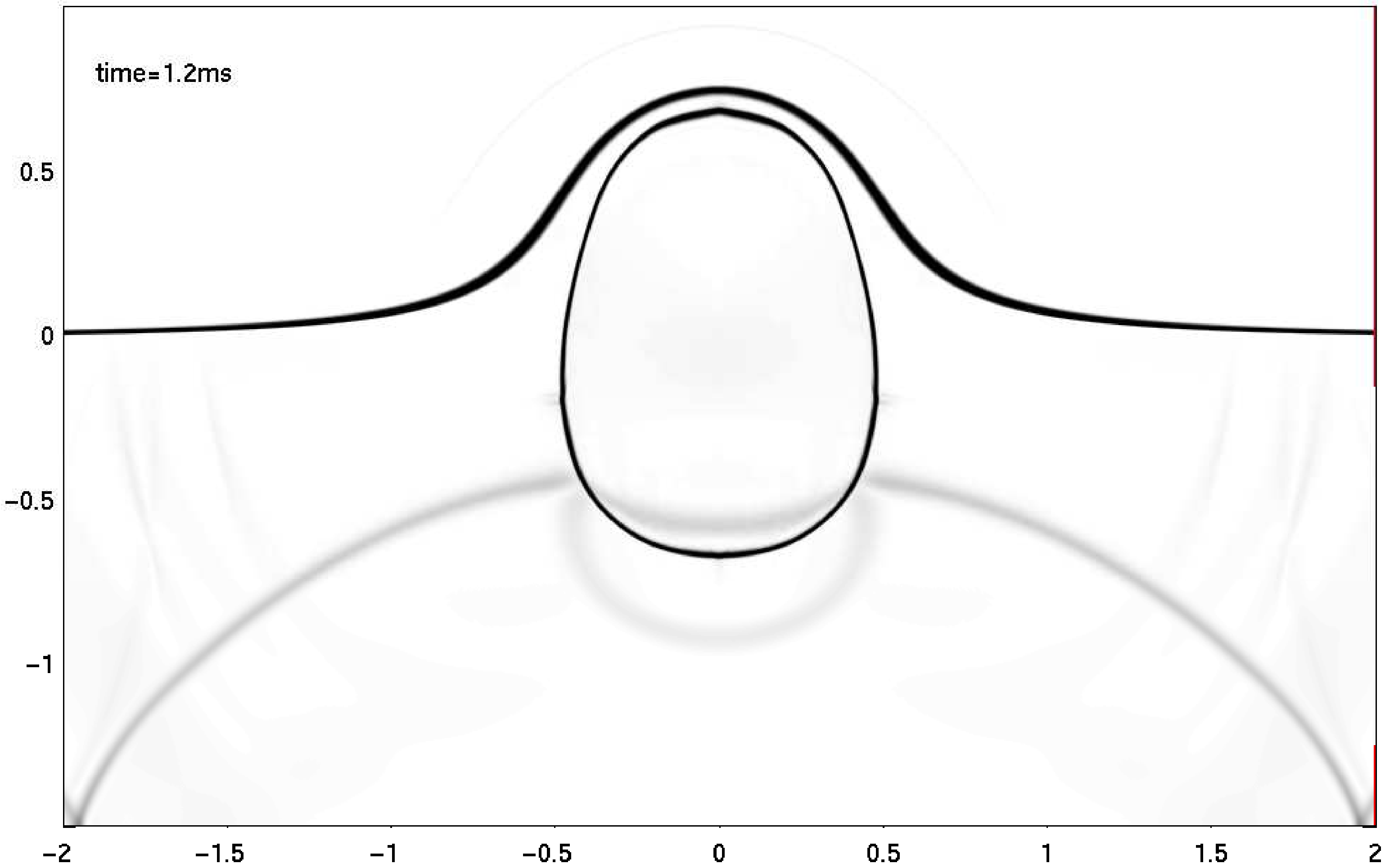
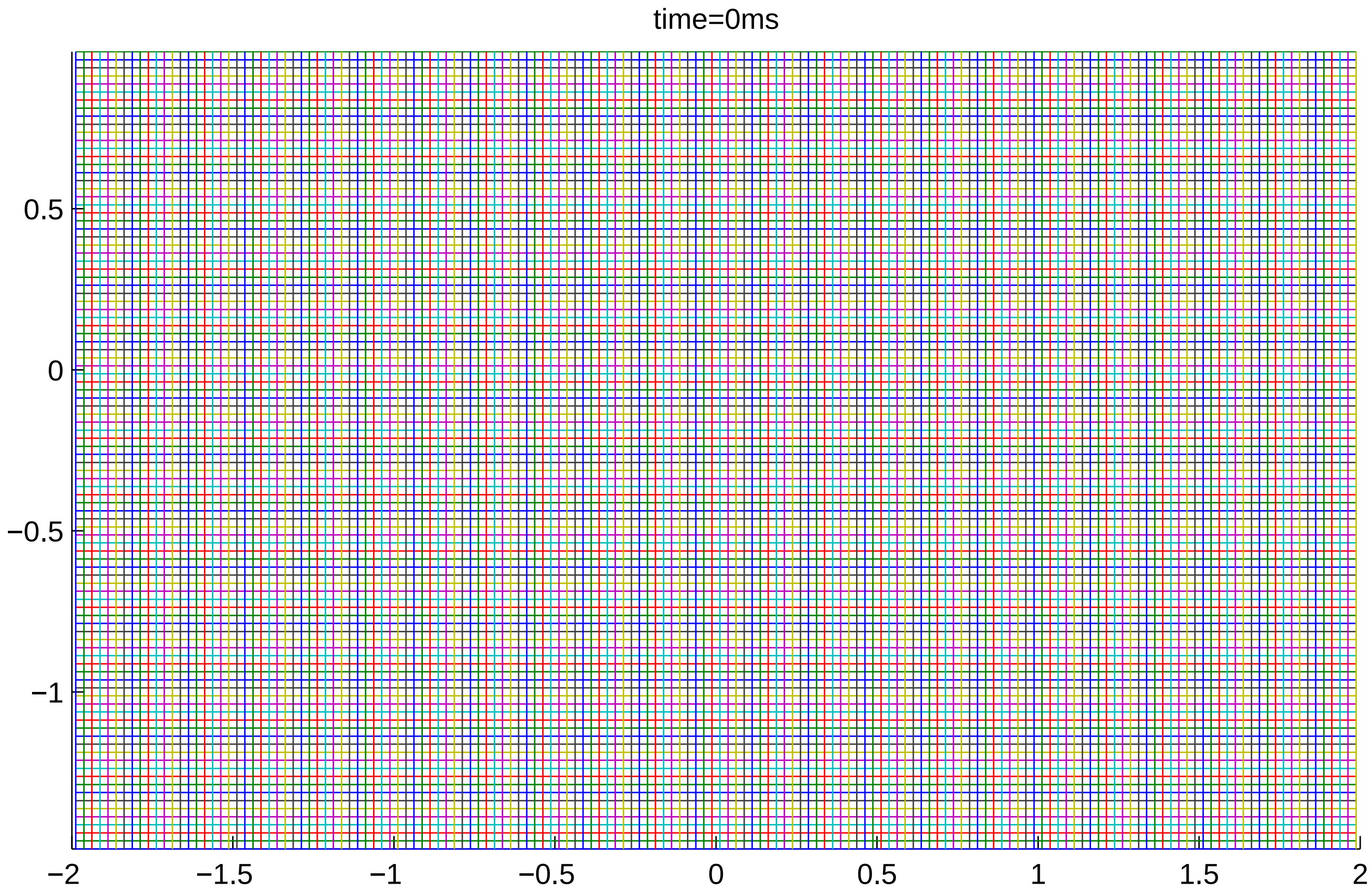


Underwater Explosions (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.9$



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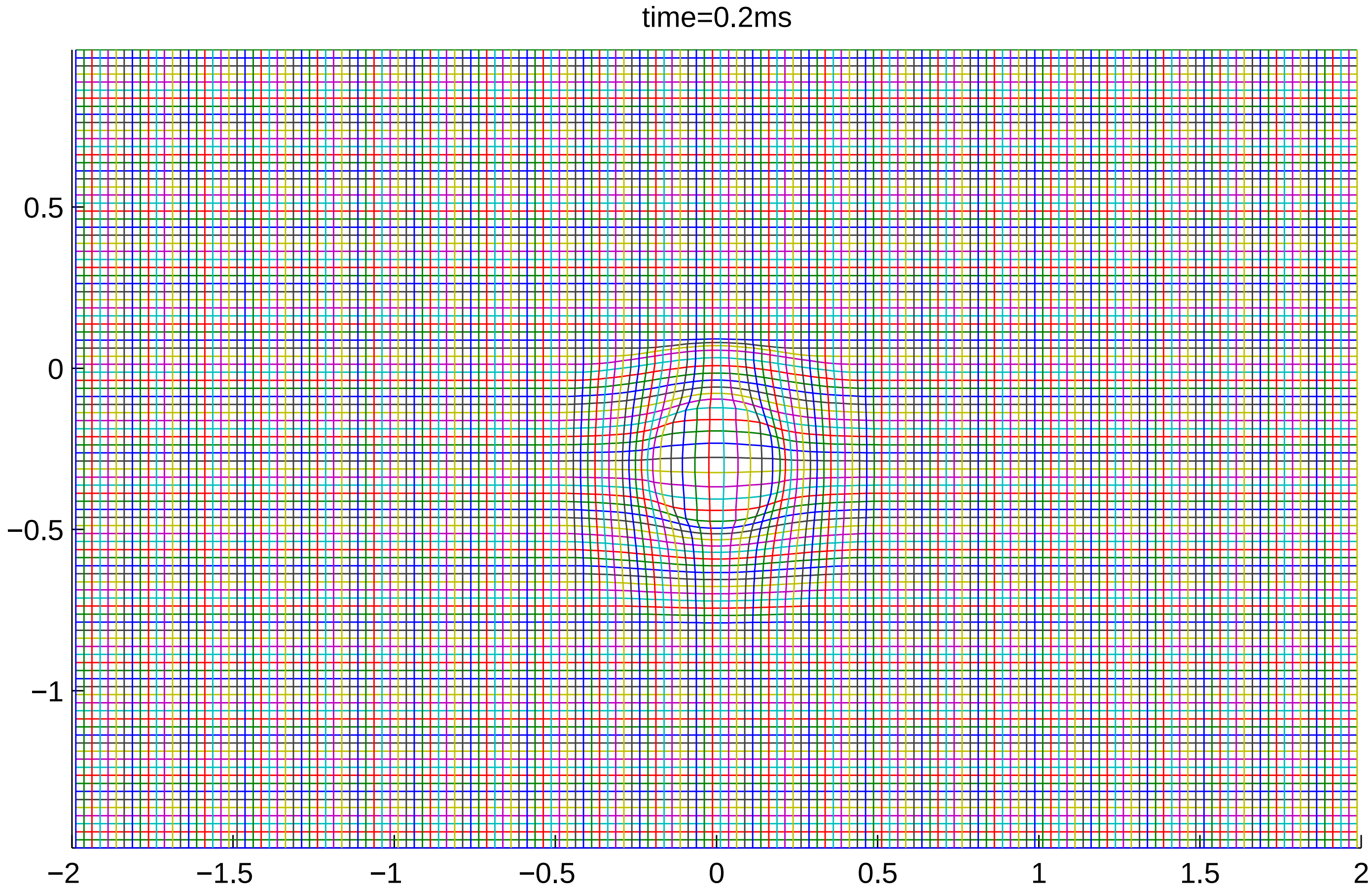

Underwater Explosions (Cont.)

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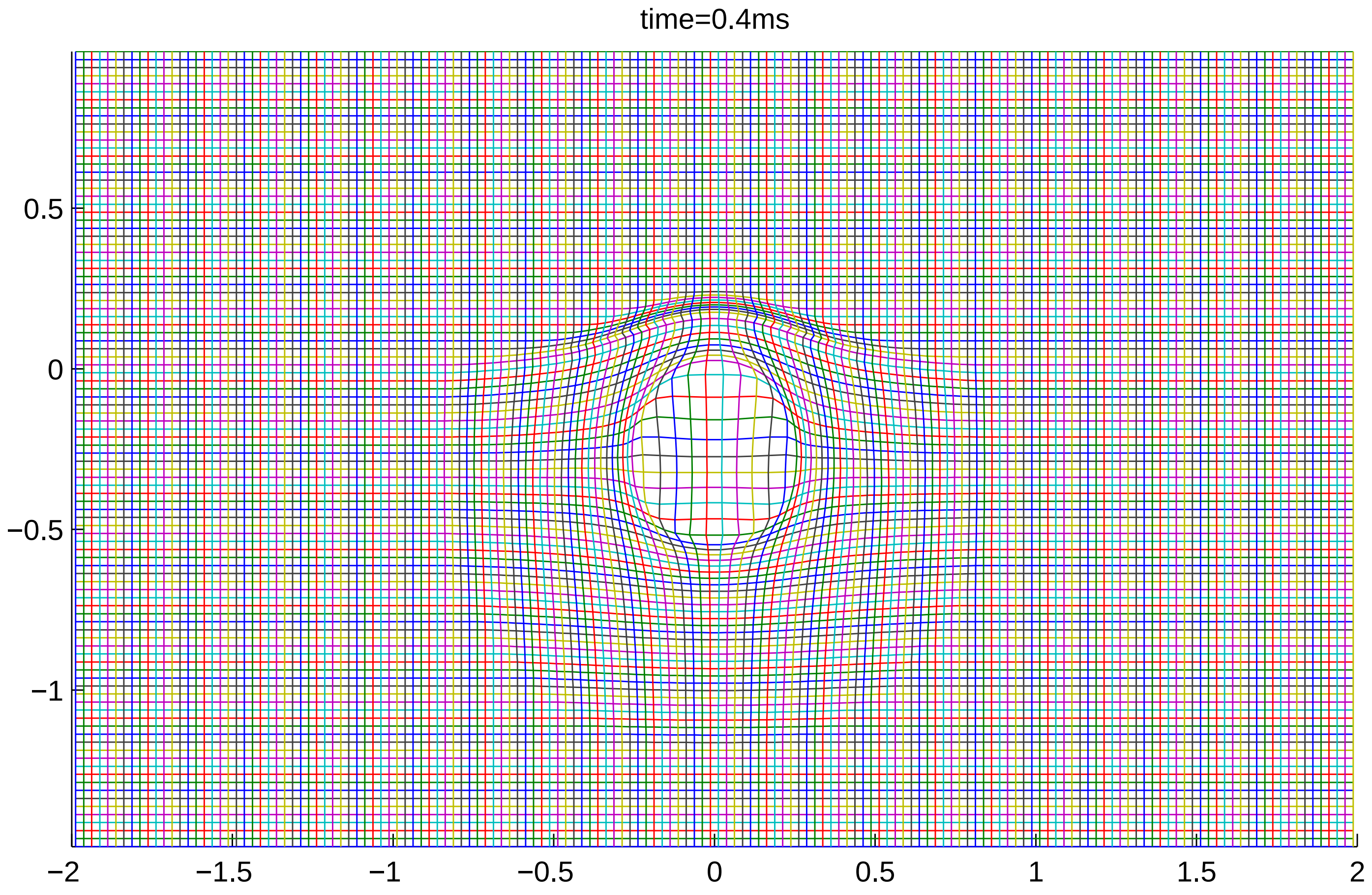

