八十九學年度第二學期

課程編號: 201 25000
科目名稱: 偏微分方程式導論

學期末考試試題

時間: 14:10-17:00 日期: 06/11/2001

- 總分: 120
- 請詳述計算過程, 無計算過程的答案不予計分
- State clearly what formulae you have used from the enclosed table when you employed them to solve problems

1. (20 points) Consider the initial value problem for the one-way wave equation with damping term:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \alpha u, \qquad t > 0, \tag{1}$$

and the initial condition: $u(x,0) = f(x)$ at $t = 0$ for $x \in \mathbb{R}$. Here both c and $\alpha(< 0)$ are (real) constants, and f is a prescribed smooth function, for example.

(a) (10 points) Use the method of Fourier transform to find solution of this problem.

(b) (10 points) Recall that if this problem is solved by using the method of characteristics, we would have (**??**) written in the form:

$$\frac{du}{dt} = \alpha u, \qquad \text{along} \quad \frac{dx}{dt} = c. \tag{2}$$

Now find the solution of the problem by integrating (**??**) directly with the same initial condition $u(x,0) = f(x)$ as before. Do you observe exactly the same result as in (a) ?

2. (13 points) Use the method of Fourier transform to find solution of the initial value problem of the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \qquad t > 0, \\ u(x,0) = f(x), \quad \dfrac{\partial u}{\partial t}(x,0) = g(x), & -\infty < x < \infty, \end{cases} \tag{3}$$

where c is a positive real constant, and f, g are the prescribed functions. Be sure to write the solution of the problem in the D'Alembert's form.

3. (32 points) Consider the initial-boundary value problem of the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < \infty, \qquad t > 0, \\ u(x,0) = f(x), & 0 \le x < \infty, \\ u(0,t) = 0, & t > 0, \end{cases} \tag{4}$$

where ε is a positive real constant.

(a) (12 points) Use the method of Fourier transform to find solution of this problem.

(b) (10 points) Note that the solution of (**??**) may be computed by the so-called "method of images" in that the initial condition is first extended to the whole line by a "suitable way" to match the boundary condition and the solution of the corresponding initial value problem, namely,

$$u(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{-(x-\xi)^2/4\varepsilon t}\, d\xi,$$

can be used to the problem, where $\hat{f}$ is the extended initial condition of the problem. Questions: What the function $\hat{f}$ should be in this case with homogeneous Dirichlet boundary condition, and so the form of the solution ? Do you observe exactly the same result as in (a) ?

(c) (10 points) If we replace the boundary condition of (**??**) to be a Neumann boundary $\partial u/\partial x(0,t) = 0$, what the solution of this problem would be ? (Give reasons for your result obtained by using either the method of Fourier transform or the method of images.)

4. (20 points) Given two arbitrary numbers a and α, and given a square integrable function f. Let $F(x) = e^{-i\alpha x} f(x+a)$.

(a) (10 points) Show that $\Delta_0 F = \Delta_a f$, and $\Delta_0 \widehat{F} = \Delta_\alpha \widehat{f}$, where $\Delta_a f$ denotes the uncertainty of a nontrivial function f about a point a, and is defined as

$$\Delta_a f = \frac{\int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2\, dx}{\int_{-\infty}^{\infty} |f(x)|^2\, dx},$$

and $\widehat{F}$, $\widehat{f}$ represent the Fourier transforms of F and f, respectively.

(b) (10 points) Show that $\Delta_a f \Delta_\alpha \widehat{f} \ge \frac{1}{4}$.

5. (35 points) Consider the boundary value problem for the Laplace equation in the exterior of a unit disk:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & x^2 + y^2 > 1, \\ u(x,y) = f(x,y), & x^2 + y^2 = 1. \end{cases} \tag{5}$$

(a) (5 points) Show that in polar coordinate with $(x, y) = (r\cos\theta, r\sin\theta)$ the Laplace equation in Cartesian coordinate (**??**) takes the form:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{6}$$

(b) (10 points) With (**??**) for $r > 1$ and $f(x, y) = f(\theta)$ on $r = 1$, use the method of separation of variables to find the formal solution of the problem.

(c) (10 points) Verify that (by direct substitution)

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\frac{r^2 - 1}{r^2 + 1 - 2r\cos(\theta - \phi)}\, d\phi$$

is the solution to (**??**), and show that it is the unique solution with

$$\lim_{r\to 1, \theta\to\theta_0} u(r,\theta) = f(\theta_0).$$

(d) (10 points) If we replace the boundary condition of (**??**) to be a Neumann boundary $\partial u/\partial r(1,\theta) = g(\theta)$, and consider the interior part of the region $r < 1$, verify that

$$u(r,\theta) = -\frac{1}{2\pi}\int_0^{2\pi} g(\phi)\ln\left[1 + r^2 - 2r\cos(\theta - \phi)\right] d\phi$$

solves the problem under the condition $\int_0^{2\pi} g(\theta)\, d\theta = 0$. Be aware of the fact that the solution of this problem is unique up to a constant (you are not required to show this).