PDE Midterm Exam: Sample Solutions

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1. Note that the *Tricomi* equation is often written by either the form

$$u_{yy} - yu_{xx} = 0$$

or

$$u_{xx} - \frac{1}{y}u_{yy} = 0.$$

Take the latter form of the equation, for example. It can be viewed as the multiply of two differential operators as follows:

$$\left(\frac{\partial}{\partial x} - \frac{1}{\sqrt{y}}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{1}{\sqrt{y}}\frac{\partial}{\partial y}\right)u = 0.$$

Then from the above we may have the splitting of the equation:

$$\left(\frac{\partial}{\partial y} - \frac{1}{\sqrt{y}}\frac{\partial}{\partial x}\right)u = 0 \qquad \text{or} \qquad \left(\frac{\partial}{\partial y} + \frac{1}{\sqrt{y}}\frac{\partial}{\partial x}\right)u = 0;$$

yielding a pair of the characteristic equations for the Trocomi equation. From there, it is easy to see that the associated characteristic curves can be determined by

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{y}}.$$

Hence, integrating the equation on the both sides, we find the result:

$$\int \pm\sqrt{y}dy = \int dx \qquad \Rightarrow \qquad \pm\frac{2}{3}y^{\frac{3}{2}} + C = x \qquad \Rightarrow \qquad 3x \pm 2y^{\frac{2}{3}} = C,$$

where C is an integration constant.

2. Given the d'Alembert form of the solution for the wave equation: $u(x,t) = \phi(x+ct) + \psi(x-ct)$, for some smooth functions ϕ and ψ, we may perform partial differentiation of u with respective to t, and find

$$u_t(x,t) = c\phi'(x+ct) - c\psi'(x-ct).$$

With that, by applying the initial conditions, we have

$$\phi(x) + \psi(x) = f(x), \tag{1}$$

$$c\phi'(x) - c\psi'(x) = g(x). \tag{2}$$

Now integrating (2) with respective to x, we have

$$\phi(x) - \psi(x) = \frac{1}{c}\int_{x_0}^{x} g(s)\, ds + C. \tag{3}$$

Then by the simple algebraic operation $\frac{1}{2}[(1)+(3)]$ and $\frac{1}{2}[(1)-(3)]$, it is easy to get

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\, ds + \frac{C}{2},$$
$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\, ds - \frac{C}{2},$$

where C is constant. Hence, we obtain the result:

$$\begin{aligned} u(x,t) &= \phi(x+ct) + \psi(x-ct) \\ &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{c}\int_{x-ct}^{x+ct} g(s)\, ds. \end{aligned}$$

3. (a) Assume that $u(x,t) = X(x)T(t)$ is a solution. We then substitude it into the equation, after simple algebraic manipulation, we have

$$\frac{T'' + 2kT'}{c^2 T} = \frac{X''}{X} = r,$$

where r is a constant, yielding easily the decoupled system of equations:

$$X'' - rX = 0, \tag{4}$$
$$T'' + 2kT' - rc^2 T = 0. \tag{5}$$

Now, if $r = \mu^2 > 0$, then from (4), we find

$$X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}. \tag{6}$$

In the other case, when $r = 0$, we have

$$X(x) = c_3 x + c_4. \tag{7}$$

For nontrivial solution, from the boundary conditions:

$$u(0,t) = X(0)T(t) = 0, \quad u(L,t) = X(L)T(t) = 0,$$

we should have $X(0) = X(L) = 0$. Applying them to both (6) and (7), we will have

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\mu L} + c_2 e^{-\mu L} = 0 \end{cases} \quad \text{or} \quad \begin{cases} c_4 = 0 \\ c_3 L + c_4 = 0. \end{cases}$$

From them, it is easy to check that we will have $c_1 = c_2 = 0, c_3 = c_4 = 0$, and so trivial solutions when r is non-negative. Having this in mind, we therefore take $r = -\mu^2 < 0$. After some work, we find

$$X_n(x) = \sin(\mu_n x) = \sin\left(\frac{n\pi}{L}x\right), \qquad n = 1, 2, \cdots.$$

