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Singular solutions of the BBM equation: analytical and numerical study

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Abstract
We show that the Benjamin–Bona–Mahony (BBM) equation admits stable travelling wave solutions representing a sharp transition from a constant state to a periodic wave train. The constant state is determined by the parameters of the periodic wave train: the wave length, amplitude and phase velocity, and satisfies both the generalized Rankine–Hugoniot conditions for the exact BBM equation and for its wave averaged counterpart. Such stable shock-like travelling structures exist if the phase velocity of the periodic wave train is not less than the solution wave averaged. To validate the accuracy of the numerical method, we derive the (singular) solitary limit of the Whitham system for the BBM equation and compare the corresponding numerical and analytical solutions. We find good agreement between analytical results and numerical solutions.

Keywords: nonlinear dispersive equations, Whitham's modulation equations, solitary limit
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(Some figures may appear in colour only in the online journal)

1. Introduction

The Benjamin–Bona–Mahony (BBM) equation was proposed as a unidirectional model of weakly nonlinear waves in shallow water [5]:

$$v_t + v_x + vv_x - v_{txx} = 0,$$

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involving one dependent variable $v(t, x)$ and two independent variables t (time) and x (space coordinate). The last term v_{txx} is responsible for the nonlocal nature of the BBM equation. After the change of variables $v = u - 1$ one gets the equation:

$$u_t + uu_x - u_{txx} = 0. \tag{1}$$

Olver [32] justified that (1) admits only three independent conservation laws:

$$(u - u_{xx})_t + \left(\frac{u^2}{2}\right)_x = 0, \tag{2}$$

$$\left(\frac{u^2}{2} + \frac{u_x^2}{2}\right)_t + \left(\frac{u^3}{3} - uu_{tx}\right)_x = 0, \tag{3}$$

$$\left(\frac{u^3}{3}\right)_t - \left(u_t^2 - u_{xt}^2 + u^2 u_{xt} - \frac{u^4}{4}\right)_x = 0, \tag{4}$$

and proposed a Hamiltonian formulation of the BBM equation [33]. In particular, the Lagrangian for the BBM equation is:

$$\mathcal{L} = -\frac{\varphi_t \varphi_x}{2} + \frac{\varphi_t \varphi_{xxx}}{2} - \frac{\varphi_x^3}{6}, \qquad u = \varphi_x. \tag{5}$$

The conservation law (2) is the Euler–Lagrange equation for (5). The conservation laws (3) and (4) correspond to the invariance of the Lagrangian under space and time translations (Noether's theorem).

A number of important qualitative results have been obtained for the BBM equation: in [44] the modulation equations were derived; the well (ill)-posedness of the Cauchy problem for the BBM equation was studied in [2]; the modulational instability of short periodic waves has been proven in [31].

The Riemann problem for the BBM equation is the Cauchy problem

$$u(0, x) = \begin{cases} u^-, & x < 0, \\ u^+, & x > 0. \end{cases} \tag{6}$$

with constant values of $u^\pm$. Such a problem is often called Gurevich–Pitaevskii problem, who were the first to give its asymptotic solution for the Korteweg–de Vries (KdV) equation [19]. This approach has been further developed and applied to both integrable and non-integrable dispersive equations [3, 4, 10–12, 20, 21, 24]. The Riemann problem for (1) was recently investigated in [8]. The authors analytically and numerically studied the influence of the initial step data and of a smoothing parameter (the stepwise initial data was replaced by the hyperbolic tangent having this parameter as a characteristic width of the transition zone) on the solution structure. The fact that the solution can depend on the smoothing parameter has been also discussed in [37] for the Serre–Green–Naghdi (SGN) equations which is a nonlinear bi-directional model of shallow water flows [17, 18, 36, 39].

The BBM equation admits exact weak stationary solutions which are at the same time weak solutions to the Hopf equation $u_t + \left(u^2/2\right)_x = 0$ [12]. In particular, for the antisymmetric initial data $u^+ = -u^- < 0$ the solution is a shock satisfying Lax 'entropy condition', while $u^+ = -u^- > 0$ corresponds to an unstable shock which transforms to a rarefaction wave (which is also a solution to both the BBM and Hopf equations). Numerically, the Lax shock is accompanied by narrow zones of very short waves. The shock solution is not structurally stable under non-symmetric perturbations. For $u^+ = -u^- > 0$, a transient discontinuous structure appears algebraically decaying in time and finally degenerating into the rarefaction wave of Hopf's equation [8, 13].

A natural question arises: can we find non-transient stable discontinuous solutions to the BBM equation? Such shock-like structures were recently discovered for the SGN equations and Boussinesq equations [14]. They were obtained as solutions of the generalized Riemann problem (GRP) where constant initial states were replaced by periodic solutions of the SGN equations. In particular, the authors of [14] found such shock-like transition fronts linking a constant state to a periodic wave train. The velocity of such a shock coincides with the velocity of the periodic wave train. Across the shock considered as a dispersionless limit, generalized Rankine–Hugoniot (GRH) conditions were satisfied. These conditions are the classical conservation laws for mass and momentum augmented by an additional condition which expresses the continuity of one-sided first order derivatives of unknowns. Physically, this extra condition is nothing but the absence of oscillations at the shock front (the one-sided gradients of unknowns are vanishing). A multi-dimensional version of the GRH conditions was also derived for a class of Euler–Lagrange equations describing, in particular, the second gradient fluids, multi-dimensional SGN equations and fluids containing gas bubbles [15].

The question about the existence of shock-like transition fronts for the BBM equation is reasonable because the BBM and SGN equations share a common 'hyperbolic' feature: the phase and group velocity obtained for the corresponding linearized equations are finite for any wave number.

Smooth travelling wave solutions linking uniform levels with periodic wave trains, or even disparate wave trains were also found to the Kawahara and fifth order KdV equations [23, 43]. They are heteroclinic orbits to saddle-center type fixed points. The averaged limit states (periodic or constant) satisfy the Rankine–Hugoniot conditions for the corresponding Whitham modulation system.

Such a scenario cannot obviously appear for the BBM equation because the periodic solutions of the BBM equation are described by a low order Hamiltonian differential equation which does not admit periodic-to-periodic or periodic-to-constant connections. So, we are looking for a possibility to construct travelling wave solutions satisfying the BBM equation in a weak sense.

The aim of this paper is to give precise conditions for the existence of stable shock-like structures for the BBM equation. To validate the accuracy of numerical results, we need to test the numerical method (see a short description in appendix B) on closed form analytical solutions (e.g., travelling waves) or asymptotic solutions (e.g., the solutions of modulation equations for the BBM equation). The test based on travelling wave solution is a little bit trivial. It is interesting thus to find closed form analytical non-stationary solutions of the modulation equations (three equations model), but they do not exist in the literature. Indeed, the BBM equation is not integrable, so no hope to rewrite the modulation equations in the form of Riemann invariants, as it was done for the KdV equation [46] and NLS equation [34]. One of the possibilities is to find the solitary limit of the corresponding modulation equations. For this we need to find a long wave limit of the wave action conservation law [21]. For generic Hamiltonian systems such an approach was recently developed in [6] with interesting applications to the second gradient fluids. We will derive such a solitary limit for the BBM equation and will obtain corresponding analytical solutions.

