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Front Tracking Based on High-Resolution Wave Propagation Methods

KEH-MING SHYUE

1. Introduction

In previous work by LeVeque and the author [**6**],[**7**], a simple front-tracking approach has been developed and studied that models the propagation of discontinuous solutions for nonlinear hyperbolic systems of conservation laws in both one and two space dimensions. In this approach, we use a conservative finite volume method based on the *large time step* wave propagation approach [**4**],[**5**] to overcome a major difficulty associated with the limit on the time step when small cells are created by the tracked front cutting through the grid, while maintaining conservation of the algorithm. A variety of problems have been solved to validate the method for problems involving shock waves arising in gas dynamics. In this paper, we apply this method to the simulation of compressible Rayleigh-Taylor unstable interfaces, and to an oil reservoir simulation in porous media, in which a hyperbolic conservation equation is coupled with an elliptic PDE. Some preliminary results for both problems will be shown. Further details can be found in [**7**],[**10**].

2. Method

The basic idea of our front-tracking algorithm is quite simple. We choose a uniform underlying grid with some grid cells subdivided by tracked interfaces made up of moving points in 1D and curves in 2D, approximately aligned with discontinuities in the flow field. In each time step, we introduce a new set of

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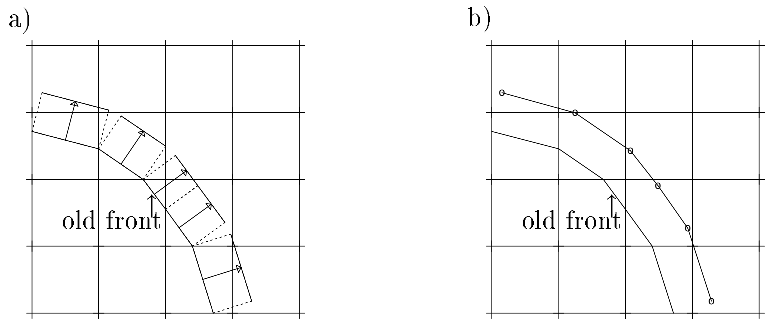

FIGURE 1. Two-dimensional front propagation. a) Tracked interfaces after advancing the original interfaces using the strong wave speeds obtained from solving the normal Riemann problems under the current time step. b) New tracked interfaces after taking an average of two neighboring points on the new interface location shown in a), and connecting the resulting points (indicated by large dots) consecutively by straight lines.

interfaces that are approximately aligned with the expected locations of discontinuities in the solution at the end of the time step. A high-resolution finite volume method is then applied on the resulting nonuniform grid to update the cell values. If we have chosen the new interface locations well, the resulting solution will remain sharp and be smooth away from these new interfaces. The old interfaces can then be eliminated by recombining the adjacent cells; see Figure 1.

To advance the tracked interfaces from the current time step to the next, we solve a one-dimensional Riemann problem at the tracked interface using the values from the adjacent cells as data, and follow strong waves to determine the new locations at the end of a time step. Since each tracked interface is the boundary between two cells, we can use the values from the adjacent cells as data to solve a one-dimensional Riemann problem in the direction normal to the interface. We expect the solution to this Riemann problem to consist of only one strong wave, corresponding to the shock or interface (contact discontinuity or slip line) being tracked, and other weaker waves.

Since we want the new tracked interfaces to form a continuous piecewise linear curve, as shown in Figure 1, we need to use the solutions of neighboring Riemann problems in some coordinated manner to determine the new interfaces. There are various ways that this can be done *via* some sort of curve fitting through points determined by the strong waves from the Riemann solutions; see Figure 1, for example. Our basic philosophy of tracking, however, gives us some flexibility on this score – we do not view the interface we introduce as being the definitive location of a tracked front, but rather as a grid interface that is sufficiently well located and aligned that the solution can be well captured on the resulting grid.

Note that for interface problems, the tracked front should be advanced not only in the normal direction to the front, as illustrated in Figure 1a, but also in the tangential direction. This can be done quite easily by moving points tangential to the front using, for example, an average tangential velocity from the data of the normal Riemann problem. It is easy to show that doing so gives exact front propagation for linear advection equations and for simultaneously diagonalizable linear hyperbolic systems. For general linear hyperbolic systems, or for nonlinear equations, this front-moving procedure gives a good approximation to the front motion. (See [**7**] for some test results.)

Once the new grid is constructed, the solution is then advanced using a fully conservative shock-capturing method. This method must be able to deal with the irregular cells near the tracked interface. In particular, we must maintain stability even if some of these cells are very small relative to the underlying mesh size used to determine the time step. We also hope to maintain second-order accuracy in the smooth flow on either side. This is accomplished using a high-resolution method based on the *large time step* wave propagation approach, developed by LeVeque [**4**],[**5**]. The main idea is that waves arising from the solution of Riemann problems at the cell boundaries are propagated the appropriate distance determined by the wave speed and time step, and used to update cell averages in any grid cell they encounter. The wave may affect more than one cell if the neighboring cell is very small. In this manner the stencil of the method adjusts automatically so that the CFL condition is always satisfied regardless of the configuration of the grid.

