幾何學的昨日與今日

王金龍

Chin-Lung Wang

國立臺灣大學

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- Euclid 歐幾里得 直線
- Newton 牛頓 二次曲線 (圓錐曲線)
- Gauss 高斯 一般曲線 (曲面)

- 古典幾何學

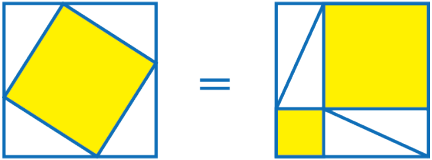
  Pythagoras (畢達哥拉斯, 570-495 B.C.)

  Euclid's Elements (歐幾里得原本, 300 B.C.)

- 公理化數學 / Logic

- 尺規作圖 (三大難題)

畢氏定理 (幾何的起源)

1. 

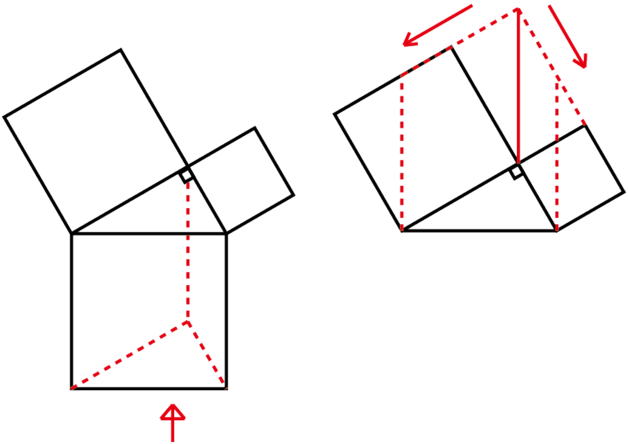
2.

$$(a+b)^2 = a^2 + 2ab + b^2$$

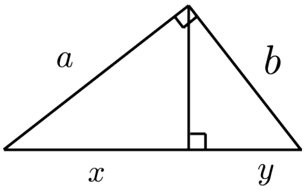
$$\|$$

$$c^2 + 4 \times \frac{ab}{2} \quad \Rightarrow a^2 + b^2 = c^2$$

Dynamical Proof 動畫式證明



3.



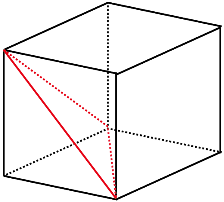
$$\frac{y}{b} = \frac{b}{c} \Rightarrow b^2 = yc$$

$$\frac{x}{a} = \frac{x}{c} \Rightarrow a^2 = xc$$

$$a^2 + b^2 = (x+y)c$$
$$= c^2$$

方法的延伸

1



$$\text{Vol} = \frac{1}{6}$$

$$\text{錐體積} = \frac{1}{3}\ \text{底面積} \times \text{高}$$

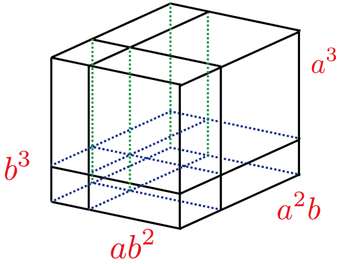
Key formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1)$$

How can one " find this "?

2.

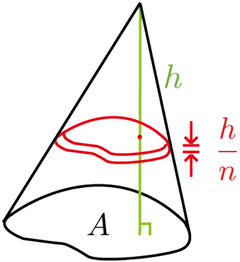
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$



$$
\begin{aligned}
2^3 &= (1+1)^3 = 1^3 + 3\cdot 1^2 + 3\cdot 1 + 1 \\
3^3 &= (2+1)^3 = 2^3 + 3\cdot 2^2 + 3\cdot 2 + 1 \\
4^3 &= (3+1)^3 = 3^3 + 3\cdot 3^2 + 3\cdot 3 + 1 \\
&\vdots \\
(n+1)^3 &= n^3 + 3n^2 + 3n + 1
\end{aligned}
$$

$$
\Rightarrow (n+1)^3 = 1 + 3\left(1^2 + 2^2 + \cdots + n^2\right) + 3\frac{n(n+1)}{2} + n
$$

$$
\Rightarrow 1^2 + \cdots + n^2 = \frac{1}{3}\left((n+1)^3 - 1 - \frac{3}{2}n(n+1) - n\right)
$$