2. Periodic solutions of the BBM equation

The travelling wave solutions of the BBM equation $u = u(\xi),\ \xi = x - Dt$ satisfy the equation:

$$-D(u - u'') + \frac{u^2}{2} = c_1, \quad c_1 = \text{const.} \tag{7}$$

Here 'prime' means the derivative with respect to ξ. It implies the first integral:

$$D\frac{u'^2}{2} = -\frac{u^3}{6} + D\frac{u^2}{2} + c_1\, u + c_2 = \frac{1}{6}(u - u_1)(u - u_2)(u_3 - u), \quad c_2 = \text{const.}, \tag{8}$$

where new constants $u_1 \leqslant u_2 \leqslant u_3$ are introduced. They related with D, c_1, and c_2:

$$D = \frac{1}{3}(u_1 + u_2 + u_3), \quad c_1 = -\frac{1}{6}(u_1u_2 + u_1u_3 + u_2u_3), \quad c_2 = \frac{1}{6}u_1u_2u_3. \tag{9}$$

Another form of the equation is:

$$(u_1 + u_2 + u_3)u'^2 = P(u), \quad P(u) = (u - u_1)(u - u_2)(u_3 - u). \tag{10}$$

In the following, we will consider only positive solutions ($0 < u_1 < u_2 < u < u_3$) (the negative solutions can be found by the symmetry $u \to -u$ and $D \to -D$). Such a restriction is not necessary: the only condition is $D \neq 0$. However, this will allow us to avoid every time remarks on the sign of the travelling wave velocity. The periodic solution $u(\xi)$ is:

$$u(\xi) = u_2 + a\ cn^2\,(\eta, m)\,, \tag{11}$$

where

$$m = \frac{u_3 - u_2}{u_3 - u_1}, \quad a = u_3 - u_2, \quad \eta = \frac{\xi + \xi_0}{2\sqrt{3D}}\sqrt{\frac{a}{m}}, \quad \xi_0 = \text{const.} \tag{12}$$

Here $cn\,(\eta, m) = \cos(\varphi(\eta, m))$, where φ is defined implicitly from

$$\eta = \int_0^{\varphi(\eta,m)} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}. \tag{13}$$

The wave length is given as

$$L = 4\sqrt{3}\sqrt{\frac{Dm}{a}}\,K(m). \tag{14}$$

In particular, the solitary wave solution obtained in the limit $L \to \infty$ and for the values $u_1 = u_2 > 0, a = u_3 - u_2$ is in the form

$$u(\xi) = u_2 + \frac{a}{\cosh^2(\eta)}, \quad \eta = \frac{\xi + \xi_0}{2\sqrt{1 + \frac{3u_2}{a}}}, \quad D = u_2 + \frac{a}{3}, \quad \xi_0 = \text{const.} \tag{15}$$

We will define the wave averaged of any function $f(u)$ as

$$\overline{f(u)} = \int_{u_2}^{u_3} \frac{f(u)\mathrm{d}u}{\sqrt{P(u)}} \Bigg/ \int_{u_2}^{u_3} \frac{\mathrm{d}u}{\sqrt{P(u)}}. \tag{16}$$

In particular, the wave averaged of u (denoted below by $\overline{u}$) is given by:

$$\overline{u} = \frac{\int_{u_2}^{u_3} \frac{u\mathrm{d}u}{\sqrt{P(u)}}}{\int_{u_2}^{u_3} \frac{\mathrm{d}u}{\sqrt{P(u)}}} = u_1 + (u_3 - u_1)\frac{E(m)}{K(m)} = u_2 + \frac{a}{m}\left(\frac{E(m)}{K(m)} + m - 1\right). \tag{17}$$

Here the complete elliptic integrals of the first and second type are defined as [1]:

$$K(m) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}, \qquad E(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2\theta}\,\mathrm{d}\theta. \tag{18}$$

The inverse formulas expressing u_1, u_2 and u_3 as functions of $\overline{u}$, a and m are given by

$$u_1 = \bar{u} - \frac{a}{m}\frac{E(m)}{K(m)}, \quad u_2 = \bar{u} - \frac{a}{m}\left(\frac{E(m)}{K(m)} + m - 1\right), \quad u_3 = \bar{u} - \frac{a}{m}\left(\frac{E(m)}{K(m)} - 1\right). \tag{19}$$

One can check that the change of variables is invertible, i.e., its Jacobian matrix has its inverse because

$$\det\left(\frac{\partial(u_1, u_2, u_3)}{\partial(\overline{u}, a, m)}\right) = \frac{a}{m^2} \neq 0. \tag{20}$$

The velocity D is given by the formula

$$D = \frac{1}{3}(u_1 + u_2 + u_3) = \bar{u} + \frac{a}{m}\left(\frac{2-m}{3} - \frac{E(m)}{K(m)}\right). \tag{21}$$

We will show further the importance of a special case $D = \bar{u}$: the phase velocity coincides with the characteristic of the Hopf equation for the homogeneous state $\overline{u}$. The corresponding value of m is the solution of:

$$\frac{2-m}{3} = \frac{E(m)}{K(m)}. \tag{22}$$

This value is unique: $m = m_c \approx 0.961\,149$.

3. Whitham modulation equations for the BBM system

Two equivalent methods can be used to obtain the modulation equations: the averaging of the conservation laws [7, 45] and Whitham's method of averaged Lagrangian [46]. Both methods are complementary in the analysis of the modulation equations. The first one assures the initial conservative structure of the governing equations, while the second one can give an idea about the choice of 'appropriate' variables for the theoretical study of the modulation equations [24, 46].

The method of conservation laws for the BBM equation was used, in particular, in [44]. The essence of the method is as follows. We are looking for the solution $u(\xi, X, T, \varepsilon)$ which is periodic with respect to ξ and varies slowly with respect to time and space, with $\xi = \frac{X-DT}{\varepsilon} = x - Dt,\ X = \varepsilon x,\ T = \varepsilon t$, ε being a small parameter. The solution period L is thus also a slowly varying function. Commutating the averaging with respect to ξ, and time and space derivatives, we obtain from the first two conservation laws (2) and (3) the equations:

$$(\overline{u})_t + \left(\frac{\overline{u^2}}{2}\right)_x = 0,$$

$$\left(\frac{\overline{u^2}}{2} + \frac{\overline{u'^2}}{2}\right)_t + \left(\frac{\overline{u^3}}{3} - D\overline{u'^2}\right)_x = 0.$$

We used here the relation $\overline{u''} = 0$, $\overline{(uu')'} = 0$. The averaging of the third equation is equivalent to the phase conservation law [44]:

$$k_t + (Dk)_x = 0, \quad k = \frac{1}{L}.$$

For simplicity, we defined here the wave number k as $1/L$ and not as $2\pi/L$. Also, instead of the slow variables T, X we returned back to the variables t, x.

Using (10), one can write the modulation equations in an equivalent form:

$$(\overline{u})_t + \left(\frac{\overline{u^2}}{2}\right)_x = 0, \tag{23}$$

$$\left(\frac{\overline{u^2}}{2} + \frac{\overline{P(u)}}{6D}\right)_t + \left(\frac{\overline{u^3}}{3} - \frac{\overline{P(u)}}{3}\right)_x = 0, \tag{24}$$

$$(1/L)_t + (D/L)_x = 0. \tag{25}$$

We choose the variables $\overline{u}$, a and m as unknowns. One can find:

$$\overline{u} = u_2 + a\,A_1, \tag{26a}$$

$$\overline{u^2} = u_2^2 + 2u_2 a\,A_1 + a^2 A_2 = \overline{u}^2 + a^2(A_2 - A_1^2), \tag{26b}$$

$$\begin{aligned}\overline{u^3} &= u_2^3 + 3u_2^2 a A_1 + 3u_2 a^2 A_2 + a^3 A_3 \\ &= \overline{u}^3 + 3\overline{u}a^2(A_2 - A_1^2) + a^3(A_3 - 3A_1A_2 + 2A_1^3),\end{aligned} \tag{26c}$$

$$\overline{P(u)} = \frac{a^3}{m} P_2(m), \tag{26d}$$

with

$$A_k(m) = \int_0^{\pi/2} \frac{\cos^{2k}\theta\,\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}} \Bigg/ \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}},$$

$$P_2(m) = \int_0^{\pi/2} \sin^2\theta\cos^2\theta\,\sqrt{1 - m\sin^2\theta}\,\mathrm{d}\theta \Bigg/ \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - m\sin^2\theta}}.$$

The integrals $A_k(m)$ and $P_2(m)$ can also be expressed in terms of $E(m)$ and $K(m)$ (see appendix A). Still, even if the equations can now be explicitly written in terms of $a, \overline{u}, m$, it is difficult to extract from (23)–(25) 'reasonably simple' closed form solutions to compare with numerical solutions of the exact BBM equation. The idea is to simplify the equations (23)–(25) in the singular limit as the wave length goes to infnity (solitary limit) [6, 21]. Equations (23) and (24) give in this limit the Hopf equation, while the equation (25) becomes a trivial identity. We follow here the approach proposed in [6] where such a limit was obtained from the action conservation law for the averaged Lagrangian.