The high-resolution algorithm used for the calculations below is based on using rotated data to solve Riemann problems in the directions normal and tangential to the front, tangential splitting of the resulting waves to increase stability and resolution, and propagation of piecewise linear "correction waves" to increase the order of accuracy.

3. Numerical Results

We now show some preliminary results of interface tracking. As a first example, we consider a Rayleigh-Taylor unstable interface arising from the Euler equations of gas dynamics. (See [**9**] for an overview of this problem.) For comparison purposes, we choose the same initial setup as used in Gardner *et al.* [**1**], namely, we introduce a small single-mode perturbation of an isothermal equilibrium with a flat interface separating exponentially stratified flow above and below the interface. As in the laboratory experiments, the heavy fluid lies *below* the light fluid and gravity is directed *upwards*. For simplicity, we take the same gas with $\gamma = 1.4$ for both the heavy and light fluids. We use the following parameters: density ratio $D = 10$, $M^2 = 0.5$ (this parameter measures the compressibility of the fluids), $\varepsilon = 0.015$ (the amplitude of the perturbation). Figure 2 shows results at time $t = 6$ on a 40×160 grid. The growth of the interface and

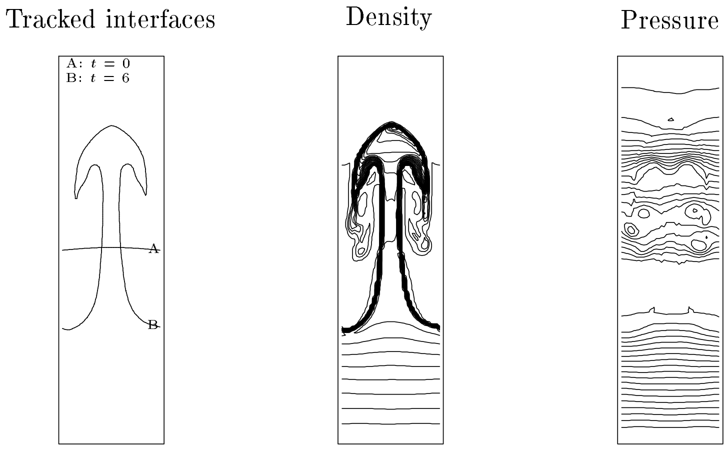

FIGURE 2. Results for the Rayleigh-Taylor instability at time $t = 6$ on a 40×160 grid. Parameters $D = 10$, $M^2 = 0.5$, and $\varepsilon = 0.015$ were used in the run.

the formation of a rising bubble and falling spike can be easily seen. Qualitatively, as far as the dominant unstable mode is concerned, our result agrees with the result shown in [**1**]. The results of [**1**] also show a secondary instability below the rising bubble. In our case, we do not observe such an instability. It is not clear which result is correct and work is continuing to clear up this discrepancy.

Our next example of interface tracking concerns a model problem for oil reservoir simulation in which the fluids are oil and solvent (such as CO_2). The governing equations for this model consist of a hyperbolic saturation equation, Darcy's law, and an equation enforcing the incompressibility of the fluids; see, *e.g.*, [**8**]. To solve this system of equations, we use the popular IMPES procedure, in which the hyperbolic saturation equation and the elliptic pressure equation are dealt with separately in two sequential steps. Here the hyperbolic saturation equation is solved using the front-tracking algorithm, and the elliptic pressure equation is solved using a standard five-point finite difference method on a uniform grid.

We consider a test problem in which the geometry is a standard five-spot pattern with fluid being injected into the center of a unit square domain and the oil being recovered from the four corners. The initial saturation we use is $s = 1$ (pure solvent) inside a perturbed circular interface and $s = 0$ (pure oil) outside the perturbed interface. In this model there is a dimensionless parameter called the frontal mobility ratio M, which characterizes the stability of the interface (for $M \leq 1$ the interface is stable, while for $M > 1$ the interface is unstable). Figure 3 shows results using a 50×50 grid with two different mobility ratios $M = 1$ and $M = 10$. For similar calculations, see [**2**],[**3**].

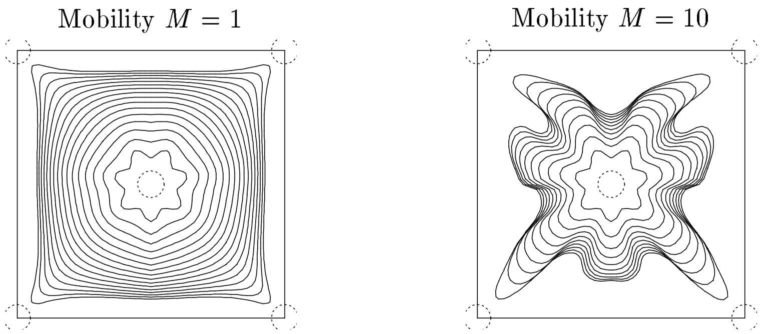

FIGURE 3. Evolution of the tracked interfaces for the miscible displacement oil reservoir simulation with mobility ratios $M = 1$ and $M = 10$, plotted every 10 time steps.

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DEPARTMENT OF APPLIED MATHEMATICS AND STATISTICS, SUNY AT STONY BROOK, STONY BROOK, NY 11794

E-mail address: shyue@ams.sunysb.edu