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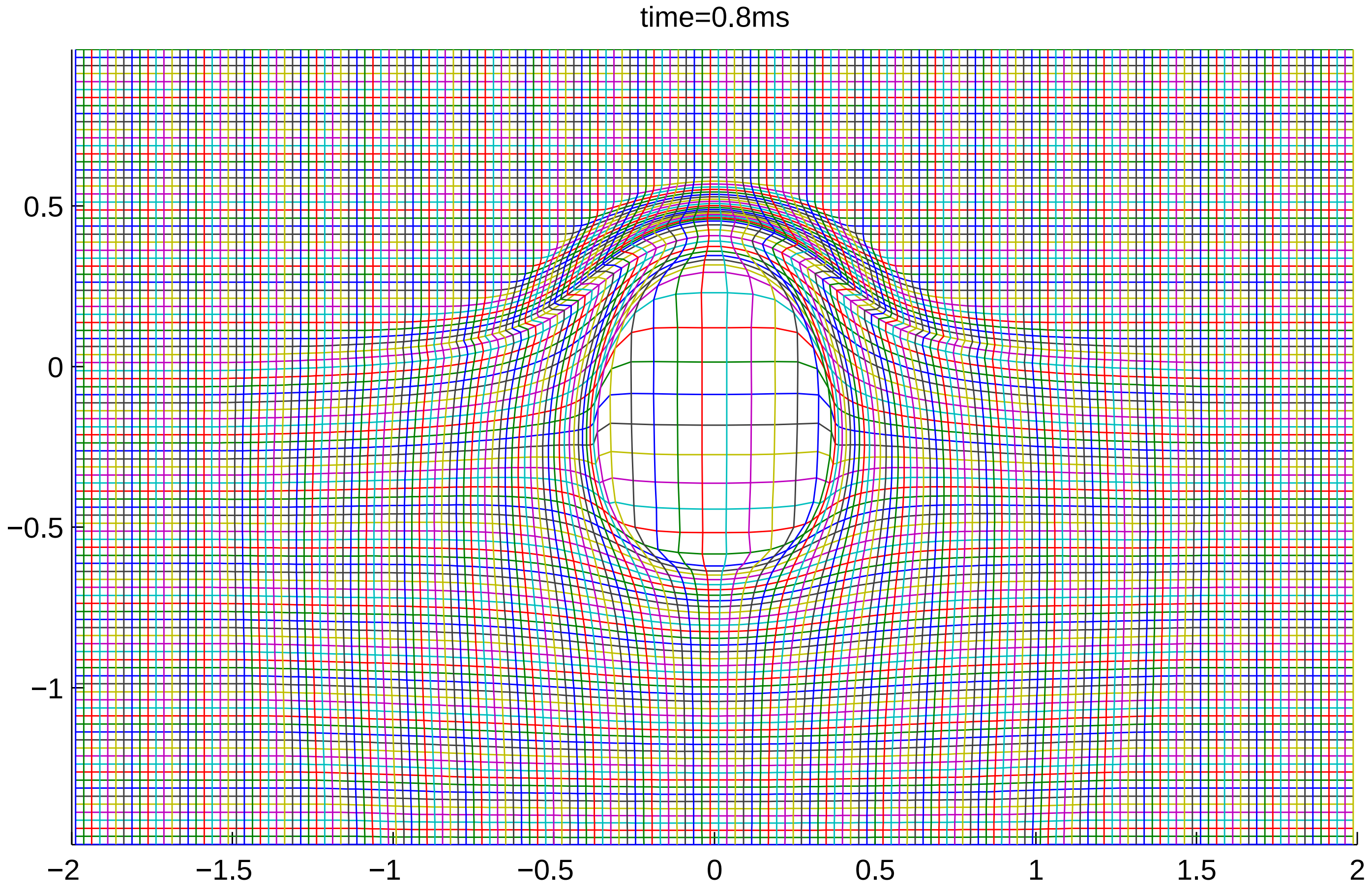

Underwater Explosions (Cont.)

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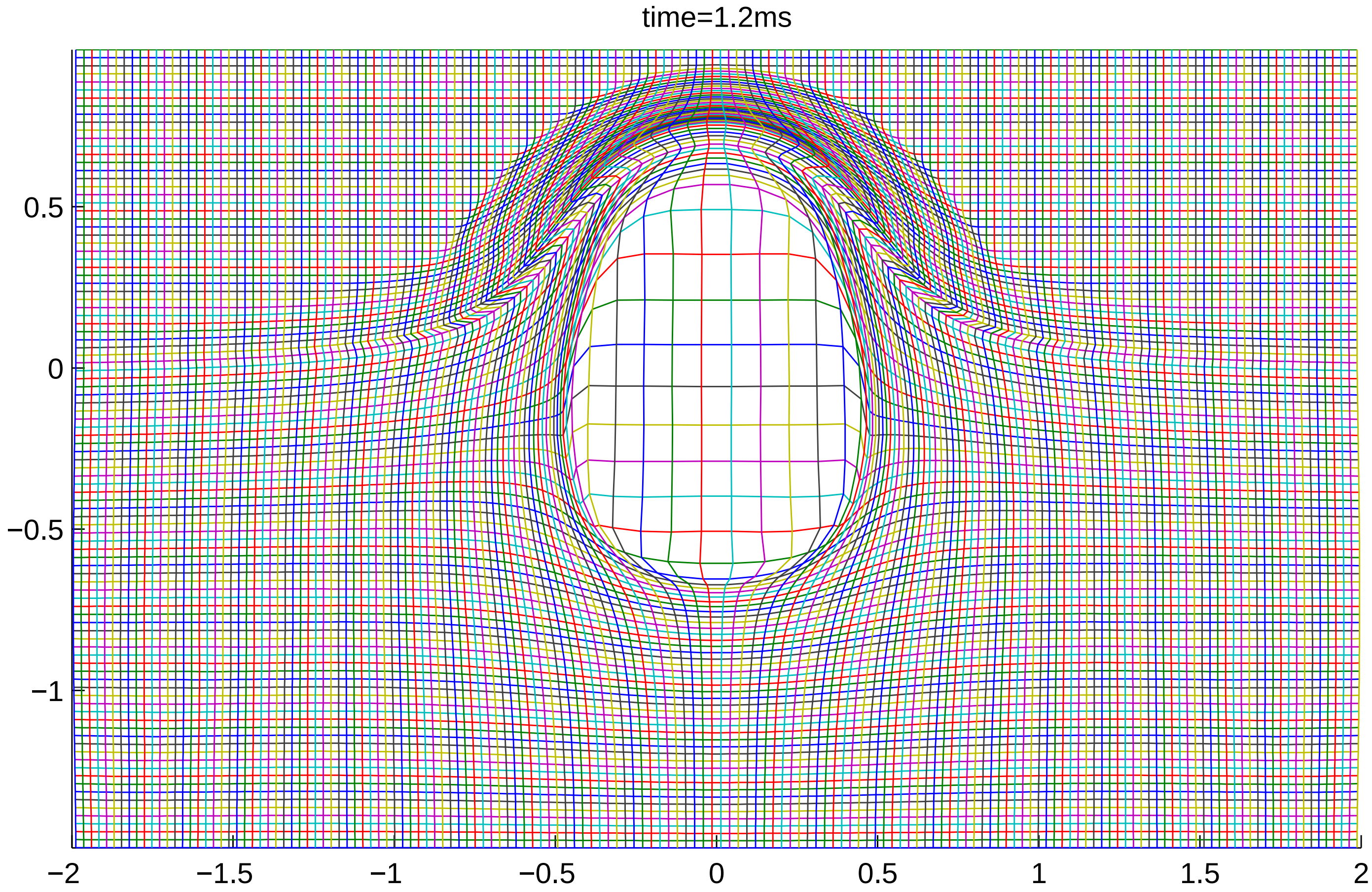
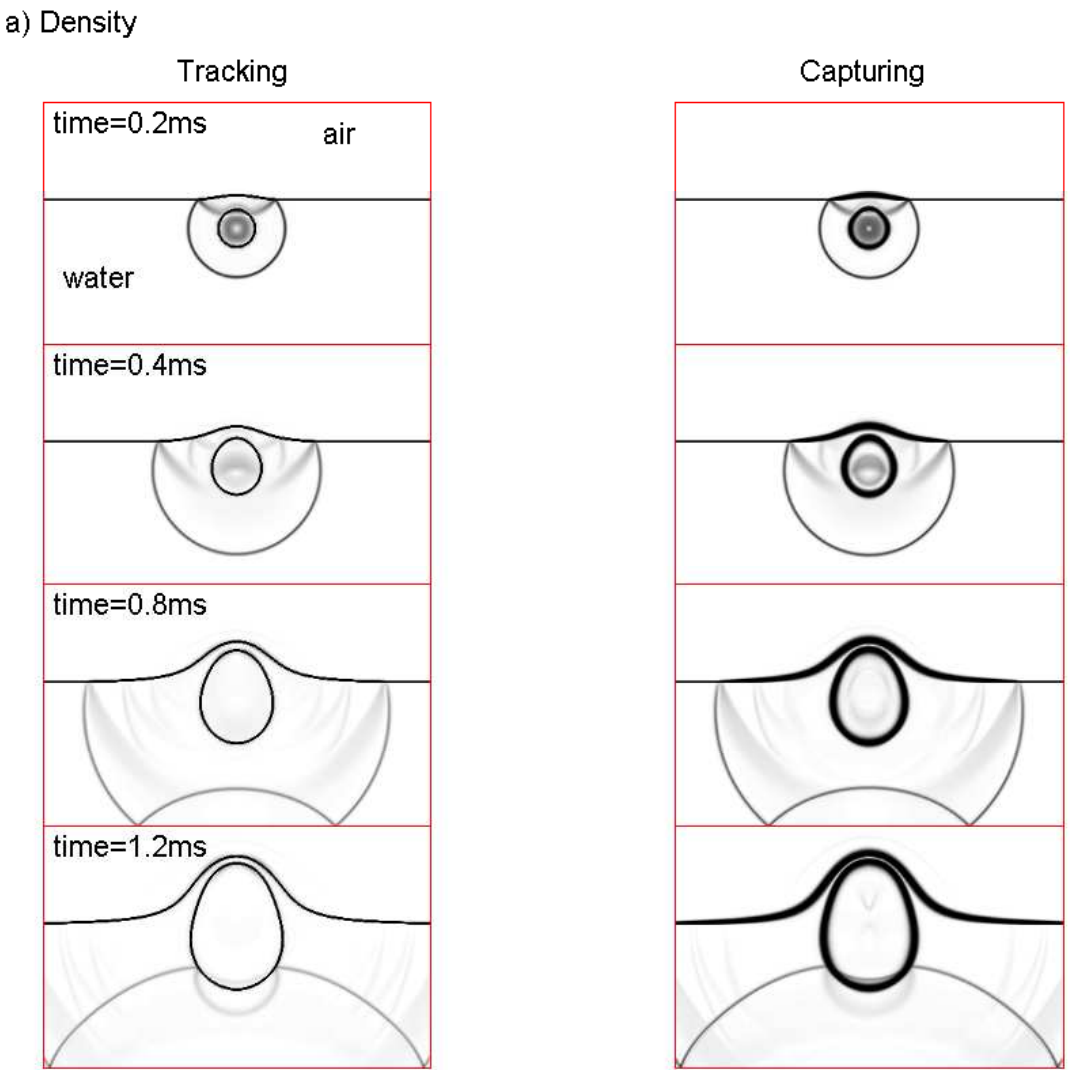


Underwater Explosions (Cont.)

- Volume tracking & interface capturing results

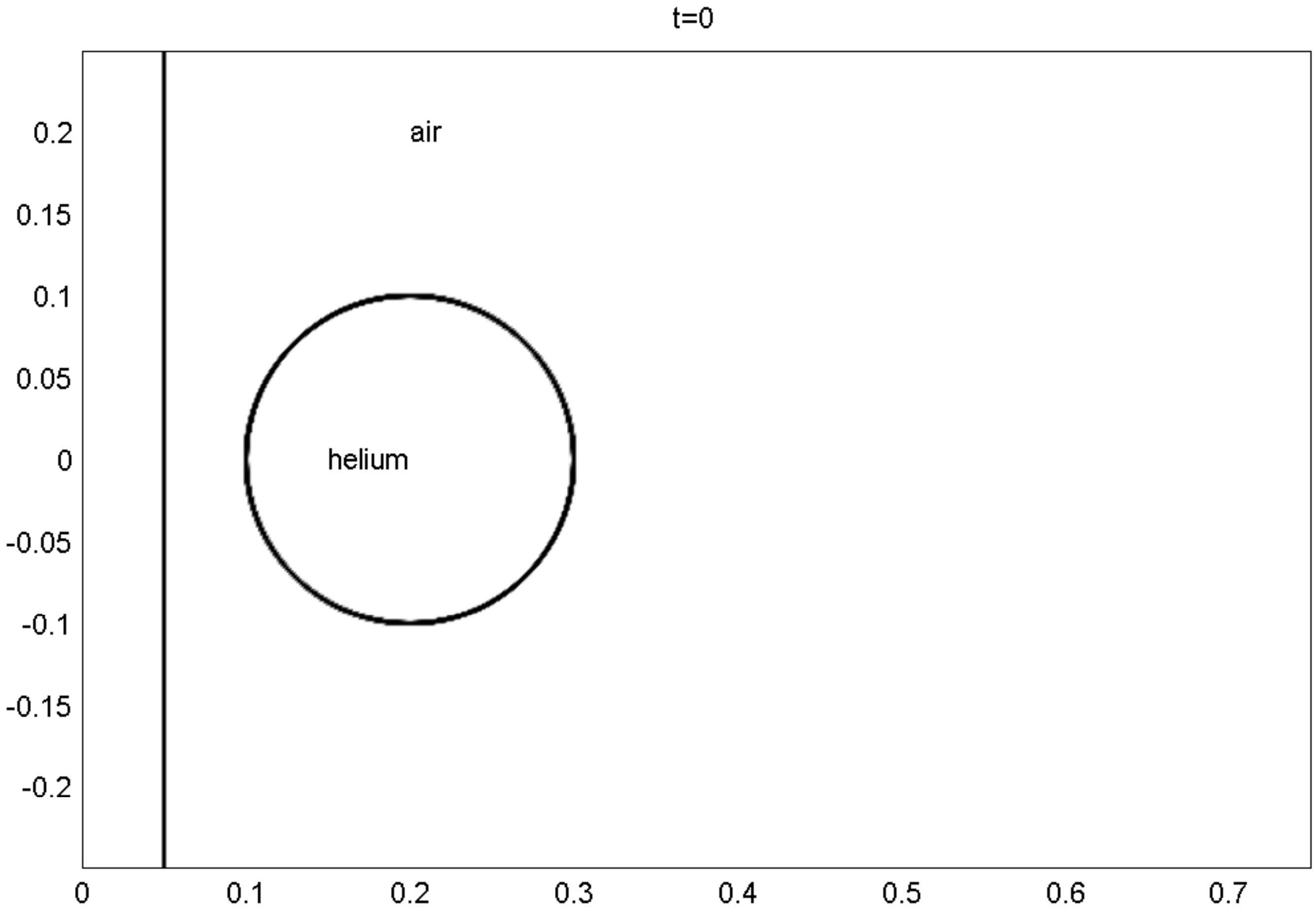


Underwater Explosions (Cont.)