The Whitham method of the averaged Lagrangian consists in looking for a solution of the Euler–Lagrange equations for (5) of the form [46]:

$$\varphi = \beta x - \gamma t + \psi(\theta), \quad \theta = kx - \omega t,$$

with β, γ, k, ω depending on T and X. The following relations are the compatibility conditions:

$$\beta_t + \gamma_x = 0, \quad k_t + \omega_x = 0. \tag{27}$$

The function $\psi(\theta, T, X, \varepsilon)$ is supposed to be one-periodic with respect to the variable θ. Since $\omega = Dk$, the variables θ and ξ are related: $\theta = k\xi$.

The unknown functions should be determined as solutions of the Euler–Lagrange equations for the averaged Lagrangian

$$\overline{\mathcal{L}} = \int_0^1 \mathcal{L}\mathrm{d}\theta, \tag{28}$$

where $\mathcal{L}$ is given by (5). The derivation is quite standard and follows directly the derivation of the modulation equations for the KdV equation (see [46], section 16.14). We present here a rapid derivation. In zero order one has:

$$\begin{aligned} u = \varphi_x &\approx \beta + k\psi_\theta, \\ \varphi_t &\approx -\gamma - \omega\psi_\theta = -\gamma - D(u - \beta), \\ \varphi_{xxx} = u_{xx} &\approx k^2 u_{\theta\theta}. \end{aligned}$$

Then the zero order Lagrangian (5) (defined up to the full derivative with respect to θ) is:

$$\mathcal{L} \approx \frac{u(\gamma - D\beta)}{2} + \frac{Du^2}{2} - \frac{u^3}{6} + \frac{D}{2}k^2 u_\theta^2.$$

The dependence of u on the rapid variable is determined from (7):

$$Dk^2 u_{\theta\theta} - Du + \frac{u^2}{2} = c_1.$$

It can be integrated once:

$$Dk^2 \frac{u_\theta^2}{2} = \frac{1}{6}\left(-u^3 + 3Du^2 + 6c_1 u + 6c_2\right) = \frac{P(u)}{6},$$

where $P(u) = -u^3 + 3Du^2 + 6c_1 u + 6c_2$. Then, the averaged Lagrangian (28) becomes

$$\overline{\mathcal{L}} \approx \frac{2k}{\sqrt{3}}\sqrt{D}\int_{u_2}^{u_3}\sqrt{P(u)}\mathrm{d}u - c_1\beta - c_2 + \frac{\beta(\gamma - D\beta)}{2}.$$

The variation with respect to c_2 gives us the dispersion relation which is equivalent to the expression (14) for the wave length:

$$\frac{1}{k} = 2\sqrt{3D}\int_{u_2}^{u_3}\frac{\mathrm{d}u}{\sqrt{P(u)}} = 4\sqrt{\frac{3Dm}{a}}K(m).$$

The variation with respect to c_1 gives us the identity $\beta = \overline{u}$. Finally, the last two Euler–Lagrange equations

$$\left(\overline{\mathcal{L}}_\gamma\right)_t - \left(\overline{\mathcal{L}}_\beta\right)_x = 0, \tag{29}$$

$$\left(\overline{\mathcal{L}}_\omega\right)_t - \left(\overline{\mathcal{L}}_k\right)_x = 0, \tag{30}$$

should be written. The equation (29) is exactly equation (23):

$$\overline{u}_t + \left(\frac{\overline{u^2}}{2}\right)_x = 0.$$

Its combination with the equation $\beta_t + \gamma_x = 0$ gives us $c_1 = \frac{\gamma - D\beta}{2}$. One also has:

$$\overline{\mathcal{L}}_\omega = \frac{\overline{u^2} - (\overline{u})^2}{2k} + \frac{\overline{P(u)}}{6Dk}, \qquad \overline{\mathcal{L}}_k = -D\left(\frac{\overline{u^2} - (\overline{u})^2}{2k} - \frac{\overline{P(u)}}{6Dk}\right).$$

Hence the wave action equation (30) is:

$$\left(\frac{\overline{u^2} - (\overline{u})^2}{2k} + \frac{\overline{P(u)}}{6Dk}\right)_t + \left(D\left(\frac{\overline{u^2} - (\overline{u})^2}{2k} - \frac{\overline{P(u)}}{6Dk}\right)\right)_x = 0. \tag{31}$$

The equation (31) can also be obtained as a consequence of the equations (23)–(25) [35].

4. Solitary limit

The solitary limit is a singular limit of the modulation equations when the wave length $L \to \infty$ (or $k \to 0$, or $m \to 1$). In this limit, one has

$$\overline{u^2} \to \overline{u}^2, \qquad \overline{P(u)} \to P(\overline{u}) = 0.$$

Thus, the equations (23) and (24) have the same limit:

$$\overline{u}_t + \overline{u}\,\overline{u}_x = 0.$$

We need thus to find the limit form of (31). Hand calculations are feasible but a bit tedious, and the calculation can best be done using a computer algebra system. One example is shown in appendix A using Matlab. One obtains the following equation:

$$\left(\frac{a^{3/2}(2a + 5\overline{u})}{\sqrt{a + 3\overline{u}}}\right)_t + \left(\frac{a^{3/2}(4a + 15\overline{u})\sqrt{a + 3\overline{u}}}{9}\right)_x = 0.$$

The final conservative system for the solitary limit of the BBM equation is thus

$$\begin{aligned} &\overline{u}_t + \left(\frac{(\overline{u})^2}{2}\right)_x = 0, \\ &F(a, \overline{u})_t + G(a, \overline{u})_x = 0, \end{aligned} \tag{32}$$

where

$$F(a,\overline{u}) = \frac{a^{3/2}(2a+5\overline{u})}{\sqrt{a+3\overline{u}}}, \quad G(a,\overline{u}) = \frac{a^{3/2}(4a+15\overline{u})\sqrt{a+3\overline{u}}}{9}.$$

The quasilinear form of (32) is:

$$\overline{u}_t + \overline{u}\,\overline{u}_x = 0, \quad a_t + D\,a_x + \frac{a}{3}\frac{14a^2+75a\overline{u}+90\overline{u}^2}{8a^2+40a\overline{u}+45\overline{u}^2}\overline{u}_x = 0, \quad D = \overline{u} + \frac{a}{3}. \tag{33}$$

The characteristics of this hyperbolic system are $\overline{u}$ and D. We will construct now closed form non-stationary solutions of (33).

5. Interaction of solitary waves with a step

Consider the Cauchy problem for (32):

$$(\overline{u}, a)(0,x) = \begin{cases} (u^-, a^-), & x < 0, \\ (u^+, a^+), & x > 0. \end{cases}$$

We are looking for self-similar continuous solutions of (32) (or (33)) for the corresponding Riemann problem in the case $0 < u^- < u^+$ (the case of 'positive' rarefaction waves). In particular, the simple-wave solutions of this system will be used to describe the interaction of an incident solitary wave of amplitude a^- with a step function for $\overline{u}$. As a result of such an interaction, an outgoing solitary wave of amplitude a^+ is formed (see figure 1). Such a problem, even in a more general framework, was analytically and numerically studied in [42] for the defocusing nonlinear Schrödinger equation, in [30] for the conduit equation, and in [38] for the modified KdV equation. We obtain here an analytical solution of this interaction problem for the BBM equation.