阿基米德—祖沖之

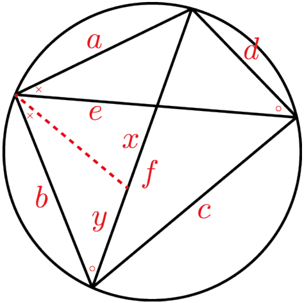
第 k 層的體積

$$A \cdot (\frac{k}{n})^2 \cdot \frac{h}{n}$$

加起來 =

$$Ah\frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) \underset{n\to}{\rightarrow} \frac{1}{3}Ah$$

❸ Ptolemy (托勒密, AD 90 ~ 168)



$$ac + bd = ef$$

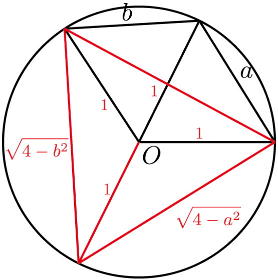
$$\frac{y}{b} = \frac{d}{e}$$

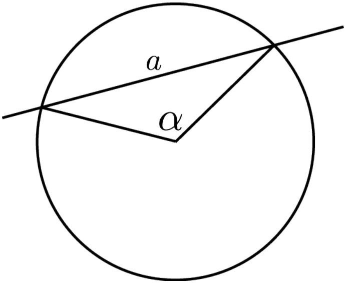
$$\frac{x}{a} = \frac{c}{e}$$

$$ac + bd = (x + y)e$$

$$= fe$$

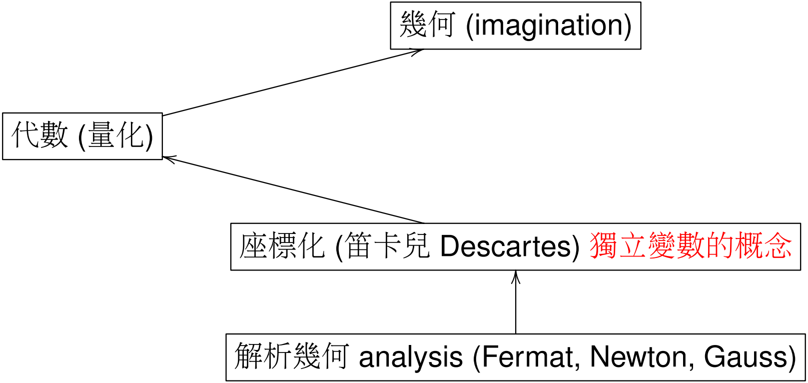
古代的正弦和角公式（天文觀測）

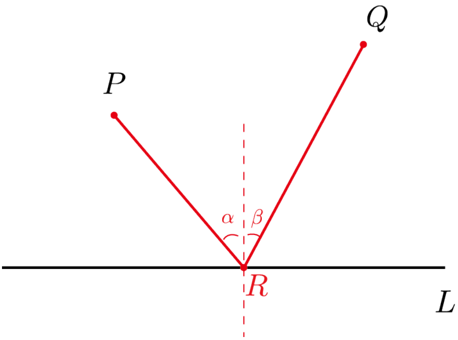




$$2\,\text{sine}(\alpha+\beta)$$
$$= a\sqrt{4-b^2}+b\sqrt{4-a^2}$$

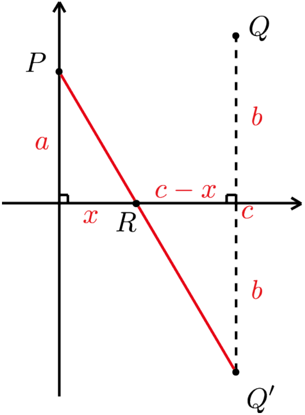
$$a = \text{sine}(\alpha)$$
$$= 2\sin(\frac{\alpha}{2})$$


幾何 (imagination)
代數 (量化)
座標化 (笛卡兒 Descartes) 獨立變數的概念
解析幾何 analysis (Fermat, Newton, Gauss)