To find $T_n(t)$, assuming that $T_n = e^{m_n t}$, from (5), we find the characteristic equation

$$m_n^2 + 2km_n + c^2\mu_n^2 = 0,$$

yielding the characteristic roots

$$m_n = -k \pm \sqrt{k^2 - c^2\mu_n^2}.$$

For different $\mu_n = \frac{n\pi}{L}$, let

$$\lambda_n = \sqrt{|k^2 - c^2\mu_n^2|}.$$

We find that:

$$\begin{aligned}
&\text{if} \quad k^2 > c^2\left(\frac{n\pi}{L}\right)^2, \quad T_n(t) = e^{-kt}[a_n \cosh(\lambda_n t) + b_n \sinh(\lambda_n t)],\\
&\text{if} \quad k^2 = c^2\left(\frac{n\pi}{L}\right)^2, \quad T_n(t) = e^{-kt}(a_n + b_n t),\\
&\text{if} \quad k^2 < c^2\left(\frac{n\pi}{L}\right)^2, \quad T_n(t) = e^{-kt}[a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)].
\end{aligned}$$

Note that $k - (cn\pi/L) > (<,=)0$ implies $n < (>,=)(kL)/(\pi c)$. By superposition principle , we have formal solution of the problem as

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} X_n(x)T_n(t)\\
&= e^{-kt}\left\{\sum_{1\le n<\frac{kL}{\pi c}} \sin\left(\frac{n\pi}{L}x\right)[a_n \cosh(\lambda_n t) + b_n \sinh(\lambda_n t)] + \right.\\
&\qquad \sum_{n>\frac{kL}{\pi c}} \sin\left(\frac{n\pi}{L}x\right)[a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)] +\\
&\qquad \left. \sin\left(\frac{k}{c}x\right)\left(a_{\frac{kL}{\pi c}} + b_{\frac{kL}{\pi c}} t\right)\right\}, \qquad (8)
\end{aligned}$$

where the last term is added when $\frac{kL}{\pi c}$ is an integer.
To determine a_n and b_n, we apply the initial condition at $t = 0$. In the case when $u(x,0) = f(x)$, we have

$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) = f(x), \qquad (9)$$

and so

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\, dx, \qquad n = 1, 2, \cdots, \qquad (10)$$

according to the basic formular for the fourier-sine series expansion of $f(x)$.
While in the other case when $\partial u/\partial t(x,0) = 0$, we have

$$\sum_{n=0}^{\infty}(-ka_n + \lambda_n b_n)\sin\left(\frac{n\pi}{L}x\right) + \left[\sin\left(\frac{k}{c}x\right)(-ka_{\frac{kL}{\pi c}} + b_{\frac{kL}{\pi c}})\right] = 0.$$

Thus we may compute b_n as follows:

$$b_n = \begin{cases} ka_n & \text{if } n = \frac{kL}{\pi c} \text{ is a positive integer}\\ ka_n/\lambda_n & \text{otherwise}\end{cases} \qquad (11)$$

with a_n determined by (10), $n = 1, 2, \cdots$.

(b) If $k = 0$, we have only the case $k^2 - c^2(\frac{n\pi}{L})^2 < 0$, and so $n > \frac{kL}{\pi c}$ for all n. Since $\lambda_n = \mu_n c = \frac{n\pi c}{L} \neq 0$, from (11) we know that $b_n = 0$ for all n . Hence from (8), we arrive at

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\lambda_n t\right) \\ &= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi c}{L}t\right) \\ &= \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin\left[\frac{n\pi}{L}(x+ct)\right] + \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin\left[\frac{n\pi}{L}(x-ct)\right]; \end{aligned}$$

this is clearly the d'Alembert form of the solution.

(c) To show that the energy $E(t)$ is a nonincreasing function with respective to time t, we compute dE/dt as follows:

$$\begin{aligned} \frac{dE(t)}{dt} &= \frac{1}{2}\int_0^L \frac{d}{dt}[(u_t)^2 + (cu_x)^2)]dx \\ &= \int_0^L \left(u_t u_{tt} + c^2 u_x u_{xt}\right) dx \\ &= \int_0^L u_t u_{tt}\, dx + \int_0^L c^2 u_x u_{tx}\, dx \\ &= \int_0^L u_t u_{tt}\, dx + \left[\left(c^2 u_x u_t\right)|_0^L - \int_0^L c^2 u_t u_{xx}\, dx\right]. \end{aligned} \tag{12}$$

Note that we have the homogeneous boundary conditions: $u(0,t) = u(L,t) = 0$, and so $u_t(0,t) = u_t(L,t) = 0$. Thus, together with the damped wave equation, the above expression can be written as

$$\begin{aligned} \frac{dE(t)}{dt} &= \int_0^L u_t\left(u_{tt} - c^2 u_{xx}\right)\, dx \\ &= -\int_0^L 2k(u_t)^2\, dx \leq 0, \end{aligned}$$

for any positive constant k.