The Hopf equation implies: $\overline{u} = s = x/t$, $u^- < s < u^+$. For the function $a(s) = a(\overline{u})$ one obtains from (33) the following ODE:

$$\frac{\mathrm{d}a}{\mathrm{d}\overline{u}} = -\frac{G_{\overline{u}} - \overline{u}F_{\overline{u}}}{G_a - \overline{u}F_a} = -\frac{14a^2+75a\overline{u}+90\overline{u}^2}{8a^2+40a\overline{u}+45\overline{u}^2}. \tag{34}$$

It admits the group transformation $a \to ba$, $\overline{u} \to b\overline{u}$, $b = \text{const}$. For the corresponding invariant $z = a/\overline{u}$ one obtains the equation

$$\overline{u}\frac{\mathrm{d}z}{\mathrm{d}\overline{u}} = -f(z), \quad f(z) = \frac{14z^2+75z+90}{8z^2+40z+45} + z.$$

It allows us to obtain the relation between the incoming a^- and outgoing a^+ solitary wave amplitudes:

$$\int_{z^+}^{z^-} \frac{\mathrm{d}z}{f(z)} = \ln\frac{u^+}{u^-}, \quad z^{\pm} = \frac{a^{\pm}}{u^{\pm}}. \tag{35}$$

The relation (35) can be written as

$$p(z^-) - p(z^+) = \ln\left(\frac{u^+}{u^-}\right), \quad z^{\pm} = \frac{a^{\pm}}{u^{\pm}}, \tag{36}$$

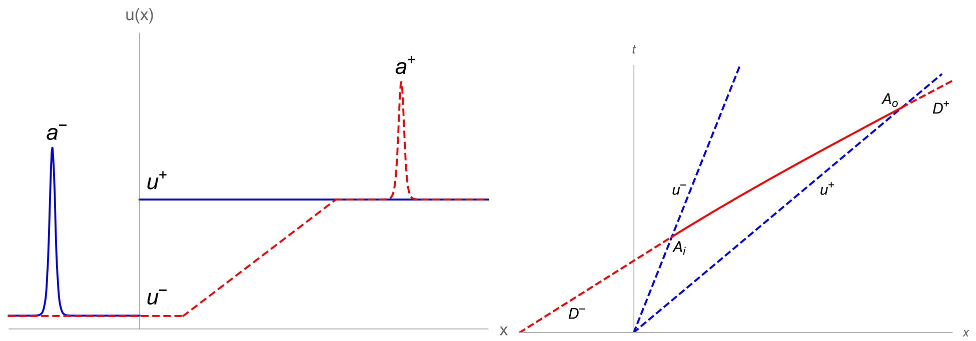

Figure 1. Left figure: a sketch of the interaction of a solitary wave of amplitude a^- with a step. Right figure: (x, t) diagram of the interaction problem. The solitary wave of velocity $D^- = u^- + a^-/3$ (dashed red line) enters the rarefaction fan bounded by the characteristics $u^\pm$ (dashed blue lines) at point A_i, interacts with it (red line between points A_i and A_o), and finally comes out of it at point A_o with the velocity $D^+ = u^+ + a^+/3$ (dashed red line). Such a configuration can exist if and only if the amplitude a^- of an incident solitary wave is greater than some critical value $a_{\min} = z_{\min}u^-$, where $z_{\min}$ is the root of (38). Otherwise, the solitary wave is trapped by the rarefaction fan.

with

$$p(z) = \frac{1}{24}\left(-2\sqrt{15}\arctan\left(\frac{15+8z}{\sqrt{15}}\right) - 6\ \ln(3+z) + 15\ \ln(15+15z+4z^2)\right). \quad (37)$$

The condition for the solitary wave trapping is $z^+ = 0$. To have a solitary wave which is capable to pass the initial step function, we have to take z^- larger than the minimal value $z^-_{\min}$ which is a unique root of the equation

$$p(z^-_{\min}) - p(0) - \ln\left(\frac{u^+}{u^-}\right) = 0. \quad (38)$$

In figure 2 we show the comparison of the theoretical curve (36) between incoming-outgoing amplitudes $a^\pm$, and numerical results for the exact BBM equation ('dots') for particular values of $u^\pm, u^- < u^+$ and different values of the incoming amplitude a^+. The idea of such a simple solution of the interaction problem was originally proposed in [42] and was applied there to the KdV equation, and in a companion paper [30] to the conduit equation. One of the key points of such an approach is a possibility to obtain an analytical expression for the Riemann invariants.

The maximum of the initial solitary wave was placed at $x_0 = -400$, the initial discontinuity was replaced by the hyperbolic tangent:

$$u(0, x) = u^+ + (u^+ - u^-)\tanh\left(\frac{x - x_0}{l}\right), \quad (39)$$

with $l = 100$. The numerical results do not depend on the choice of x_0 and l, if $x_0 \gg l \gg 1$. A very good agreement between the theoretical and numerical results can be observed. In figure 3 a solitary wave having the incoming amplitude $a^- \approx 2.248\,13$ is taken. For $u^- = 1/3$ and $u^+ = 1$ the amplitude a^+ of the outgoing wave fits perfectly the theoretical value $a^+ = 1$. In the case of several solitary waves having the same amplitude a^- one obtains the solitary wave train of the same amplitude a^+.

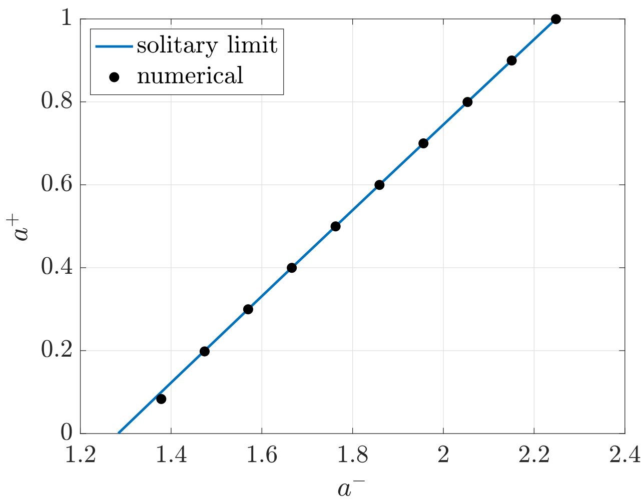

Figure 2. The case $u^- = 1/3$ and $u^+ = 1$ is illustrated. The amplitude $a^+ \in [0, 1]$ of the outgoing solitary wave as a function of the incoming wave amplitude a^- is shown. In particular, the condition (38) for the wave trapping ($a^+ = 0$) gives us $a^- \approx 1.283\,212\,944$. To have $a^+ = 1$ we need to take $a^- \approx 2.248\,131\,44$ (for this, one needs to solve (36)). The theoretical relation (36) and (37) (continuous line) is compared with the corresponding numerical computations for the exact BBM equation (shown by 'dots'). A very good agreement is observed.

One can also remark that the equation (32) can be rewritten in terms of the Riemann invariants:

$$\overline{u}_t + \overline{u}\,\overline{u}_x = 0, \quad r_t + D\,r_x = 0, \quad D = \overline{u} + \frac{a}{3}, \tag{40}$$

with

$$r = \ln(\overline{u}) + p\left(\frac{a}{\overline{u}}\right),$$

where $p(z)$ is given by (37). Thus, the condition (36) is the conservation of the Riemann invariant r.

6. Generalized Riemann problem for dispersive equations

We call a GRP the Cauchy problem

$$u(0, x) = \begin{cases} u_{\mathrm{L}}(x), & x < 0, \\ u_{\mathrm{R}}(x), & x > 0, \end{cases} \tag{41}$$

where $u_{\mathrm{L,R}}(x)$, are different periodic travelling wave solutions of the corresponding dispersive equations (in particular, of the BBM equation). Such a problem was studied in [14] for the SGN equations and Boussinesq equations with linear dispersion, and in [43] for the fifth order KdV equation. In particular, in the first reference new stable shock-like travelling wave solutions were found linking a constant solution to a periodic wave train. The shock-like transition zone

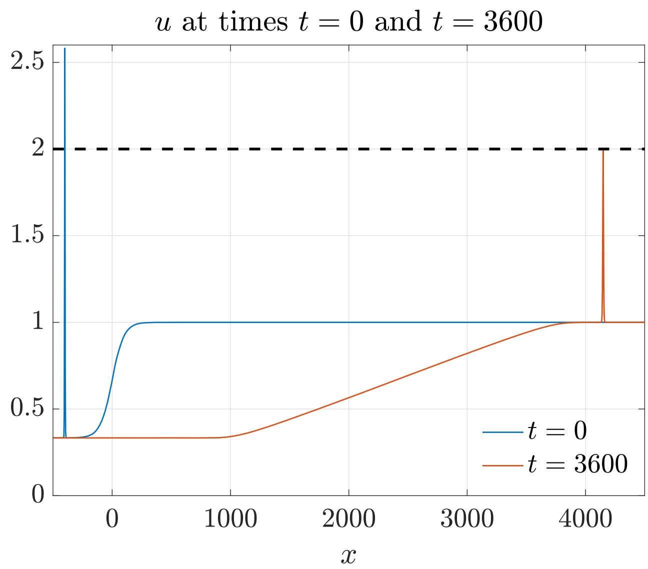

Figure 3. The incoming solitary wave of amplitude $a^- \approx 2.248\,13$ produces the outgoing solitary wave of amplitude $a^+ = 1$ (for $u^- = 1/3$ and $u^+ = 1$).

between the constant state and the wave train was well described by the half of solitary wave having the wave crest at the maximum of the nearest periodic wave.