- Generalized curvilinear grid: single bubble animation
- Cartesian grid: multiple bubble animation

Shock-Bubble (Helium) Interaction

- Numerical schlieren images: $h_0 = 0.5$, 600×400 grid



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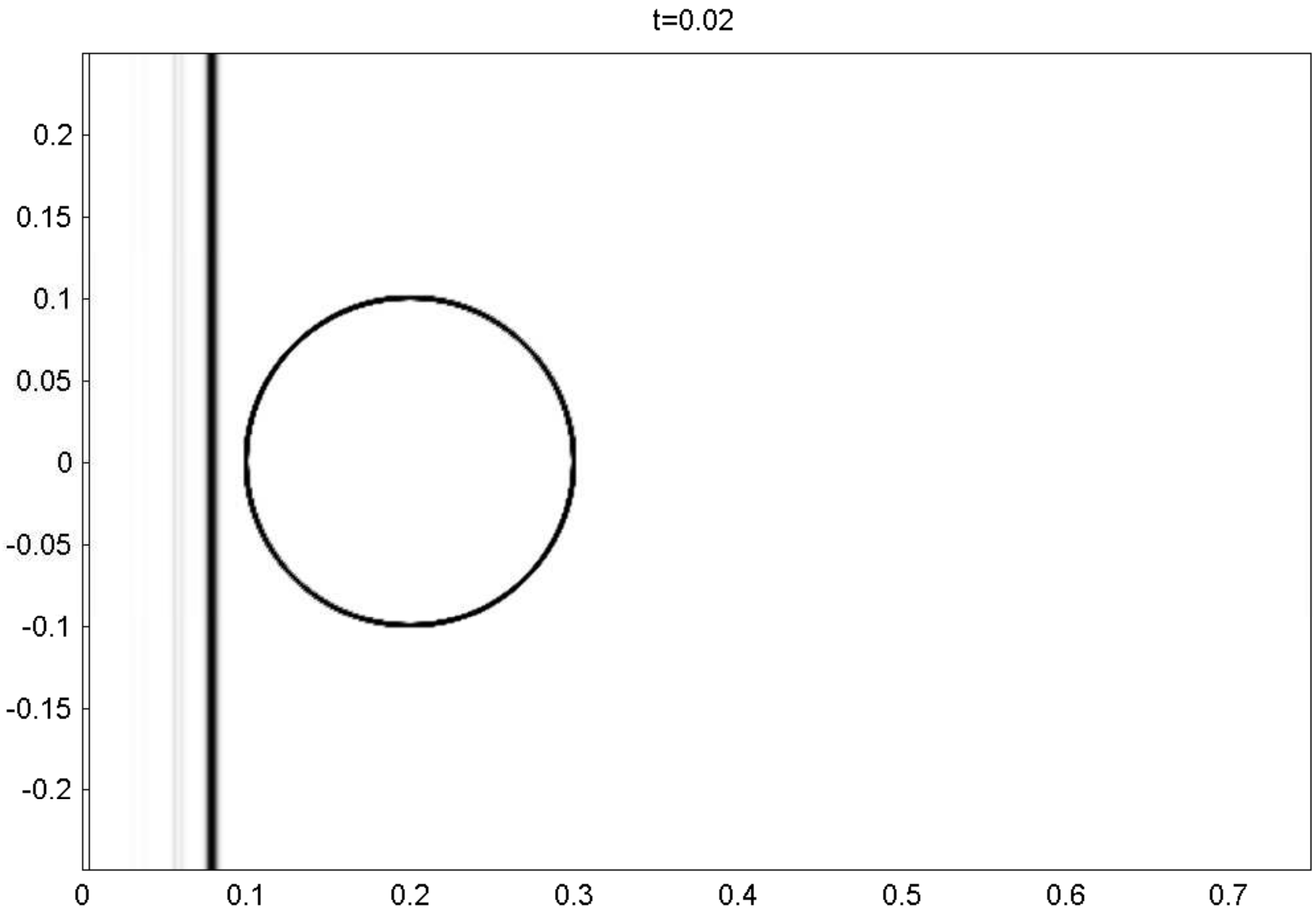

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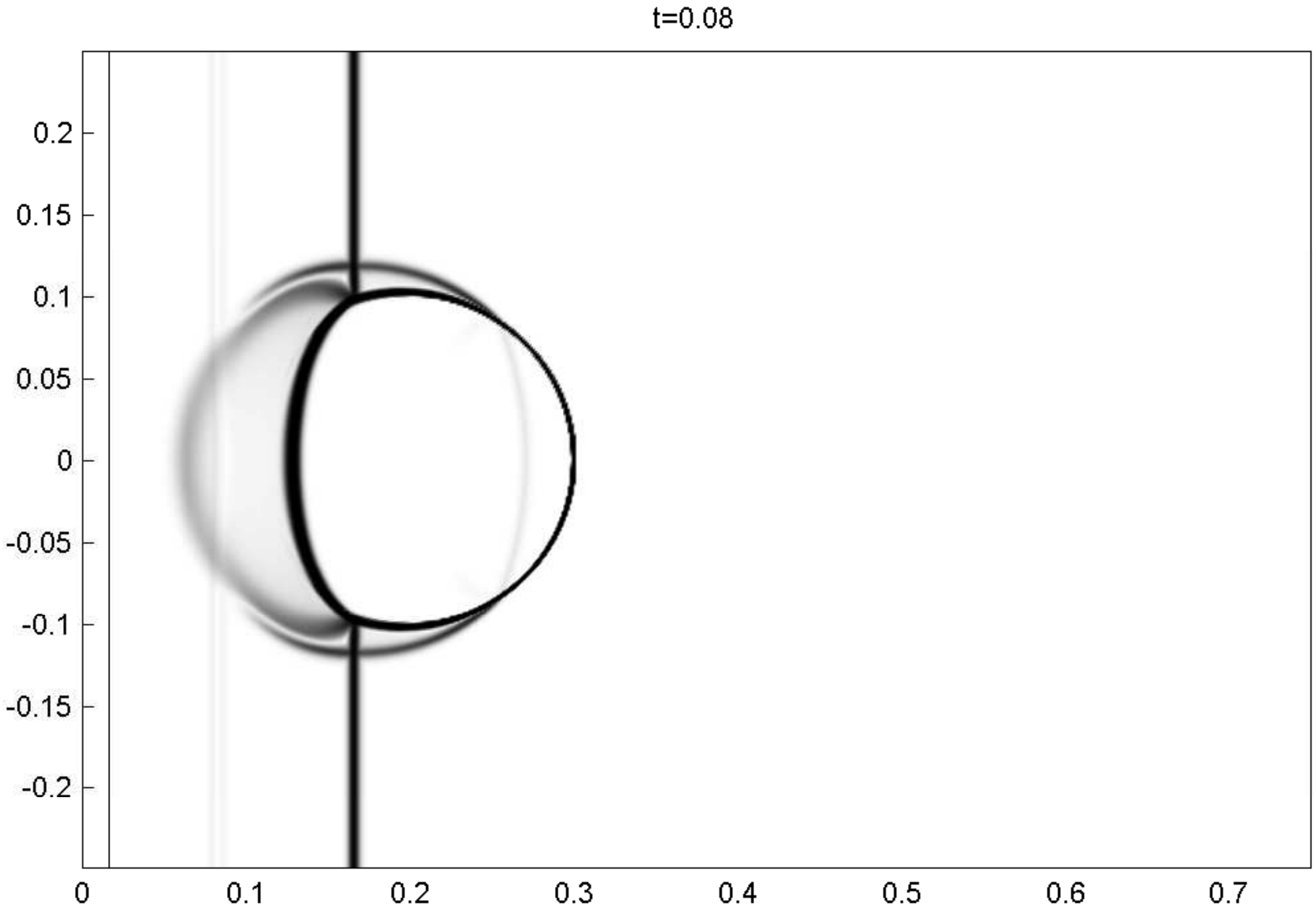

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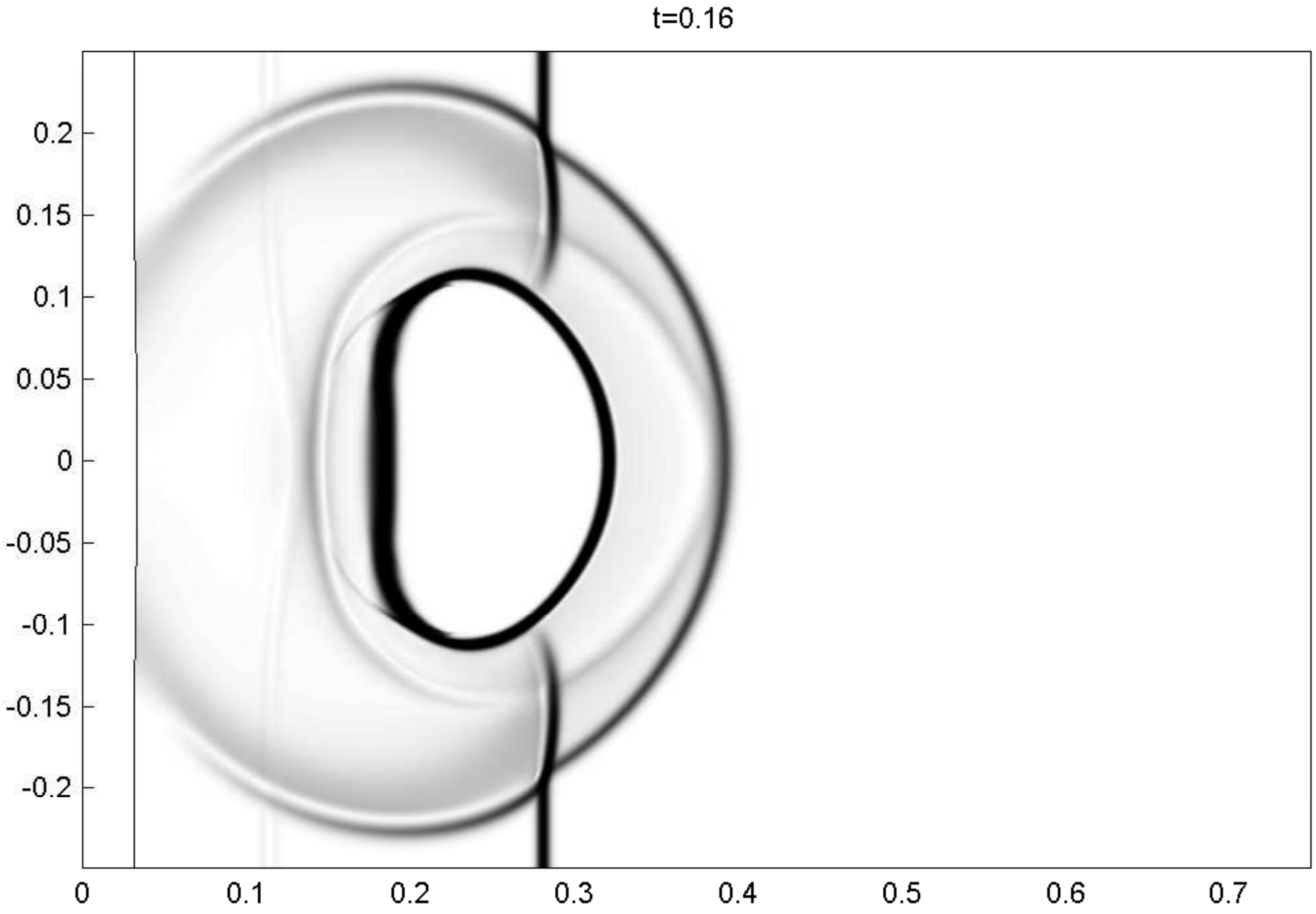