To show the uniqueness of the solution, we assume that there are two different solutions: $u_1(x,t)$ and $u_2(x,t)$, for the problem. Now let $w(x,t) = u_1(x,t) - u_2(x,t)$. Since the equation is linear, it is easy to see that $w(x,t)$ satisfies the original damped wave equation, but with zero initial and boundary conditions. Thus from the definition of the energy $E(t)$ and $dE/dt \leq 0$, clearly we have $E(t) \geq 0$ and $E(t) \leq E(0) = 0$. This leads easily to the conclusion: $E(t) = 0$ for all time t. Because of this, we conclude that $w_x = 0$ and $w_t = 0$, and so $w(x,t) = C$ (a constant). But from the zero initial and boundary conditions, we find that $C = 0$, and so $w(x,t) = 0$ which is the uniqueness of the solution $u_1 = u_2$.

(d) To see that our formal solution (8) converges uniformly under the assumed conditions, for $0 \le t \le T$, it is sufficient to look at the convergence behavior of the series:

$$e^{-kt} \sum_{n > \frac{kL}{\pi c}} \sin\left(\frac{n\pi}{L}x\right) [a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)]. \tag{13}$$

To do so, we begin by using the triangle inequality, the boundness of the trigonometric function to one, and $e^{-kt} \le 1$ for $0 \le t \le T$ to the coefficient in (13), and obtain

$$\begin{aligned} & \left| e^{-kt} \sin\left(\frac{n\pi}{L}x\right) [a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t)] \right| \\ \le & \left| a_n \sin\left(\frac{n\pi}{L}x\right) \cos(\lambda_n t) \right| + \left| b_n \sin\left(\frac{n\pi}{L}x\right) \sin(\lambda_n t) \right| \\ \le & |a_n| + |b_n| \\ = & |a_n| + \frac{k}{\lambda_n}|a_n| \qquad \text{(from (11))} \\ = & \left(1 + \frac{k}{\lambda_n}\right) |a_n|. \end{aligned}$$

Note that since

$$\lim_{n\to\infty} \frac{k}{\lambda_n} = \lim_{n\to\infty} \frac{k}{\sqrt{|k^2 - \frac{c^2 n^2 \pi^2}{L^2}|}} = 0,$$

for convergence of the series (13), it amounts to examining the convergence of the series $\sum_{n > \frac{kL}{\pi c}} |a_n|$.

Recall that from (9) and (10), we have

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) \, dx,$$

where $f(x)$ is defined to be an odd function on $[-L, L]$. Thus the derivative of $f(x)$ is an even function on $[-L, L]$, and has a fourier-cosine series representation of the form:

$$f'(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right),$$

where

$$\begin{aligned} A_0 &= \frac{1}{L} \int_0^L f'(x) dx = f(L) - f(0) = 0 \\ A_n &= \frac{2}{L} \int_0^L f'(x) \cos\left(\frac{n\pi}{L}x\right) dx \\ &= \frac{2}{L} \int_0^L \cos\left(\frac{n\pi}{L}x\right) d(f(x)) \\ &= \frac{2}{L} \left[\cos\left(\frac{n\pi}{L}x\right) f(x) \Big|_0^L + \int_0^L f(x) \frac{n\pi}{L} \sin\left(\frac{n\pi}{L}x\right) dx \right] \\ &= \frac{n\pi}{L} \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx \\ &= \frac{n\pi}{L} a_n. \end{aligned}$$

Since $f'(x)$ is square integrable for any N, from the Bessel's inequality, we have

$$\sum_{n=1}^{N} (A_n)^2 \le \frac{1}{L} \int_{-L}^{L} [f'(x)]^2 \, dx < \infty,$$

and so from the above relation between A_n and a_n we obtain

$$\sum_{n=1}^{N} (na_n)^2 \le \left(\frac{L}{\pi}\right)^2 \frac{2}{L} \int_{0}^{L} [f'(x)]^2 \, dx < \infty.$$

Now we use the Cauchy-Schwarz inequality, yielding

$$\left(\sum_{n=1}^{N} |a_n|\right)^2 \le \left(\sum_{n=1}^{N} (na_n)^2\right) \left(\sum_{n=1}^{N} \frac{1}{n^2}\right) < \infty,$$

and so

$$\sum_{n > \frac{kL}{\pi c}} \left(1 + \frac{k}{\lambda_n}\right) |a_n| < \infty.$$

When taking $n \to \infty$, we may use the Weierstrass M-test to show the uniform convergence of the above series, and so establish the fact that the formal solution (8) is a uniformly convergent one.