Such a configuration was stable under certain conditions. The aim of this section is to describe in details the analogous solutions for the BBM equations and propose an explicit criterion for the existence of such stable solutions.

For numerical purposes, we restrict our attention to a modified version of (41) in the form

$$u(0,x) = \begin{cases} u(x), & x_0 < x < x_1, \\ \overline{u}, & \text{if } x \text{ is outside of } (x_0, x_1). \end{cases} \tag{42}$$

Here (x_0, x_1) is the interval which contains a quite large number of entire periods (figure 4).

Indeed, since the BBM equation has a 'hyperbolic' property (the waves propagate with a finite speed), it is much easier to implement the numerical methods for the BBM equation when the solution tends to a constant value at infinity (see a short description of the method in appendix B). Thus, the 'hyperbolic' property allows us to study separately the evolution of the left and right boundaries of the wave train until the moment when the corresponding waves coming from the boundaries start to interact. To smooth discontinuous initial data (42) we used the same smoothing procedure as in [14].

6.1. Generalized RH conditions for the BBM equation and shock conditions for the Whitham system

Travelling wave solution $u(x)$ for the BBM equation is a smooth extremal curve of the functional

$$a[u] = \int \mathcal{L}(u, u')\mathrm{d}x, \quad \mathcal{L}(u, u') = \frac{Du'^2}{2} + \frac{P(u)}{6},$$

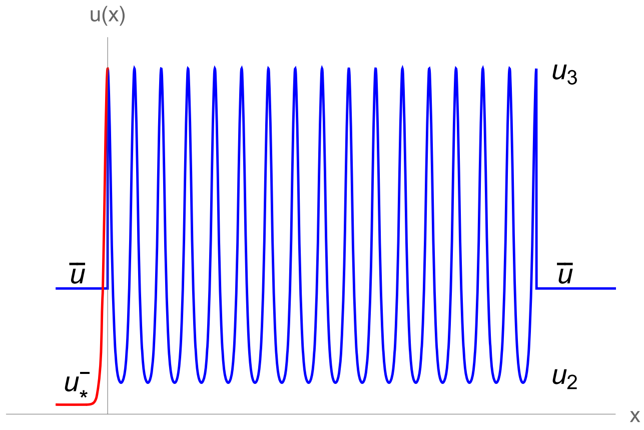

Figure 4. Sketch of the initial configuration (42) consisting of a periodic wave train having the property $D \geqslant \overline{u}$ and bounded on the left and on the right by the constant state $\overline{u}$. If, initially, instead of $\overline{u}$, one takes on the left the state $u_\star^-$ (see the definition (46)) linked with the wave train by the half-solitary wave (red curve), the left boundary of the wave train remains invariable in time.

where the third order polynomial $P(u)$ is given by (10), and the integral is taken over the basic period of $u(x)$. The variation of a can be written as:

$$\delta a = \int \left(\frac{\delta \mathcal{L}}{\delta u}\delta u + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial u'}\delta u \right) \right) \mathrm{d}x, \quad \frac{\delta \mathcal{L}}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \mathcal{L}}{\partial u'} \right).$$

Using the definition (8) of $P(u)$, it can be written as

$$\delta a = \int \left(\left(-Du'' - \frac{u^2}{2} + Du + c_1 \right) \delta u + \frac{\mathrm{d}}{\mathrm{d}x}\left(Du'\delta u \right) \right) \mathrm{d}x.$$

In the case of non-smooth ('broken') extremal curves, the same Euler–Lagrange equation should be satisfied for each smooth part of the extremal curve:

$$Du'' + \frac{u^2}{2} - Du = -c_1 = \text{const}.$$

Using the square brackets to designate the jump of variables, one can rewrite it at the 'broken' point as:

$$-D[u] + [u^2 + Du''] = 0 \tag{43}$$

This equation is nothing but a formal RH relation for the conservation law (2) considered on the travelling wave solutions (so, the derivative $-u_{tx}$ becomes Du''). But, together with (43), an additional condition coming from the term $\frac{\mathrm{d}}{\mathrm{d}x}\left(Du'\delta u \right)$ should also be satisfied at the 'broken' point:

$$[u'] = 0, \tag{44}$$

i.e. u' is continuous at the 'broken' point. This condition is usually called Weierstrass–Erdmann condition, or 'corner' condition [16]. In particular, if a piecewise C^2-solution $u(x)$ is constant on some interval of x, but is not constant on a neighboring interval, this last should have a zero slope at the 'broken' point. Thus, the classical Rankine–Hugoniot condition (43) coming from the conservation law (2) should be supplemented by condition (44). We call this set of conditions (43) and (44) GRH conditions. Such weak solutions describing shock-like transition fronts and satisfying GRH conditions have been also constructed for the SGN equations [14].

We will look now for a possibility to link a generic constant state ('cold' state) $u_\star$ with a generic periodic wave train ('hot' state) by the Rankine–Hugoniot conditions through the shock having the same velocity as the phase velocity D of the wave train (see figure 4).

The GRH condition (43) for travelling waves connecting the constant state $u_\star$ and travelling wave train is:

$$-D(u_3 - u''|_{u=u_3} - u_\star) + \left(\frac{u_3^2}{2} - \frac{u_\star^2}{2}\right) = 0.$$

Here we linked a 'cold' state $u_\star$ with the maximum u_3 of the periodic wave train. Indeed, numerical experiments show that such a linkage with the minimum u_2 is not feasible. Replacing the second derivative at $u = u_3$, the GRH condition can be written also as

$$-D\left(u_3 + \frac{(u_3 - u_1)(u_3 - u_2)}{2(u_1 + u_2 + u_3)} - u_\star\right) + \left(\frac{u_3^2}{2} - \frac{u_\star^2}{2}\right) = 0. \tag{45}$$

This quadratic equation has two real roots, $u_\star^\pm$, $0 \leqslant u_1 < u_\star^- < u_2 < \overline{u} < u_\star^+ < u_3$, given explicitly as:

$$u_\star^\pm = D \pm \frac{\sqrt{u_1^2 + u_2^2 + u_3^2 - u_1u_2 - u_1u_3 - u_2u_3}}{3}, \quad D = \frac{u_1 + u_2 + u_3}{3}. \tag{46}$$

The numerical study shows that in the case of positive u, it is the state $u_\star^-$ which is linked with the maximum of the travelling wave, i.e., u_3. A possible reason for this will be discussed in section 6.2.

Proposition. *The solutions $u_\star$ obtained from both, the RH condition coming from the wave averaged conservation law* (23)

$$-D(\overline{u} - u_\star) + \left(\frac{\overline{u^2}}{2} - \frac{u_\star^2}{2}\right) = 0, \tag{47}$$

and the GRH condition given by (43) *and* (44) *coincide*.

Proof. Subtracting (47) from (45), one obtains:

$$-D\,(u_3 - \overline{u}) + \left(\frac{u_3^2}{2} - \frac{\overline{u^2}}{2}\right) = \frac{(u_3 - u_1)(u_3 - u_2)}{6}.$$

It is sufficient to prove that this is an identity. To show this, one can use the inverse formulas (19) for u_1, u_2, u_3 and (26b) for $\overline{u^2}$ to express them in terms of $\overline{u}$, a and m. Then the proof is direct. Again, a mathematical software package can be used to carry out these analytical computations.

