八十九學年度第二學期

課程編號: 201 25000
科目名稱: 偏微分方程式導論

學期中考試試題

時間: 10:00-12:00 日期: 04/21/2001

- 總分:
- 請詳述計算過程, 無計算過程的答案不予計分

1. (10 points) The well-known *Tricomi* equation takes the form

$$\frac{\partial^2 u}{\partial x^2} - y\frac{\partial^2 u}{\partial y^2} = 0, \qquad \text{for} \qquad (x, y) \in \mathbb{R}^2. \tag{1}$$

If $y > 0$, find the characteristic equations and the associated characteristic curves for (1).

2. (15 points) Consider the initial value problem of the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}, & \text{for} \quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x), \quad \dfrac{\partial u}{\partial t}(x,0) = g(x), & \text{for} \quad -\infty < x < \infty, \end{cases} \tag{2}$$

where c is a positive real constant. The d'Alembert's solution of this problem takes the form

$$u(x,t) = \phi(x+ct) + \psi(x-ct), \tag{3}$$

for some function ϕ and ψ. Use (3) to derive the explicit expression of the functions ϕ and ψ for (2).

3. (45 points) Consider the initial-boundary value problem of the damped wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + 2k\dfrac{\partial u}{\partial t} = c^2\dfrac{\partial^2 u}{\partial x^2}, & \text{for} \quad 0 < x < L, \quad t > 0, \\ u(x,0) = f(x), \quad \dfrac{\partial u}{\partial t}(x,0) = 0, & \text{for} \quad 0 \le x \le L, \\ u(0,t) = 0, \quad u(L,t) = 0, & \end{cases} \tag{4}$$

where k and c are positive real constants.

(a) (15 points) Use the method of separation of variables to find the *formal* solution of the problem.

(b) (10 points) Verify that, when $k = 0$, the *formal* solution you obtained in (a) can be written in the d'Alembert form (3) for the ordinary wave equation.

(c) (10 points) For this problem, define the total energy E as

$$E(t) = \frac{1}{2}\int_0^L \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(c\frac{\partial u}{\partial x}\right)^2\right] dx,$$

show that the energy is decreasing, *i.e.*, $dE/dt \leq 0$. Use this fact to prove the uniqueness of the solution for (4).

(d) (10 points) Assume that $f(x)$ is continuous, $f(0) = f(L) = 0$, and $\int_0^L (f'(x))^2\, dx$ finite. Show that the *formal* solution you obtained in (a) for $u(x,t)$ converges uniformly for $0 \leq t \leq T$, for any T.

4. (30 points) Consider the initial-boundary value problem of the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \dfrac{\partial^2 u}{\partial x^2}, & \text{for} \quad 0 < x < L, \quad t > 0, \\ u(x,0) = 0, & \text{for} \quad 0 \leq x \leq L, \\ u(0,t) = A(t), \quad u(L,t) = B(t), & \text{for} \quad t > 0, \end{cases} \tag{5}$$

where ε is a positive real constant.

(a) (5 points) Reformulate (5) as a new problem with homogeneous boundary conditions, but with appropriate initial condition and nonhomogeneous source term.

(b) (15 points) Use the method of eigenfunction expansion to find the *formal* solution of the reformulated problem as obtained in (a).

(c) (10 points) State the maximum principle and use it to show the uniqueness of the solution for (5).