Find $R \in L$ such that

$\overline{PR} + \overline{QR}$ 最小



$$\ell(x) = \ell_1(x) + \ell_2(x)$$

$$b = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c - x)^2}$$

R: 仲介者

極值發生時必有

$$\ell'(x) = \lim_{h \to 0} \frac{\ell(x+h) - \ell(x)}{h}$$

$$= 0$$

$\ell_1'(x)$:

$$
\begin{aligned}
&\frac{1}{h}\left(\sqrt{(x+h)^2+a^2}-\sqrt{x^2+a^2}\right) \\
&= \frac{1}{h}\frac{(x+h)^2+a^2-x^2-a^2}{(\sqrt{(x+h)^2+a^2}+\sqrt{x^2+a^2}} \\
&= \frac{1}{h}\frac{2xh+h^2}{(\sqrt{(x+h)^2+a^2}+\sqrt{x^2+a^2}} \underset{h\to 0}{\longrightarrow} \frac{x}{\sqrt{x^2+a^2}}
\end{aligned}
$$

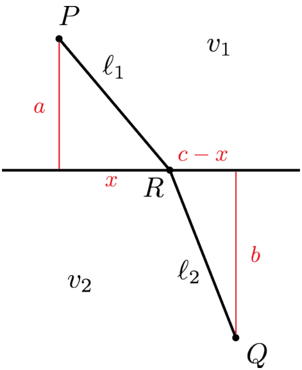
同理:

$$\ell'(x) = \frac{x}{\sqrt{x^2+a^2}} - \frac{c-x}{\sqrt{(c-x)^2+b^2}}$$

故

$$\ell'(x) = 0 \iff \sin\alpha = \sin\beta \iff \alpha = \beta$$

折射定律



Fermat's Least Action Principle

最小作用原理

$$T(x) = \frac{\ell_1(x)}{v_1} + \frac{\ell_2(x)}{v_2}$$

$$T'(x) = \frac{\ell_1'(x)}{v_1} + \frac{\ell_2'(x)}{v_2}$$

$$T'(x) = 0 \Longleftrightarrow \frac{\sin\alpha}{v_1} = \frac{\sin\beta}{v_2}$$

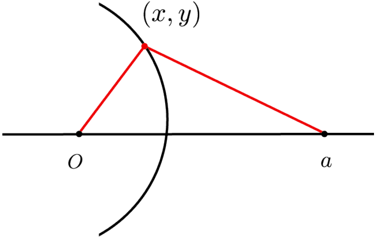
How about $\overline{PR}/\overline{QR}$?

$$\left[\frac{\ell_1(x)}{\ell_2(x)}\right]': \quad \frac{1}{h}\left[\frac{\sqrt{(x+h)^2+a^2}}{\sqrt{(x+h-c)^2+b^2}} - \frac{\sqrt{x^2+a^2}}{\sqrt{(x-c)^2+b^2}}\right]$$

$$= \frac{1}{h}\left(\frac{\sqrt{(x+h)^2+a^2}\sqrt{(x-c)^2+b^2} - \sqrt{x^2+a^2}\sqrt{(x+h-c)^2+b^2}}{\sqrt{(x+h-c)^2+b^2}\sqrt{(x-c)^2+b^2}}\right)$$

$$=???$$

Apollonius (BC 260 $\sim$ 190) 圓

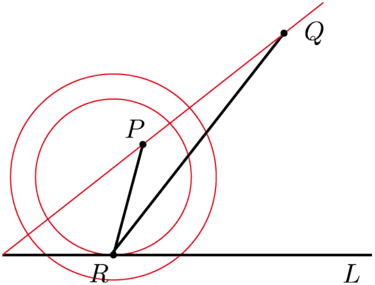


求 $\frac{\sqrt{x^2+y^2}}{\sqrt{(a-x)^2+y^2}}$ = 定值 e 之軌跡

$$x^2+y^2=e^2(x^2-2ax+a^2+y^2)$$

$$(1-e^2)x^2+(1-e^2)y^2+2ae^2x=e^2a^2$$

$\rightarrow$ circle



Homework: $R = ?$

ℓ_1/ℓ_2 之極值發生於當 L 為

Apollonius circle 之切線時

解析法

$$\left(\frac{\ell_1}{\ell_2}\right)' = \frac{\ell_1'\ell_2 - \ell_1\ell_2'}{{\ell_2}^2} = 0$$