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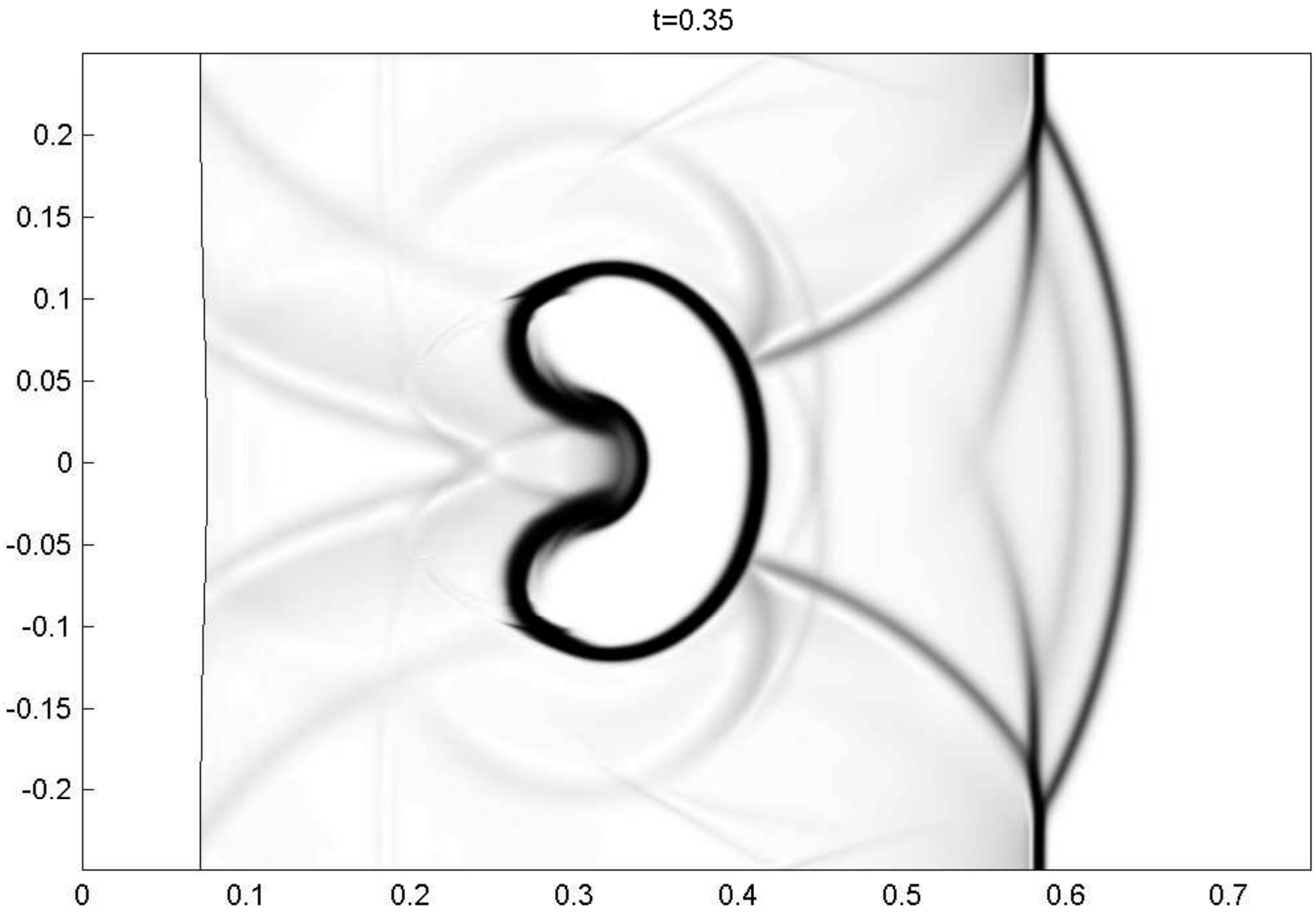
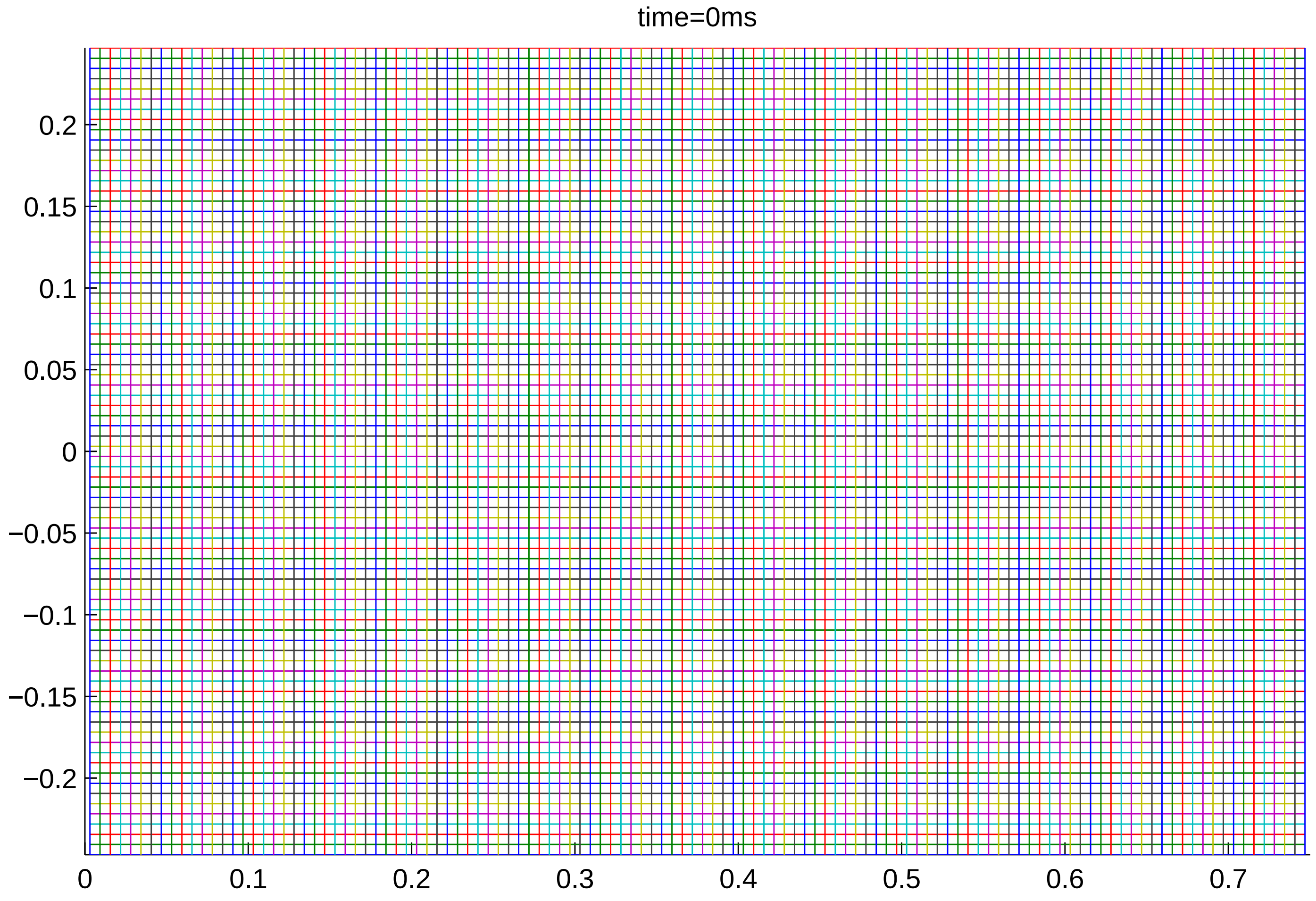


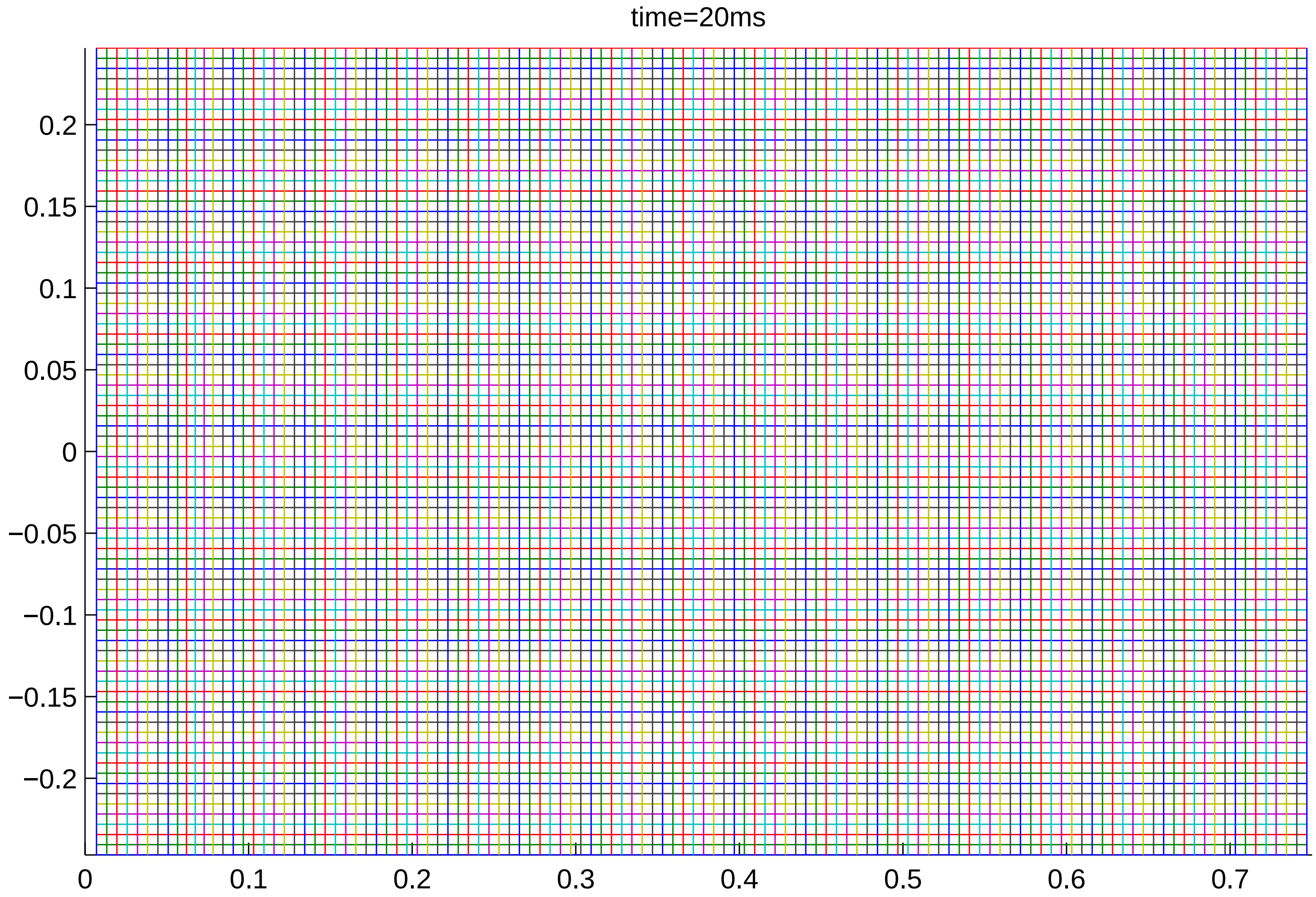
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



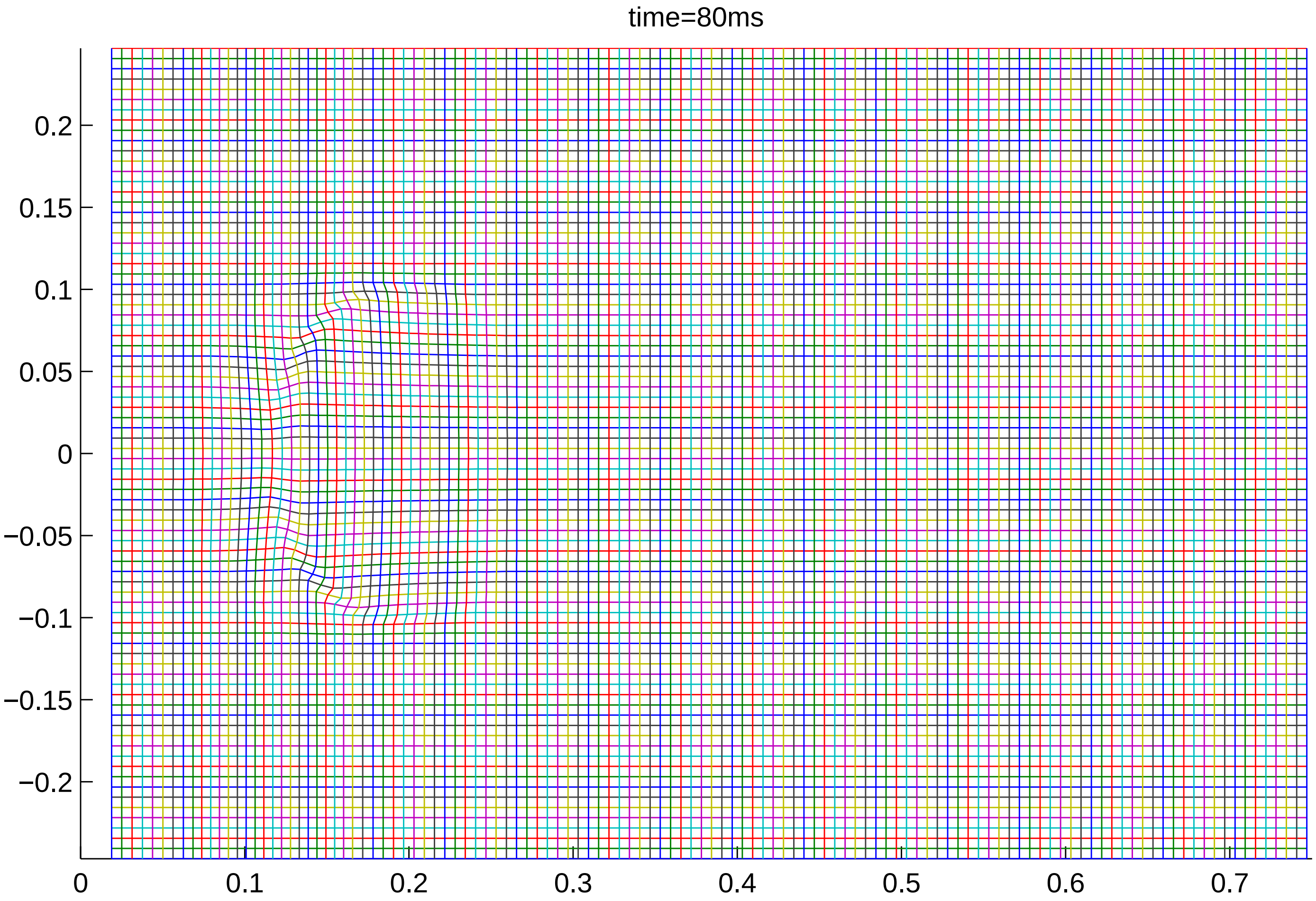
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



Shock-Bubble (Helium) (Cont.)