The other conservation laws of the Whitham system (averaged laws corresponding to (3) and (4)) are not satisfied, i.e. it is not a weak solution of the Whitham system (not a true 'Whitham shock' as it was termed in [43] where such a linkage of the wave train and a uniform level was found for the fifth order KdV equation and Kawahara equation). Indeed, the last equations can admit travelling wave solutions linking different periodic orbits. In our case, such a periodic-to periodic connection does not exist. The solutions we constructed are weak solutions to the

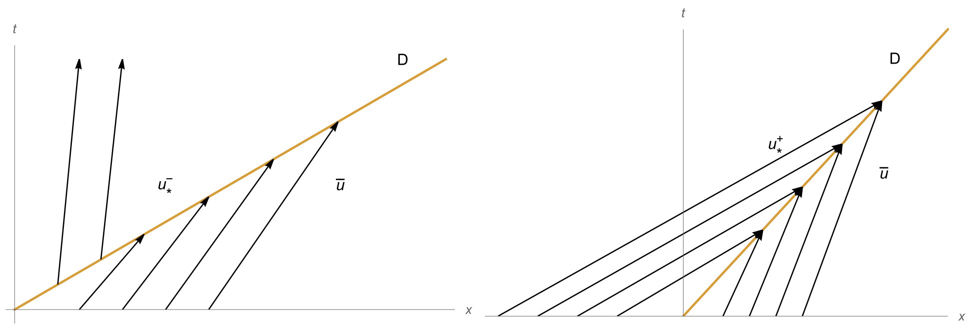

Figure 5. The relation between the characteristics of the homogeneous states $u_\star^\pm$, $\overline{u}$ and shock velocity D. Left figure: stable configuration linking constant state $u_\star^-$ to a periodic wave train with wave mean $\overline{u}$. Right figure: unstable configuration linking constant state $u_\star^+$ to a periodic wave train with wave mean $\overline{u}$.

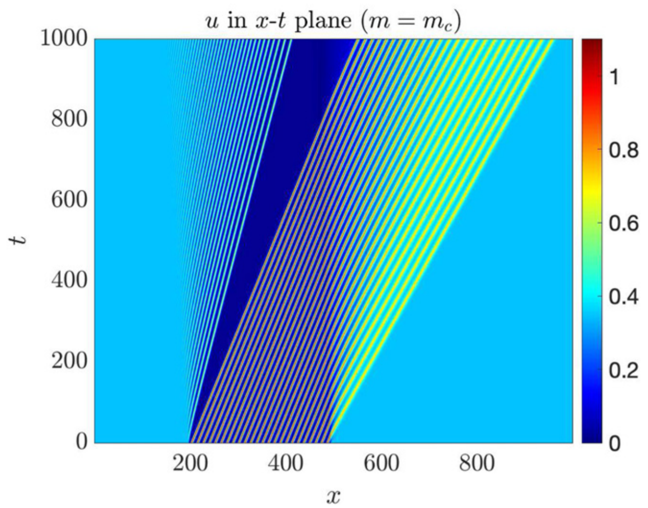

Figure 6. The general structure of the solution of the Cauchy problem (42) is shown in (x, t)-plane. The initial parameters of the wave train are: $u_1 = 0$, $u_3 = 1$ and $u_2 = 1 - m_c$, $m_c \approx 0.961\,149$. On the right side, a right facing (with respect to the state $\overline{u}$) dispersive shock is formed followed by a left facing rarefaction wave with a clearly visible front. A constant state $u_\star^-$ is formed on the left side of the initial wave train followed by a right facing (with respect to the velocity $u_\star^-$) dispersive shock having smaller amplitude than the right dispersive shock. At time t_i about 750 the rarefaction front crosses the left side of the wave train and then perturbs the constant state $u_\star^-$.

exact BBM equation (in the sense of calculus of variations). The 'miracle' is that at least one equation of the Whitham system is satisfied in a weak sense, and it corresponds exactly to the GRH conditions for the exact BBM equation in conservative form (2).

6.2. Stable shock-like transition fronts

Numerical solution of the Cauchy problem (42) shows that in the case $D \geqslant \overline{u}$, the 'cold' state $u_\star^-$ rapidly forms on the left of the periodic wave train. The structure of characteristics corresponding to the homogeneous state $u_\star^-$ and that of the wave train considered as a homogeneous

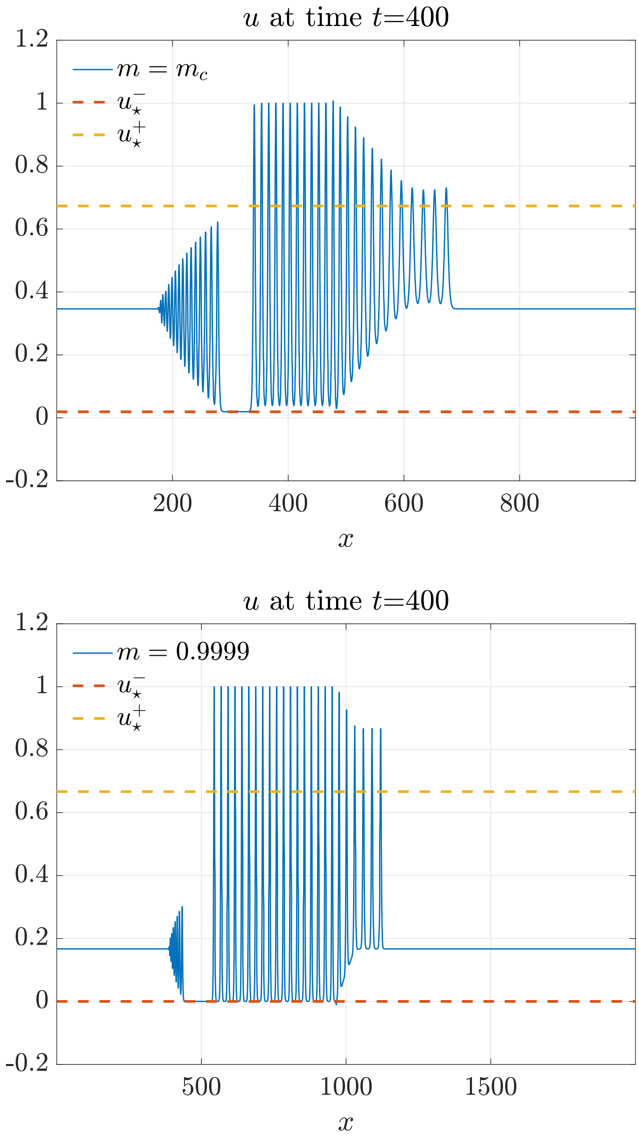

Figure 7. Initially, we consider a periodic wave train with $m \in [m_c, 1)$, with the constant states $\overline{u}$ on the left and on the right. We have chosen $u_1 = 0$, $u_3 = 1$ and $u_2 = u_3(1 - m)$, with parameter $m \in [m_c, 1)$. Then, on the left, a cold state $u_\star^-$ is formed, $u_1 < u_\star^- < u_2$, linked to the periodic wave train by GRH condition (43) and (44). Such a configuration linking the state $u_\star^-$ to the wave train is stable.

state $\overline{u}$ is shown in figure 5. The inequality $D \geqslant \overline{u}$ is equivalent to $m \geqslant m_c$ (see (21) and (22)). Physically, the condition $D \geqslant \overline{u}$ means that the periodic waves are 'almost' solitary waves. Indeed, for the solitary waves their phase velocity is given by the formula $D = \overline{u} + \frac{a}{3}$, i.e., $D > \overline{u}$ is equivalent to $a > 0$.