4.(*a*) Define the function $w(x,t) = \frac{B(t)-A(t)}{L} x + A(t)$, and let $v(x,t) = u(x,t) - w(x,t)$, where u is the solution of the problem. Then we will have the reformulated problem as

$$\begin{cases} \dfrac{\partial v}{\partial t} = \varepsilon \dfrac{\partial^2 v}{\partial x^2} + q(x,t) \\ v(x,0) = f(x) \\ v(0,t) = 0, \quad v(L,t) = 0, \end{cases} \tag{14}$$

where $q(x,t) = \frac{B'(t)-A'(t)}{L} x + A'(t)$ and $f(x) = \frac{A'(0)-B'(0)}{L} x - A'(0)$.

(*b*) To solve the reformulated problem as described in (a), we consider the expansion of v, q, and f in the forms:

$$v(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi}{L} x\right), \tag{15}$$

$$q(x,t) = \sum_{n=1}^{\infty} q_n \sin\left(\frac{n\pi}{L} x\right), \tag{16}$$

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi}{L} x\right). \tag{17}$$

Denote $K(t) = (B'(t) - A'(t))/L$. The coefficients q_n and f_n are determined by

$$\begin{aligned}
q_n(x) &= \frac{2}{L}\int_0^L q(x,t)\sin\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{2}{L}\int_0^L [K(t)x + A'(t)]\sin\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{-2K(t)}{n\pi}\left[\left(x\cos\left(\frac{n\pi}{L}x\right)\right)\Big|_0^L - \int_0^L \cos\left(\frac{n\pi}{L}x\right)dx\right] + \frac{2}{n\pi}A'(t)[1-(-1)^n] \\
&= \frac{2}{n\pi}[K(t)L(-1)^{n+1} + A'(t)(1-(-1)^n)], \\
f_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{2}{L}\int_0^L (-K(0)x - A'(0))\sin\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{-2}{n\pi}[K(0)L(-1)^{n+1} + A'(0)(1-(-1)^n)].
\end{aligned}$$

Substitude (15) and (16) into (14), after some simple algebraic manipulations, we then have the relation

$$\sum_{n=1}^{\infty}\left[\frac{dB_n(t)}{dt} + \varepsilon B_n(t)\left(\frac{n\pi}{L}\right)^2 - q_n(t)\right]\sin\left(\frac{n\pi}{L}x\right) = 0,$$

and so

$$\frac{dB_n(t)}{dt} + \varepsilon\lambda_n B_n(t) - q_n(t) = 0$$

for $n = 1, 2, \cdots;\ \lambda_n = (n\pi/L)^2$. It is easy to see that the solution of the above ODE takes the form

$$B_n(t) = e^{-\varepsilon\lambda_n t}\left[B_n(0) + \int_0^t q_n(s)e^{\varepsilon\lambda_n s}\,ds\right],$$

where $B_n(0) = f_n$; a result obtained from the initial condition. Thus we have the formal solution for (14) as written by

$$v(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin\left(\frac{n\pi}{L}x\right),$$

and so $u = v + w$ for the original problem.

(c) Consider the initial-boundary value problem for the heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < L, \quad t > 0 \\ u(x,0) = f(x), & 0 < x < L, \\ u(0,t) = g_1(t), & u(L,t) = g_2(t), \quad t \geq 0. \end{cases} \tag{18}$$

The "Maximum Principle" for the problem (18) states as:

If $u(x,t)$ satisfies (18) and there are two numbers M and m such that

$$m \le f(x) \le M, \quad m \le g_1(t) \le M, \quad m \le g_2(t) \le M.$$

Then the solution will satisfy

$$m \le u(x,t) \le M.$$

Now if there are two solutions $u_1(x,t)$ and $u_2(x,t)$ for (18), then it is easy to show that $w(x,t) = u_1(x,t) - u_2(x,t)$ is a solution of the same equation with zero boundary value and zero initial value. Clearly, by the maximum principle, we can choose $M = 0$ and $m = 0$ for $w(x,t)$. Then we have

$$0 \le u_1(x,t) - u_2(x,t) \le 0,$$

and hence $u_1(x,t) = u_2(x,t)$; the uniqueness of the result.