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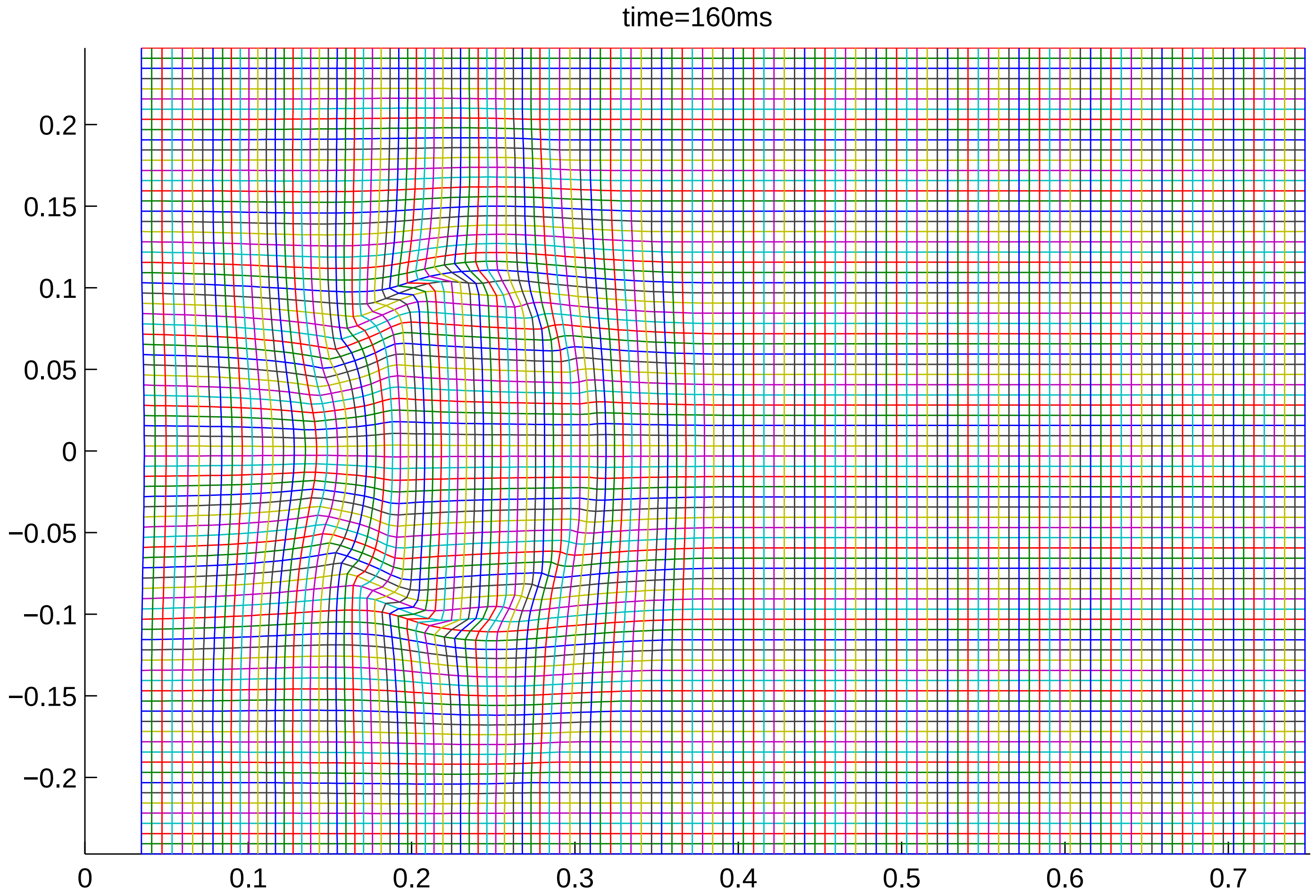
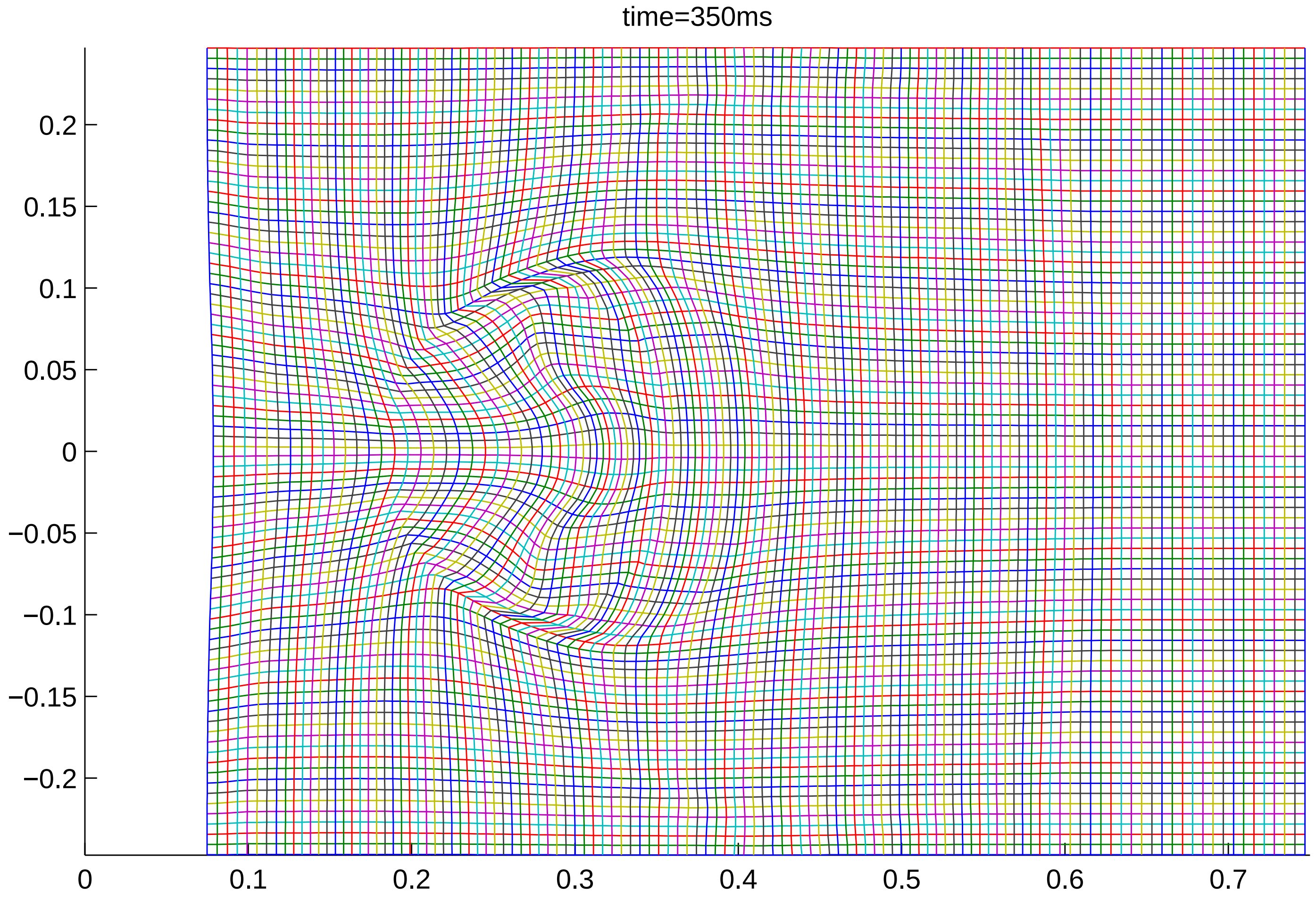


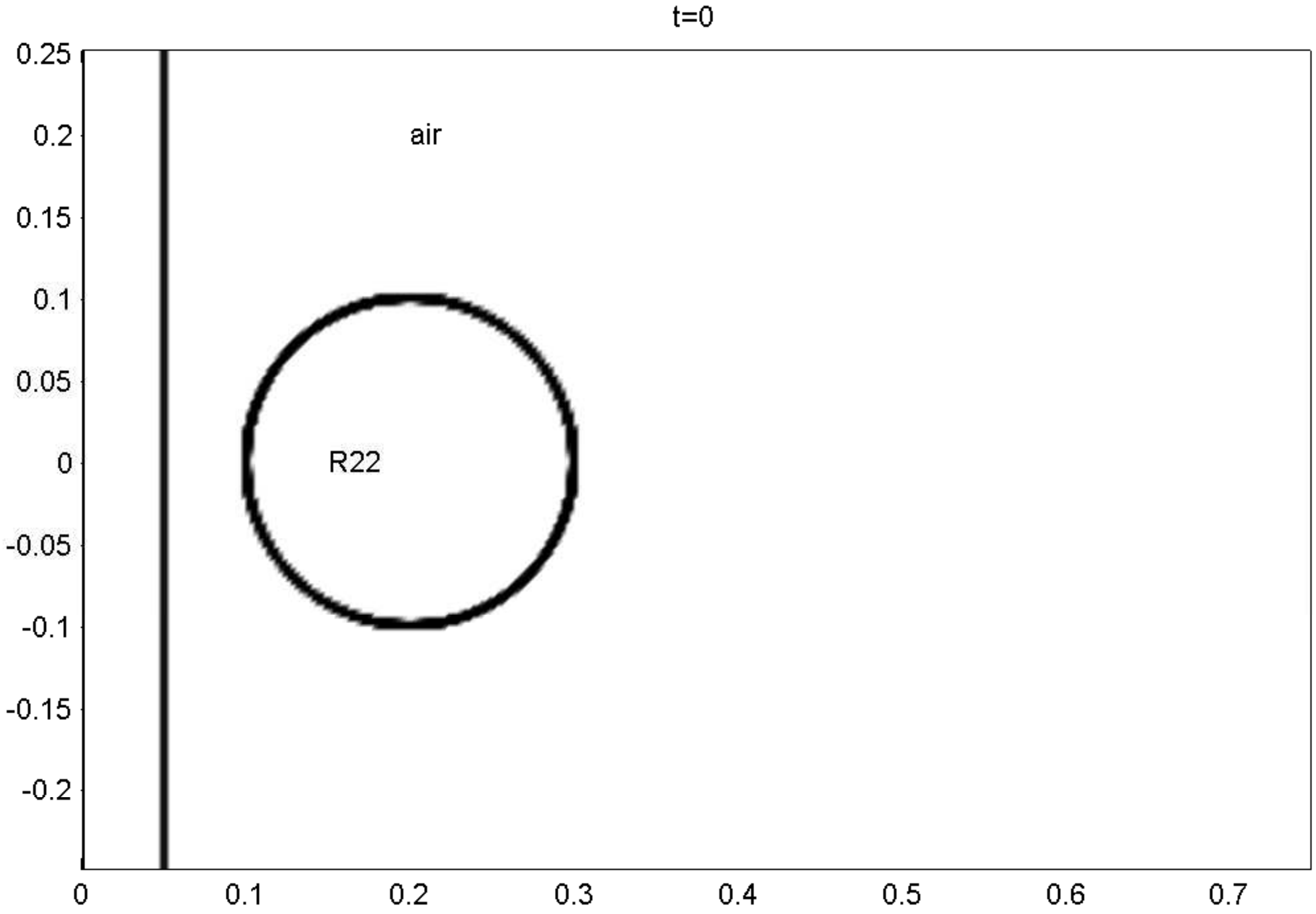
Shock-Bubble (Helium) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