Large time behavior of the solution in (x, t)-plane is shown in figure 6. On the right side, a right facing (with respect to the state $\overline{u}$) dispersive shock is formed followed by a left facing

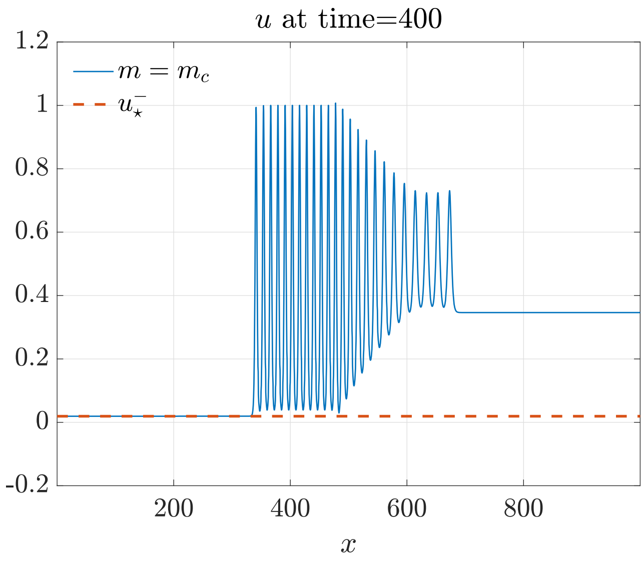

Figure 8. Initially, we take on the left of the wave train the state $u_\star^-$, and $\overline{u}$ on the right. The left part of the wave train remains invariable in time (until the left facing with respect to the wave front velocity rarefaction wave arrives). Thus, if the domain occupied by the wave train were semi-infinite to the right, it would be a true travelling wave linking the constant state $u_\star^-$ to the wave train by the GRH relations.

rarefaction wave with a clearly visible front. A constant state $u_\star^-$ is formed on the left side of the initial wave train followed by a right facing (with respect to the velocity $u_\star^-$) dispersive shock having smaller amplitude than the right dispersive shock. Since the initial wave train is finite, the rarefaction front crosses the left side of the wave train at time t_i and then perturbs the constant state $u_\star^-$. For $t < t_\mathrm{i}$ the wave train is not at all perturbed on the left: the transition front linking the state $u_\star^-$ and wave train is stable. In figure 7 the graph of u as a function to x is shown at a given time instant which is smaller than t_i for two different values of $m \geqslant m_\mathrm{c}$. Again, it can be clearly seen that the left side of the wave train is not perturbed. As in [14], one can numerically show that if, initially, we take on the left of the wave train the state $u_\star^-$ instead of $\overline{u}$ and smooth the transition zone by a half solitary wave (see figure 8) this structure remains invariable in time. If, at the beginning, such a smoothing is not performed, after a non-stationary transient process, such a sharp half-soliton structure is quickly established. Probably, such a half solitary wave resolution is quite universal. In particular, it was also found in [41] for the resolution of a Whitham shock for the Kawahara equation.

If the domain occupied by the wave train were semi-infinite to the right, it would be a true travelling wave linking the constant state $u_\star^-$ to the wave train by the GRH conditions (46). Comparison of numerical values of $u_\star^-$ and those obtained analytically from the GRH conditions (46) is shown in figure 9. A very good agreement is observed.

The mathematical reason for the stability of transition fronts linking $u_\star^-$ with the wave train is probably the following. Since the shock velocity coincides with the phase velocity, i.e., it is given *a priori*, it is sufficient to have just one characteristic entering the shock, so no need to satisfy the Lax stability condition for shocks (left figure in figure 5). This is also a reason why the state $u_\star^+$ cannot be linked to the wave train. Indeed, since the shock velocity D is already

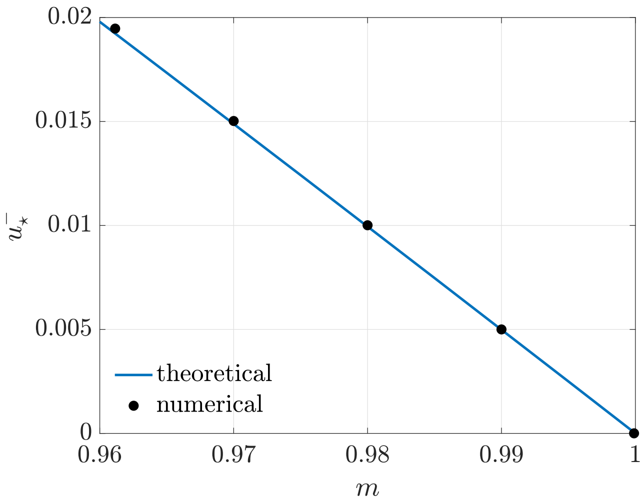

Figure 9. Comparison of numerical values of $u_\star^-$ (dots) in the case $u_1 = 0, u_3 = 1$ and $u_2 = 1 - m, m_c < m < 1$ for $m = m_c \approx 0.961\,149, 0.97, 0.98, 0.99$ and 0.9999, and the corresponding theoretical curve given explicitly from (46) is the form $u_\star^- = (2 - m - \sqrt{1 + m^2 - m})/3$ (blue line). A very good agreement is observed.

given, too much information 'arrives' on the shock front: this is not a well posed problem (right figure in figure 5).

If m is outside the interval $[m_c, 1)$ (i.e. $D < \overline{u}$), such a stationary shock-like configuration on the left does not exist. The 'cold' state appears separating the classical dispersive shock (on the left) and wave train, but the linkage is immediately destroyed by the rarefaction wave arising on the left side of the wave train (see figure 10).

Finally, for such a stable configuration, we are also able to determine the amplitude of the leading right solitary wave which is emitted on the right by the periodic wave train of finite length. The answer is surprisingly simple. Even if we cannot rigorously explain the mathematical reason of this, we can give an analytical expression for the amplitude of the leading solitary wave. Recall again that if the periodic wave train has the property $1 > m \geqslant m_c$ (or, what is equivalent, its travelling velocity is not less than $\overline{u}$), there exist a 'cold' state $u_\star^-$, $u_1 < u_\star^- < u_2 < \overline{u} < u_3$ such that the wave train is connected with the 'cold' state on the left by the half of a solitary wave having the amplitude $a_s^- = u_3 - u_\star^-$. Now, we claim that to define the amplitude of the solitary wave a_s^+ on the right, it is sufficient to solve the equation (36):

$$p(z^-) - p(z^+) - \ln\left(\frac{u^+}{u^-}\right) = 0, \tag{48}$$

with

$$z^- = \frac{a_s^-}{u^-}, \quad u^- = u_\star^-, \quad a_s^- = u_3 - u_\star^-, \quad z^+ = \frac{a_s^+}{u^+}, \quad u^+ = \overline{u}.$$

expressing the condition $r = \text{const}$. In other words, if one takes the incident solitary wave of amplitude $u_3 - u_\star^-$ (and not of $u_3 - u_2$), one obtains the leading solitary wave emitted by the wave train of amplitude a^+ defined by (48). This rather unexpected result is in very good agreement with the numerical results obtained by solving the corresponding Cauchy problem for the BBM equation (see figure 11). In particular, for $u_1 = 0, u_3 = 1$ and $u_2 = 1 - m$ one has

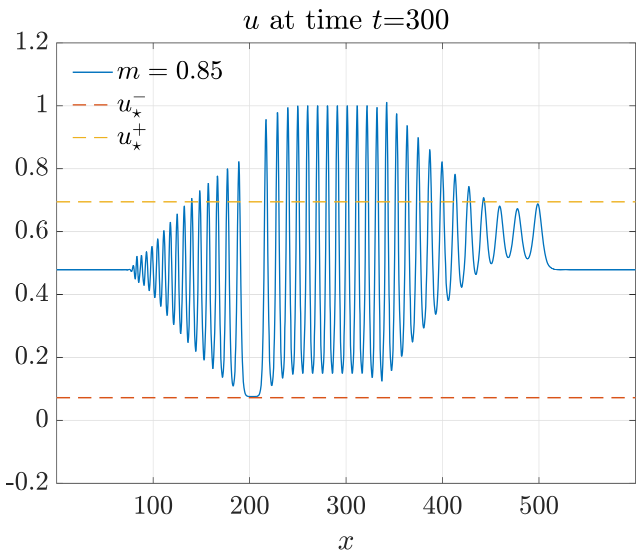

Figure 10. A periodic wave train with $u_1 = 0$, $u_3 = 1$, $u_2 = 1 - m$ and $m = 0.85 < m_c$ is taken, bounded by the constant states $\overline{u}$ on the left and on the right. A 'cold' state is formed on the left (it does not coincide with $u_\star^-$) but its linkage with the periodic wave train periodic wave train represents only a transient structure: it is immediately destroyed by the dispersive shock now additionally occurring on the left boundary of the wave train.

