91學年度第1學期

課程編號: 221 U1310
科目名稱: 數值偏微分方程式一
課程網站: http://www.math.ntu.edu.tw/~shyue/myclass/npde02

Homework ♯ 2

Assign: 10/18/2002 Due: 10/30/2002

- *Include your computer program(s), when turning the homework set*

1. The s-step Adams-Moulton method for solving the initial-value problem of the first-order ordinary differential equation: $dy/dt = f(t, y)$, can be written in the form

$$y_n = y_{n-1} + h \sum_{m=0}^{s} \gamma_m \nabla^m f_n,$$

where

$$\gamma_m = (-1)^m h^{-1} \int_{t_{n-1}}^{t_n} \binom{-u}{m} dt = \int_{-1}^{0} \binom{-u}{m} du$$

and $u = (t - t_n)/h$.

a) Show that γ_m satisfies the following recurrence relation:

$$\gamma_m + \frac{1}{2}\gamma_{m-1} + \frac{1}{3}\gamma_{m-2} + \cdots + \frac{1}{m+1}\gamma_0 = \begin{cases} 1, & m = 0 \\ 0, & m = 1, 2, 3, \cdots \end{cases}$$

b) Show that the order of accuracy of the s-step Adams-Moulton method is $s + 1$.

c) Is the method zero stable ?

2. The s-step Milne-Simpson method for solving the initial-value problem of the first-order ordinary differential equation: $dy/dt = f(t, y)$, can be written in the form

$$y_n = y_{n-2} + h \sum_{m=0}^{s} \gamma_m \nabla^m f_n,$$

where

$$\gamma_m = (-1)^m h^{-1} \int_{t_{n-2}}^{t_n} \binom{-u}{m} dt = \int_{-2}^{0} \binom{-u}{m} du$$

and $u = (t - t_n)/h$.

a) Show that γ_m satisfies the following recurrence relation:

$$\gamma_m + \frac{1}{2}\gamma_{m-1} + \frac{1}{3}\gamma_{m-2} + \cdots + \frac{1}{m+1}\gamma_0 = \begin{cases} 2, & m = 0 \\ -1, & m = 1 \\ 0, & m = 2, 3, \cdots \end{cases}$$

b) What is the order of accuracy of the s-step Milne-Simpson method ?

c) Plot the stability region of the method for $s = 1, 2, 3$, and 4, and make a comparison with that of the Adams-Moulton method.

3. The stability polynomial $\Pi(z)$ for the (one-step) s-stage explicit Runge-Kutta method is

$$\Pi(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^s}{s!}.$$

Plot the stability region of the Runge-Kutta method for $s = 1, 2, \cdots, 5$.

4. The classical 4-th order Runge-Kutta method is given by:

$$\begin{aligned}
k_0 &= hf(x_n, y_n) \\
k_1 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_0) \\
k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1) \\
k_3 &= hf(x_n + h, y_n + k_2) \\
y_{n+1} &= y_n + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)
\end{aligned}$$

where h is the step size. Implement and apply this method to the scalar problem

$$y^{'} = \sin x - 1000(y + y^2), \qquad y(0) = 1$$

on the interval $[0, 0.1]$.

a) Present numerical evidence that the method is 4-th order accurate for h small enough.

b) How small should h be so that the calculation is absolutely stable ? Show numerical results of what happens when h is not small enough.

5. This problem is concerned with a historical problem treated by Störmer in 1907: Störmer's aim was to confirm numerically the conjecture of Birkeland, who explained in 1896 the aurora borealis as being produced by electrical particles emanating from the sun and dancing in the earth's magnetic field. Suppose that an elementary magnet is situated at the origin with its axis along to the z-axis. The trajectory $[x(s), y(s), z(s)]$ of an electrical particle in this magnetic field then satisfies

$$\begin{aligned}
x^{''} &= \frac{1}{r^5}\left[3yzz^{'} - (3z^2 - r^2)y^{'}\right] \\
y^{''} &= \frac{1}{r^5}\left[(3z^2 - r^2)x^{'} - 3xzz^{'}\right] \\
z^{''} &= \frac{1}{r^5}\left[3xzy^{'} - 3yzx^{'}\right],
\end{aligned}$$

where $r^2 = x^2 + y^2 + z^2$.

a) Introducing the polar coordinates

$$x = R\cos\phi, \qquad y = R\sin\phi,$$

verify that the above system becomes equivalent to

$$\begin{aligned} R^{''} &= \left(\frac{2\gamma}{R} + \frac{R}{r^3}\right)\left(\frac{2\gamma}{R^2} + \frac{3R^2}{r^5} - \frac{1}{r^3}\right) \\ \phi^{'} &= \left(\frac{2\gamma}{R} + \frac{R}{r^3}\right)\frac{1}{R} \\ z^{''} &= \left(\frac{2\gamma}{R} + \frac{R}{r^3}\right)\frac{3Rz}{r^5}, \end{aligned}$$

where now $r^2 = R^2 + z^2$ and γ is some constant arising from the integration of $\phi^{''}$.

b) Solve the system of equations in a) numerically, with the the following initial values:

$$\begin{array}{llll} R_0 = 0.257453, & z_0 = 0.314687, & \phi_0 = 0, & \gamma = -0.5, \\ R_0^{'} = \sqrt{Q_0}\cos u, & z_0^{'} = \sqrt{Q_0}\sin u, & u = 5\pi/4, & Q_0 = 1 - (2\gamma/R_0 + R_0/r_0^3)^2, \end{array}$$

for sufficiently large time. Plot the resulting solution in the (x, y, z) space.

c) Repeat the computation done in b) but with at least 30 different neighboring initial values. What do you observe about the solution of this problem ? (see [1] p. 466 for a sample solution).

References

[1] E. Hairer, S. P. Noersett, and G. Wanner. *Solving Ordinary Differential Equations I: Nonstiff Problems*. Springer-Verlag, 2 revised edition, 1996.