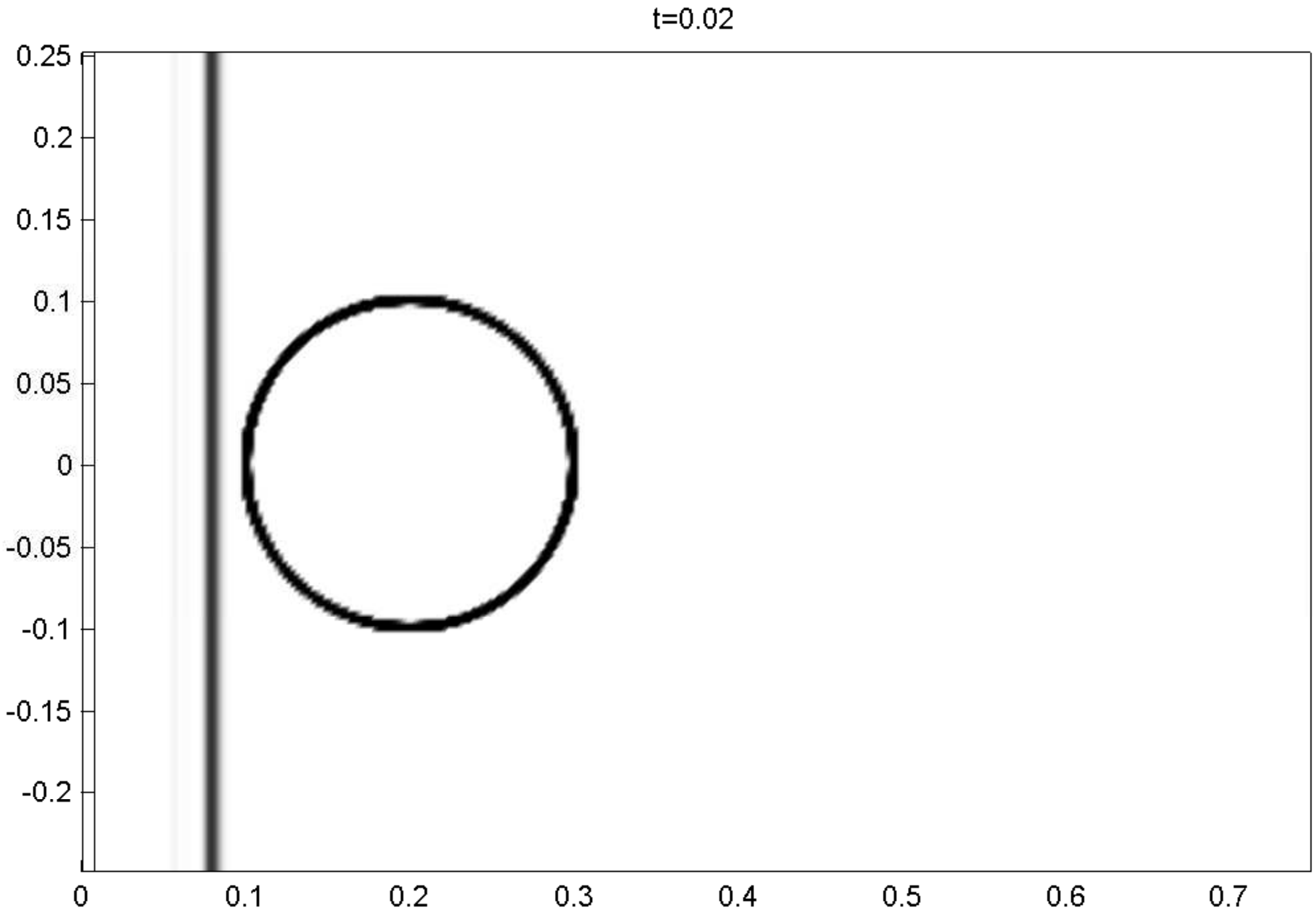
Shock-Bubble (Refrigerant) Interaction

- Numerical schlieren images: $h_0 = 0.5$, 300×200 grid



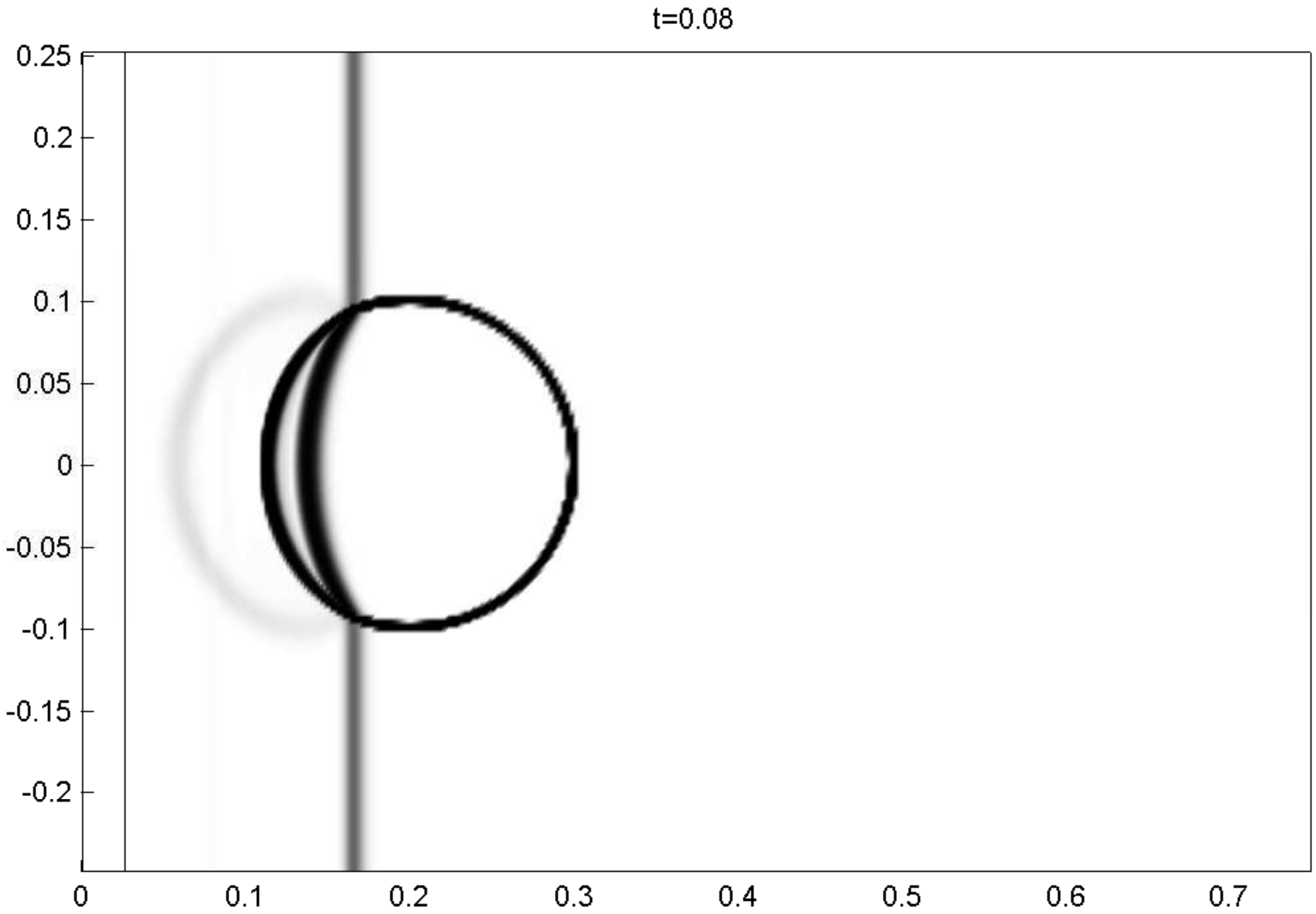
Shock-Bubble (Refrigerant) Interaction

- Numerical schlieren images: $h_0 = 0.5$, 300×200 grid



Shock-Bubble (Refrigerant) Interaction

- Numerical schlieren images: $h_0 = 0.5$, 300×200 grid



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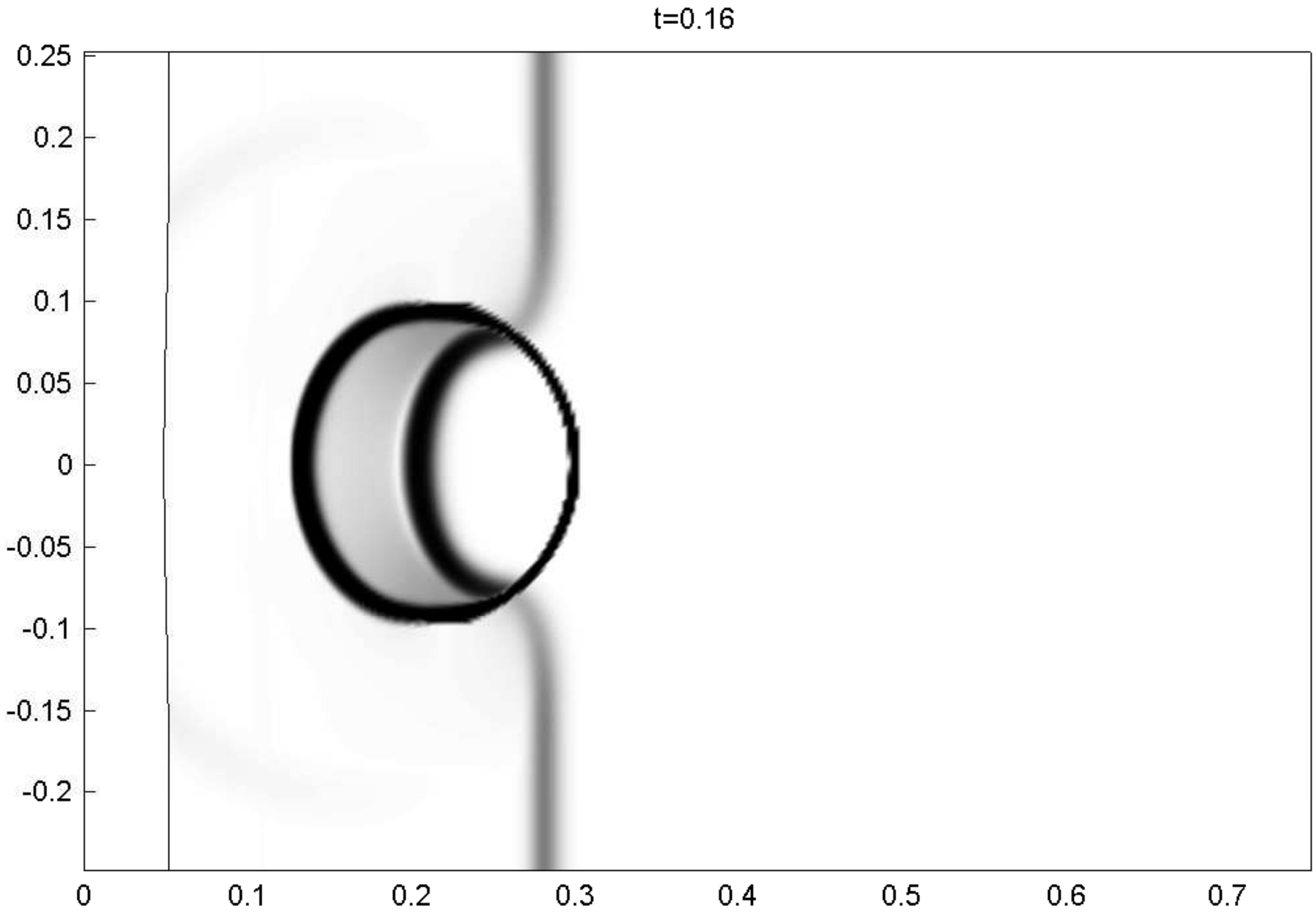
- Numerical schlieren images: $h_0 = 0.5$, 300×200 grid



Shock-Bubble (Refrigerant) Interaction

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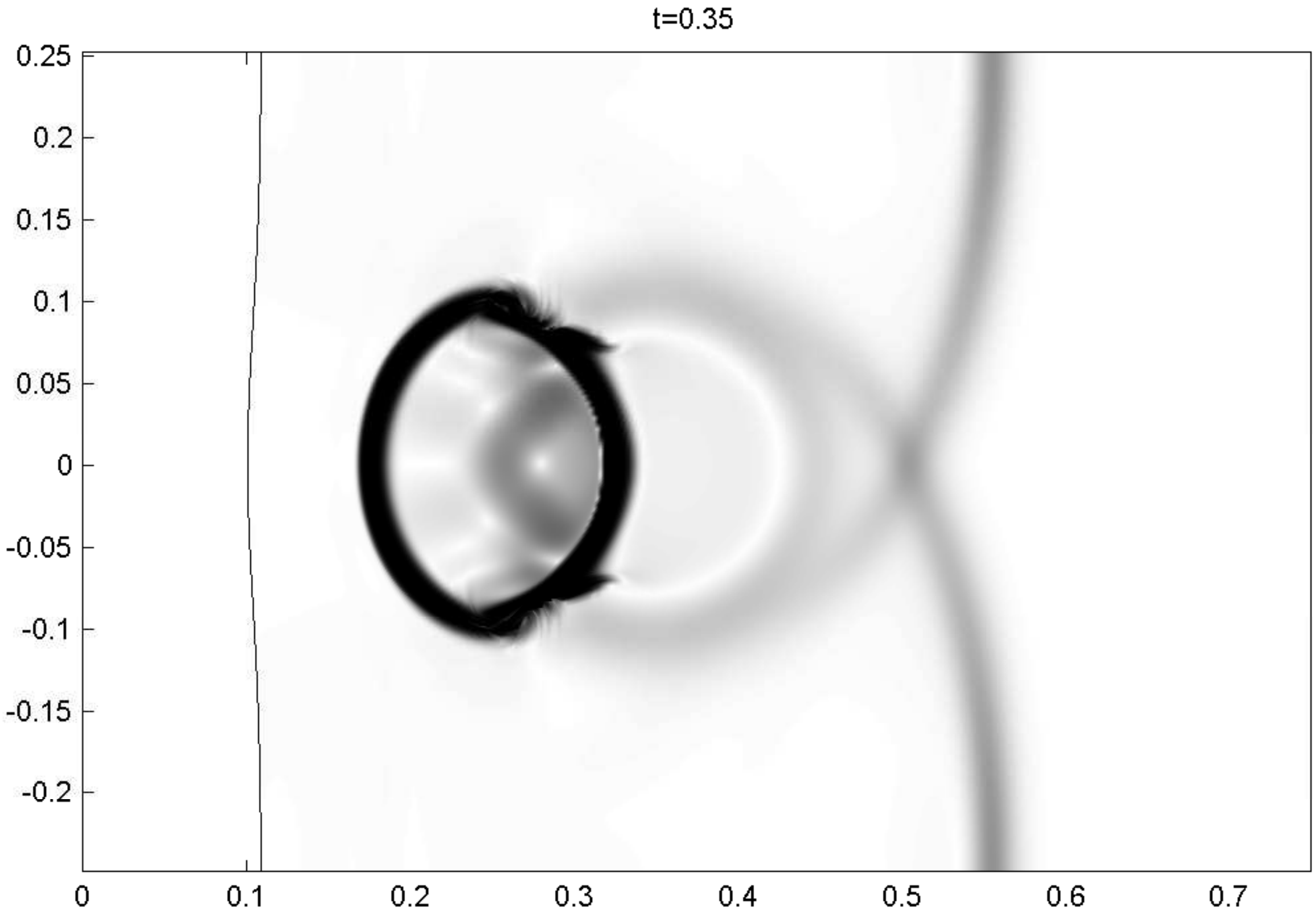
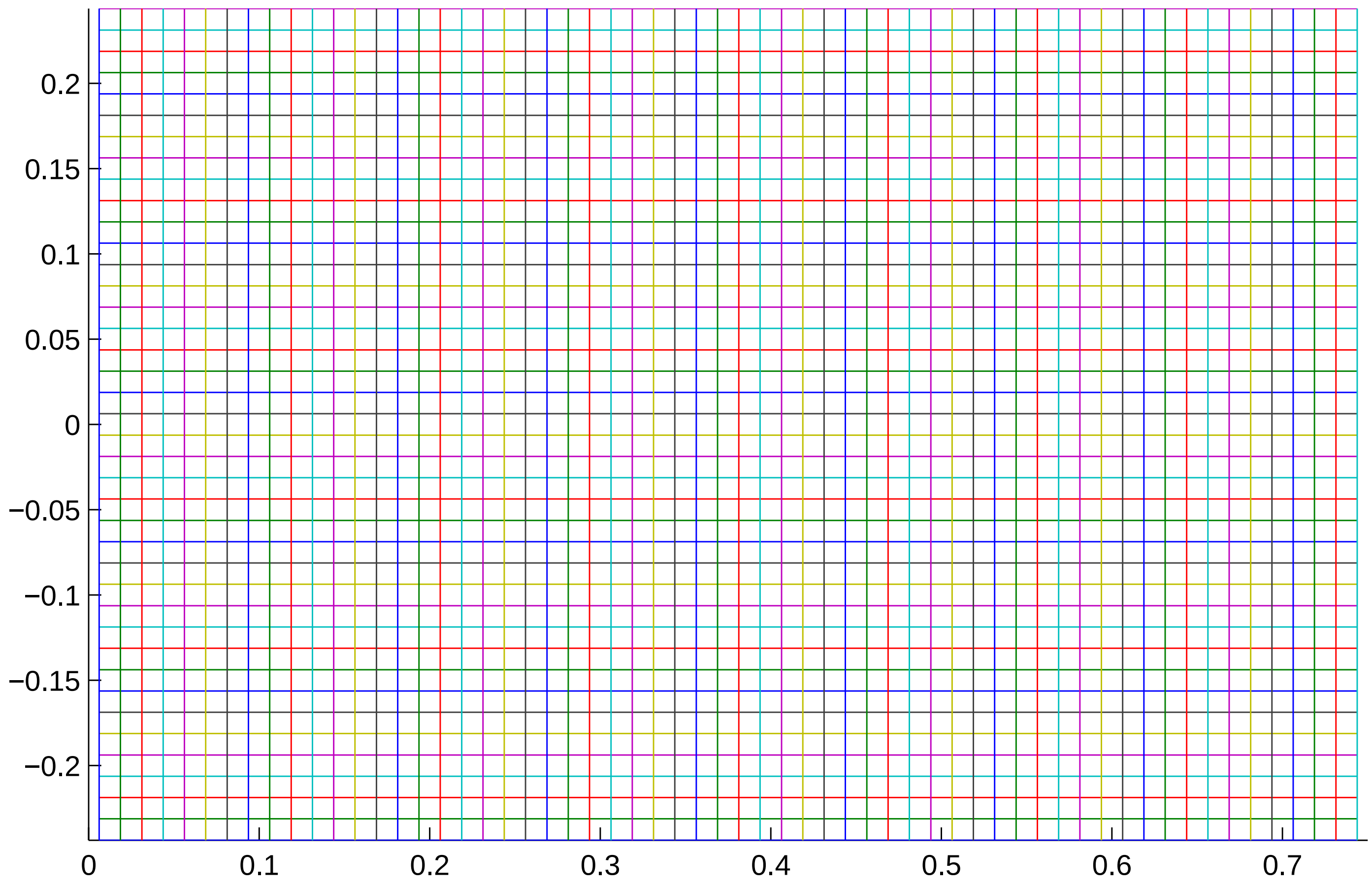


Shock-Bubble (R22) (Cont.)