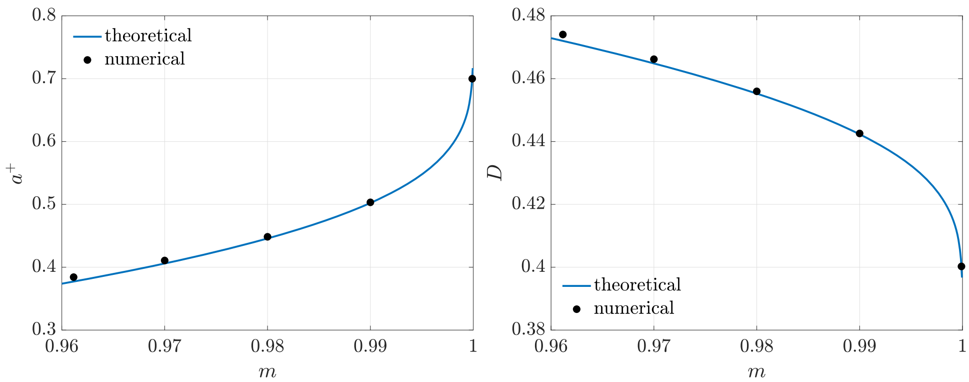

Figure 11. The amplitude and phase velocity of the right leading solitary wave emitted by L-periodic wave train ($u(0, x) = u(0, x + L)$) of finite length bounded by the constant states $\overline{u}$ on the left and on the right. The dots are numerical solutions of the Cauchy problem (42) for the BBM equation corresponding to $u_1 = 0, u_3 = 1$, $u_2 = 1 - m$, for $m = m_c \approx 0.961\,149$, 0.97, 0.98, 0.99 and 0.9999.

the following approximate values: for $m = 0.9999$ one obtains $\overline{u} \approx 0.166\,946$, $u_\star^- \approx 0.000\,05$, $a^- = 1 - u_\star^-$, and finally $a^+ \approx 0.699\,956$, $D \approx 0.400\,265$; for $m = m_c \approx 0.961\,149$, one has $\overline{u} \approx 0.346\,284$, $u_\star^- \approx 0.019\,233$, $a^- = 1 - u_\star^-$, and finally $a^+ \approx 0.377\,28$, $D \approx 0.4720$. We find good agreement between analytical results and numerical solutions (see figure 11). Probably, this can be explained by the fact that it is actually the interaction between an 'almost' solitary wave train (m is close to 1) and the rarefaction wave (see section 5).

7. Conclusion

The existence of a stable shock-like transition from a constant state to a periodic wave train was discovered in [14] for the SGN equations. Here we have established the analogous result for the BBM equation which shares with the SGN equations the same property of finite phase and group velocity for the corresponding linearized equations. The front represents the half of solitary wave linking the constant state with the periodic wave train. We formulate the condition for existence of such a shock-like structure: the phase velocity of the periodic wave train should be not less than the wave averaged solution, and the GRH conditions (43) and (44) are satisfied.

The solitary limit of the Whitham modulation equations was derived. The equations of the solitary limit are hyperbolic and admit the Riemann invariants in explicit form. This allowed us, in particular, to test the numerical method for the BBM equation on asymptotically exact solutions. For a special Cauchy problem (42), the amplitude of the right leading solitary wave has been explicitly determined by (48) which is the conservation of the Riemann invariant of the hyperbolic system (40) describing the solitary limit.

Acknowledgments

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Appendix A. MATLAB code for solitary limits of *F* and *G*

The expressions of $A_i(m)$ and $P_2(m)$ can also be given in terms of the complete elliptic integrals $K(m)$ and $E(m)$:

$$A_1(m) = \frac{E(m) - (1-m)K(m)}{mK(m)}, \tag{49a}$$

$$A_2(m) = \frac{(-2+4m)E(m) + (2-3m)(1-m)K(m)}{3m^2K(m)}, \tag{49b}$$

$$A_3(m) = \frac{(8+23m(m-1))E(m) + (-8+m(19-15m))(1-m)K(m)}{15m^3K(m)}, \tag{49c}$$

$$P_2(m) = \frac{2(1+m(m-1))E(m) + (-2+m)(1-m)K(m)}{15m^2K(m)}. \tag{49d}$$

The formulas are useful to compute approximate theoretical values of the phase velocity D, $\overline{u}$, and so on, by using a computer algebra system. One example is shown below for the computation of the solitary limits of F and G (see (32)) using Matlab (the wave averaged $\overline{u}$ is denoted below by U):

```
function solitary_limit()
syms a real; assumeAlso(a>0);
syms U real; assumeAlso(U>0);
syms m real;
syms A1(m);
syms A2(m);
syms A3(m);
syms P2(m);
syms D(a,U,m);
syms F(a,U,m);
syms G(a,U,m);
%
% define basic functions
A1(m) = ((ellipticE(m)-(1-m)*ellipticK(m))/m)/ellipticK(m);
A2(m) = (((-2+4*m)*ellipticE(m)+(2-3*m)*(1-m)*...
ellipticK(m))/(3*m^2))/ellipticK(m);
A3(m) = (((8+23*(-1+m)* m)*ellipticE(m)+(-8+(19-15*m)* m)*(1-m)*...
ellipticK(m))/(15*m^3))/ellipticK(m);
P2(m) = ((2*(1+(-1+m)* m)*ellipticE(m)+...
(-2+m)*(1-m)*ellipticK(m))/(15*m^2))/ellipticK(m);
%
% define phase velocity
D(a,U,m) = U+(a/m)*((2-m)/3-ellipticE(m)/ellipticK(m));
%
% define density F and flux G
F(a,U,m) = sqrt(D(a,U,m)*m/a)*ellipticK(m)*...
(a^2*(A2(m)-A1(m)*A1(m))/2+a^3*P2(m)/(6*m*D(a,U,m)));
G(a,U,m) = sqrt(D(a,U,m)*m/a)*ellipticK(m)*D(a,U,m)*...
(a^2*(A2(m)-A1(m)*A1(m))/2-a^3*P2(m)/(6*m*D(a,U,m)));
%
limit(F,m,1,'left');
limit(G,m,1,'left');
```

Appendix B. Numerical method

To find approximate solutions to the BBM equation, we use the hyperbolic-elliptic splitting approach developed previously in [14, 29]. This algorithm consists of two steps. In the first step, the hyperbolic step, we employ the state-of-the-art method for hyperbolic conservation laws for the numerical resolution of the equation

$$\mathcal{K}_t + \left(\frac{u^2}{2}\right)_x = 0, \quad \text{with } \mathcal{K} = u - u_{xx},$$

over a time step Δt. In the second step, the elliptic step, using the approximate solution $\mathcal{K}$ computed during the hyperbolic step, we invert numerically the elliptic operator:

$$u - u_{xx} = \mathcal{K}$$

with prescribed boundary conditions based on a fourth-order compact scheme [27].

More precisely, in the hyperbolic step, we use the semi-discrete finite volume method written in a wave-propagation form as before [14], but employ a different solution reconstruction technique, the boundary variation diminishing (BVD) principle, which is more robust than the classical one for the interpolated states ($\mathcal{K}$ for the BBM equation) at cell boundaries (cf [9] and the references cited therein). These reconstructed variables form the basis for the initial data of the Riemann problems, where the solutions of the Riemann problems (obtained from the local Lax–Friedrichs approximate solver [28] for the BBM equation) are then used to construct the fluctuations in the spatial discretization that gives the right-hand side of the system of ODEs (cf [25, 26]). To integrate the ODE system in time, the strong stability-preserving (SSP) multistage Runge–Kutta scheme [22, 40] is used. In particular, for the numerical results presented in this paper, the third-order SSP scheme was employed together with the pair of third- and fifth-order weighted essentially non-oscillatory (WENO) scheme in the BVD reconstruction process.

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