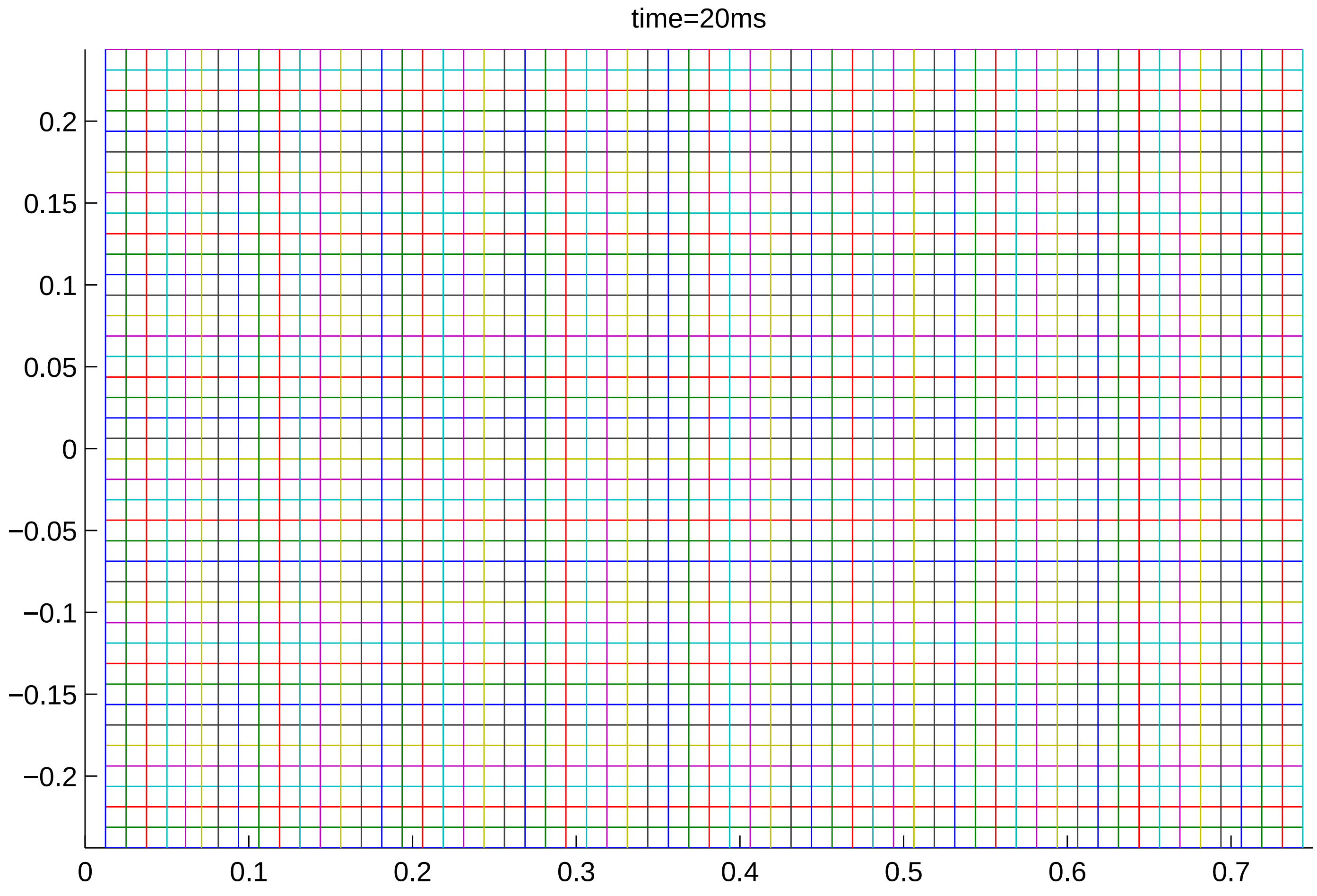
- Grid system (coarsen by factor 5) with $h_0 = 0.5$

time=0ms



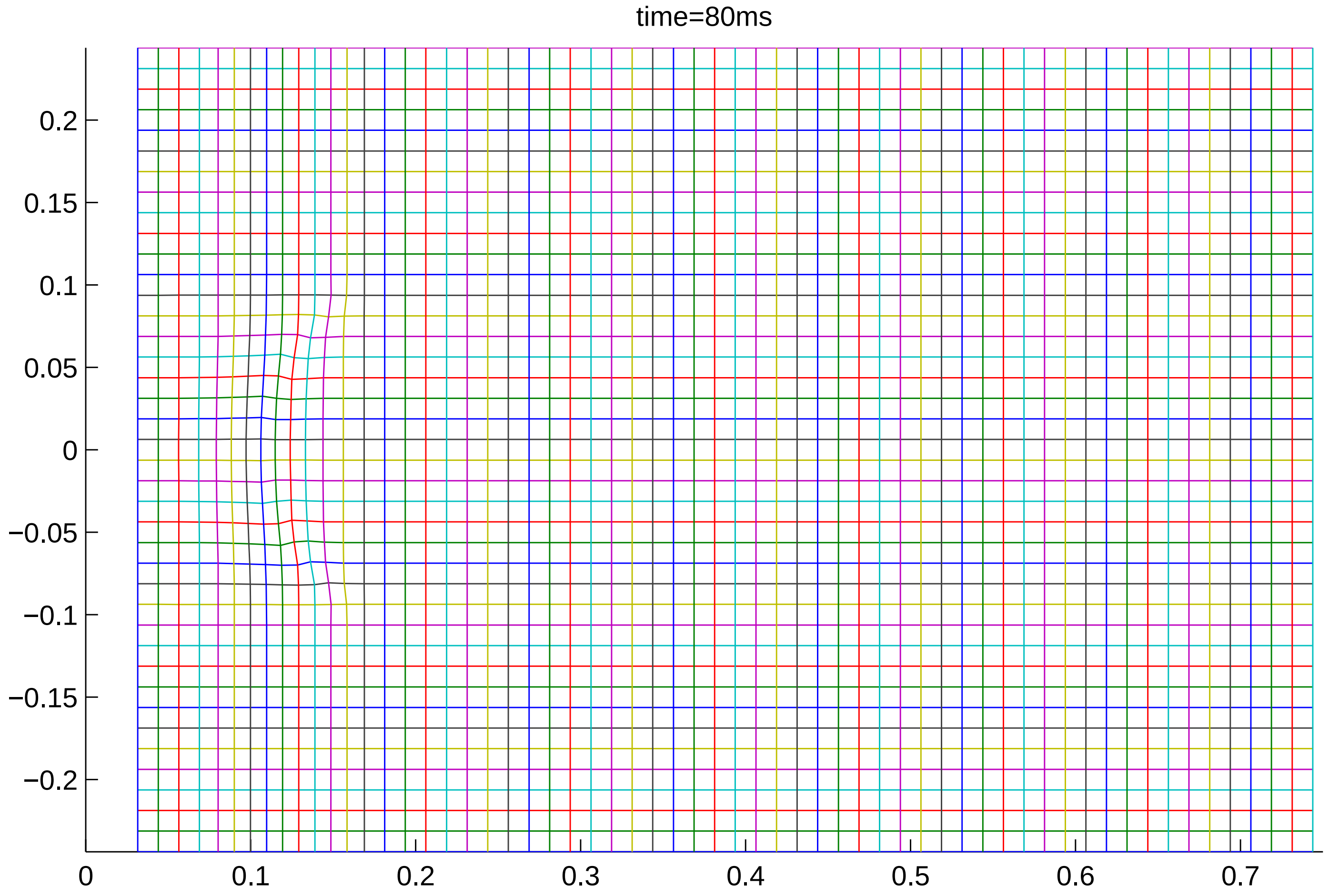
Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



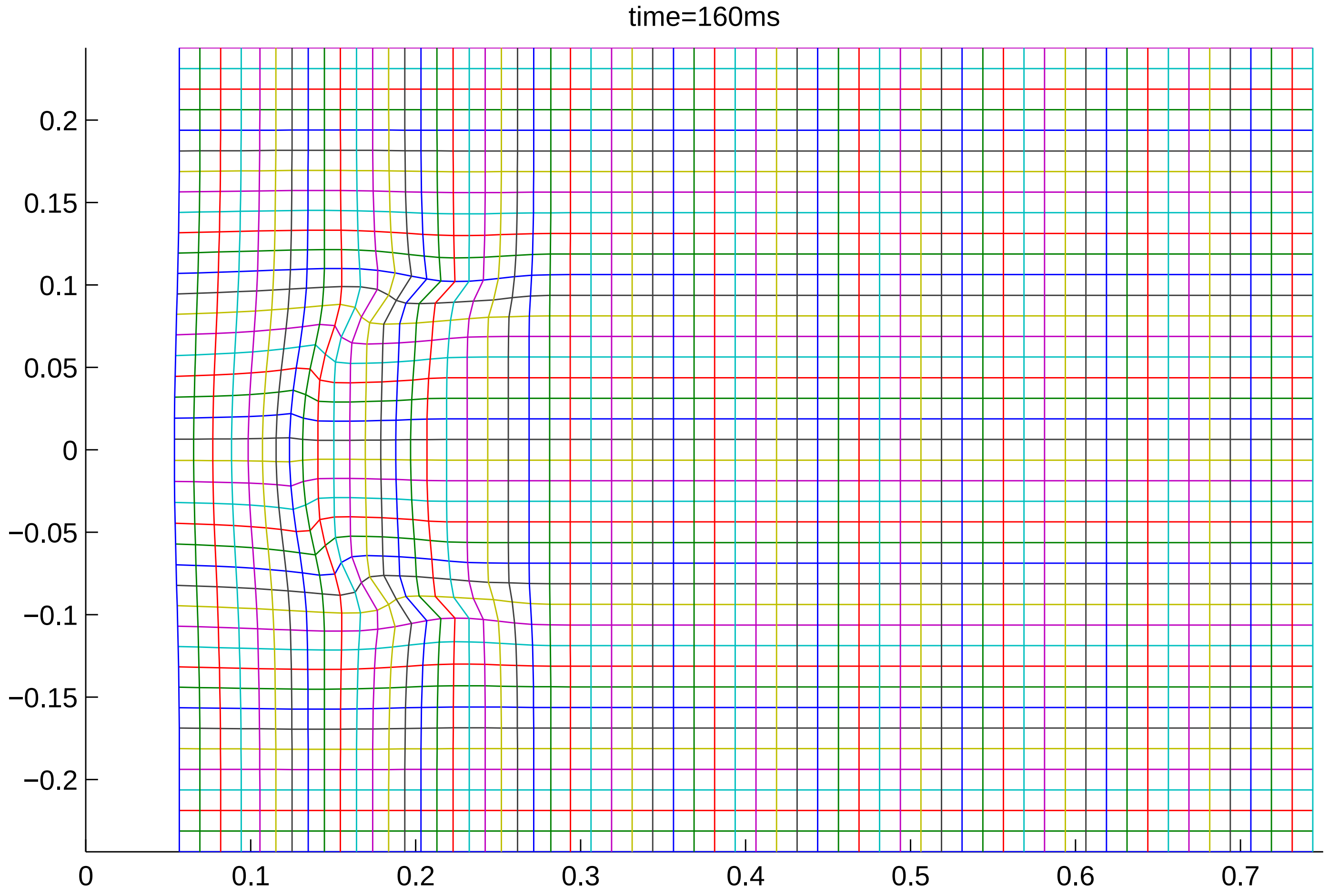
Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



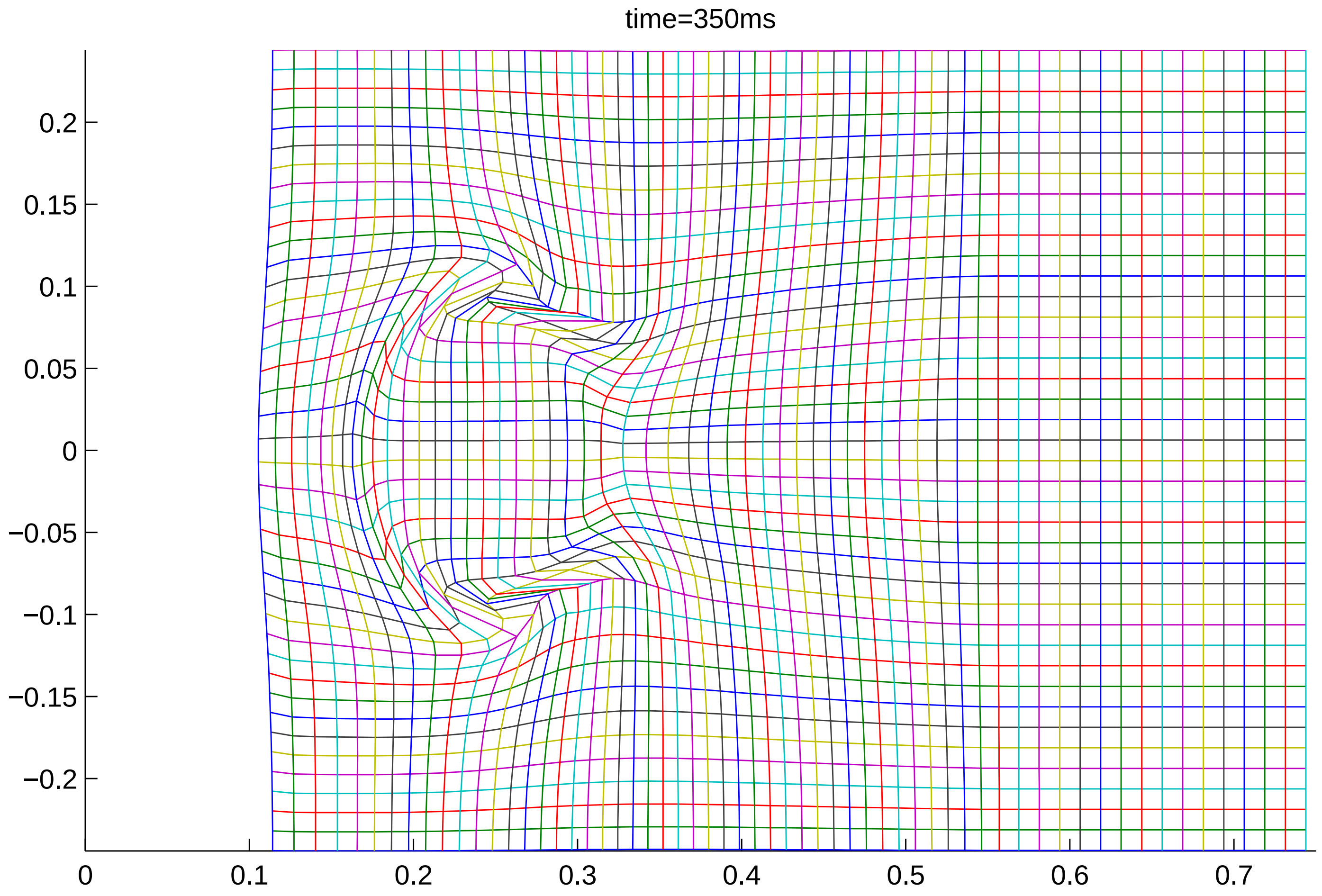
Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



Shock-Bubble (R22) (Cont.)

- Grid system (coarsen by factor 5) with $h_0 = 0.5$



Three Space Dimensions

Euler equations for inviscid compressible flow

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ \rho E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uw\\ \rho Eu+pu\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho vw\\ \rho Ev+pv\end{pmatrix}+\frac{\partial}{\partial z}\begin{pmatrix}\rho w\\ \rho uw\\ \rho vw\\ \rho w^2+p\\ \rho Ew+pw\end{pmatrix}=\psi$$

$E = e + (u^2 + v^2 + w^2)/2$, $e(\rho, p)$: internal energy
ψ: source terms (geometrical, gravitational, & so on)

Three Space Dimensions (Cont.)

Introduce transformation $(t,\ x,\ y,\ z) \to (\tau,\ \xi,\ \eta,\ \zeta)$ via

$$\begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ U & A_1 & B_1 & C_1 \\ V & A_2 & B_2 & C_2 \\ W & A_3 & B_3 & C_3 \end{pmatrix} \begin{pmatrix} d\tau \\ d\xi \\ d\eta \\ d\zeta \end{pmatrix}$$

where
$\vec{Q} = (U,\ V,\ W)$: grid velocity

- $\vec{Q} = 0$ Eulerian case
- $\vec{Q} = (u,\ v,\ w)$ Lagrangian case

A_i, B_i, C_i: geometric variables, $i = 1, 2, 3$

Three Space Dimensions (Cont.)

Inverse transformation $(\tau,\ \xi,\ \eta,\ \zeta) \to (t,\ x,\ y,\ z)$ reads

$$\begin{pmatrix} d\tau \\ d\xi \\ d\eta \\ d\zeta \end{pmatrix} = \frac{1}{J} \begin{pmatrix} J & 0 & 0 & 0 \\ J_{01} & J_{11} & J_{21} & J_{31} \\ J_{02} & J_{12} & J_{22} & J_{32} \\ J_{03} & J_{13} & J_{23} & J_{33} \end{pmatrix} \begin{pmatrix} dt \\ dx \\ dy \\ dz \end{pmatrix}, \qquad J = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

where

$$\begin{aligned} J_{11} &= B_2C_3 - B_3C_2, \quad J_{21} = C_1B_3 - B_1C_3, \quad J_{31} = B_1C_2 - C_1B_2 \\ J_{12} &= C_2A_3 - A_2C_3, \quad J_{22} = A_1C_3 - C_1A_3, \quad J_{32} = C_1A_2 - A_1C_2 \\ J_{13} &= A_2B_3 - B_2A_3, \quad J_{23} = B_1A_3 - A_1B_3, \quad J_{33} = A_1B_2 - B_1A_2 \\ J_{01} &= -\left(UJ_{11} + VJ_{21} + WJ_{31}\right), \quad J_{02} = -\left(UJ_{12} + VJ_{22} + WJ_{32}\right) \\ J_{03} &= -\left(UJ_{13} + VJ_{23} + WJ_{33}\right) \end{aligned}$$

Three Space Dimensions (Cont.)

Euler equations in generalized curvilinear coordinates

$$\frac{\partial}{\partial \tau}\begin{pmatrix} \rho J \\ \rho J u \\ \rho J v \\ \rho J w \\ \rho J E \end{pmatrix} + \frac{\partial}{\partial \xi}\begin{pmatrix} \rho \mathcal{U} \\ \rho u \mathcal{U} + p J_{11} \\ \rho v \mathcal{U} + p J_{21} \\ \rho w \mathcal{U} + p J_{31} \\ \rho E \mathcal{U} + p \mathcal{X} \end{pmatrix} + \frac{\partial}{\partial \eta}\begin{pmatrix} \rho \mathcal{V} \\ \rho u \mathcal{V} + p J_{12} \\ \rho v \mathcal{V} + p J_{22} \\ \rho w \mathcal{V} + p J_{32} \\ \rho E \mathcal{V} + p \mathcal{Y} \end{pmatrix} + \frac{\partial}{\partial \zeta}\begin{pmatrix} \rho \mathcal{W} \\ \rho u \mathcal{W} + p J_{13} \\ \rho v \mathcal{W} + p J_{23} \\ \rho w \mathcal{W} + p J_{33} \\ \rho E \mathcal{W} + p \mathcal{Z} \end{pmatrix} = \psi$$

where

$$\begin{aligned} \mathcal{U} &= (u-U)J_{11} + (v-V)J_{21} + (w-W)J_{31}, & \mathcal{X} &= uJ_{11} + vJ_{21} + wJ_{31} \\ \mathcal{V} &= (u-U)J_{12} + (v-V)J_{22} + (w-W)J_{32}, & \mathcal{Y} &= uJ_{12} + vJ_{22} + wJ_{32} \\ \mathcal{W} &= (u-U)J_{13} + (v-V)J_{23} + (w-W)J_{33}, & \mathcal{Z} &= uJ_{13} + vJ_{23} + wJ_{33} \end{aligned}$$

Three Space Dimensions (Cont.)

Geometrical conservation laws

$$\frac{\partial}{\partial \tau}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ B_1 \\ B_2 \\ B_3 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix} + \frac{\partial}{\partial \xi}\begin{pmatrix} -U \\ -V \\ -W \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{\partial}{\partial \eta}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -U \\ -V \\ -W \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{\partial}{\partial \zeta}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -U \\ -V \\ -W \end{pmatrix} = 0$$

Grid-Velocity Selection

- Pseudo-Lagrangian like

$$(U,\ V,\ W) = h_0(u,\ v,\ w), \qquad h_0 \in (0,1)$$

- Mesh-volume preserving: $\partial J/\partial t = 0$
- Grid-angle preserving
- Other novel approach

Three Space Dimensions (Cont.)

In summary, Euler equations in generalized coord. takes

$$\frac{\partial q}{\partial t} + \frac{\partial f(q,\Xi)}{\partial \xi} + \frac{\partial g(q,\Xi)}{\partial \eta} + \frac{\partial h(q,\Xi)}{\partial \zeta} = \psi$$

where

$$\begin{aligned}
q &= (\rho J,\ \rho J u,\ \rho J v,\ \rho J w,\ \rho J E,\ A_i,\ B_i,\ C_i)\\
f(q,\Xi) &= (\rho\mathcal{U},\ \rho u\mathcal{U} + pJ_{11},\ \rho v\mathcal{U} + pJ_{21},\ \rho w\mathcal{U} + pJ_{31},\ \rho E\mathcal{U} + p\mathcal{X}, \cdots)\\
g(q,\Xi) &= (\rho\mathcal{V},\ \rho u\mathcal{V} + pJ_{12},\ \rho v\mathcal{V} + pJ_{22},\ \rho w\mathcal{V} + pJ_{32},\ \rho E\mathcal{V} + p\mathcal{Y}, \cdots)\\
h(q,\Xi) &= (\rho\mathcal{W},\ \rho u\mathcal{W} + pJ_{13},\ \rho v\mathcal{W} + pJ_{23},\ \rho w\mathcal{W} + pJ_{33},\ \rho E\mathcal{W} + p\mathcal{Z}, \cdot
\end{aligned}$$

with Ξ : grid metrics & equation of state $p = p(\rho,\ e)$

Conclusion

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  - Efficient & accurate grid movement conditions
  - 3D computer program realization (have done mostly)
  - Weakly compressible flow
  - $\ldots$

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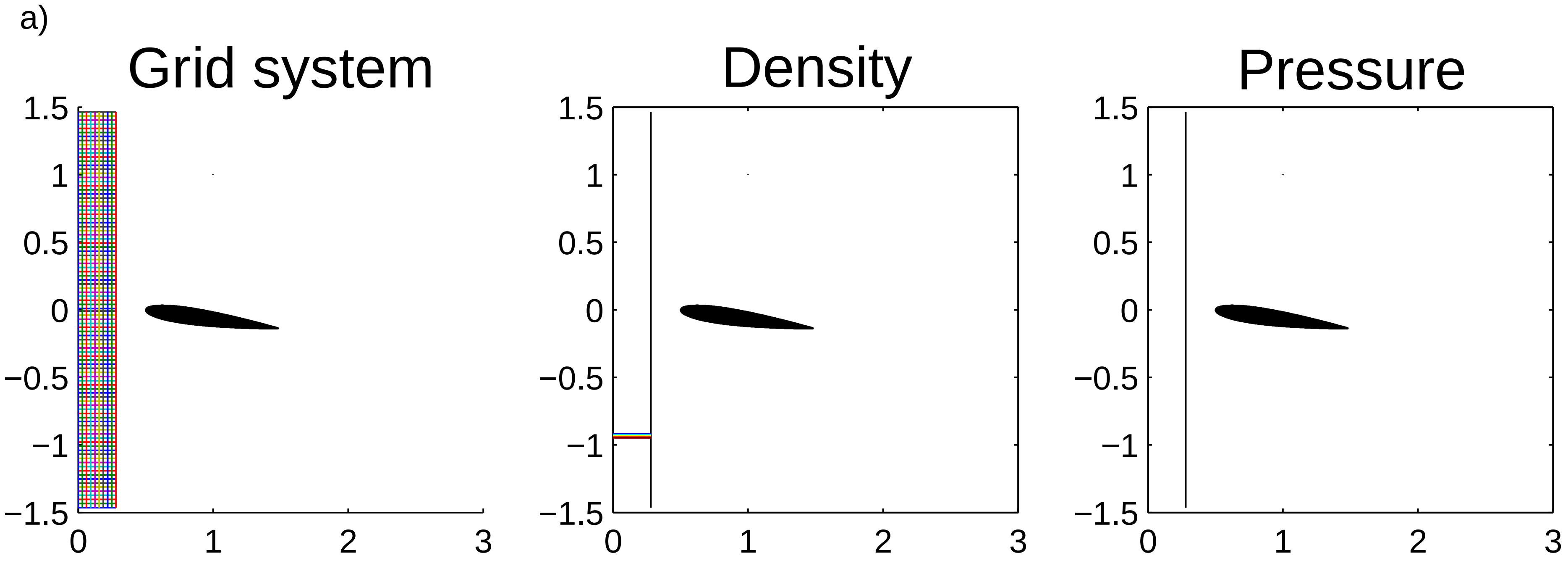
Thank You

Automatic Time-Marching Grid

- Supersonic NACA0012 over heavier gas



Automatic Time-Marching Grid

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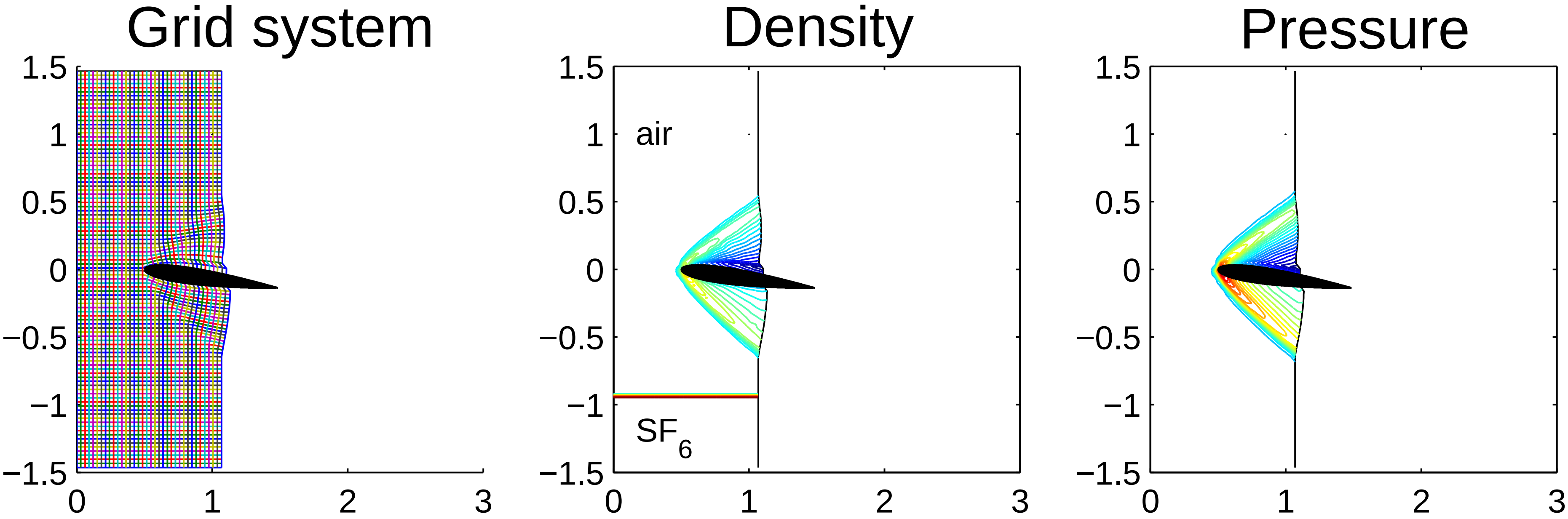
b)



Automatic Time-Marching Grid

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