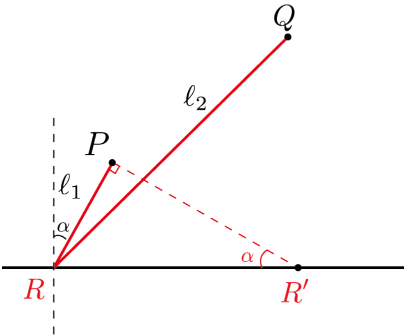
$$\iff \frac{\ell_1'}{\ell_1} = \frac{\ell_2'}{\ell_2}$$

$$i.e. \quad \frac{\sin\alpha}{\ell_1} = \frac{\sin\beta}{\ell_2}$$

Remark

Lebnitz 1646 - 1716 (萊布尼茲)

$$(fg)' = f'g + fg'$$



作 $\overline{PR'} \perp \overline{PR}$

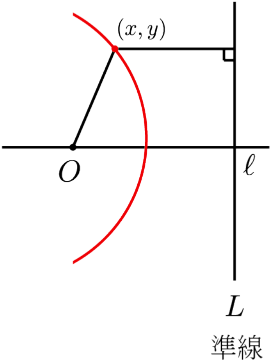
$$\frac{\ell_1}{\sin\alpha} = \overline{RR'}$$

同理, 作 $\overline{QR''} \perp \overline{QR}$

$$\frac{\ell_2}{\sin\beta} = \overline{RR''}$$

$$\Rightarrow R' = R''$$

二次曲線 (beyond circles)



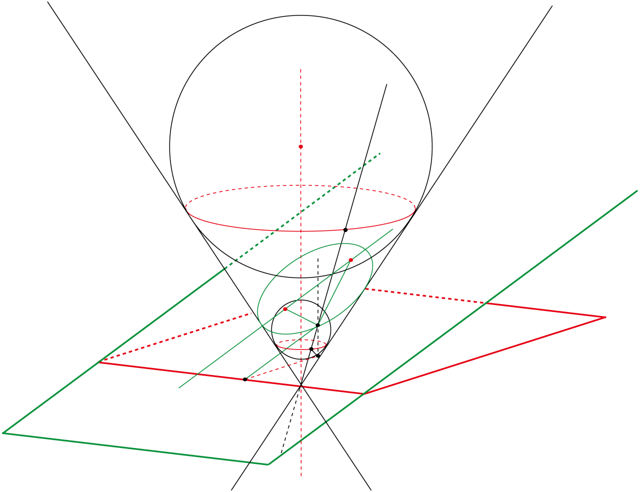
$$\frac{\sqrt{x^2+y^2}}{\ell-x} = \text{定值 } e$$

$$x^2+y^2 = e^2(x^2-2\ell x+\ell^2)$$

$$(1-e^2)x^2+y^2+2\ell e^2 x = e^2\ell^2$$

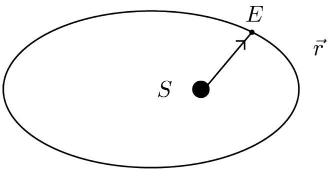
Polar coor. (r,θ) $\quad \dfrac{r}{\ell - r\cos\theta} = e \Rightarrow r = \dfrac{\ell e}{1+e\cos\theta}$

圓錐曲線 (Ancient Greek's viewpoint)



Kepler's Law (開普勒, 1619)

80 年天文觀測



Newton (牛頓, 1643-1727)

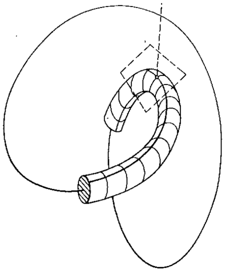
Physics:

$$\vec{F} = m\vec{a}; \qquad \vec{F} = -\frac{GMm}{r^2}\hat{r}$$

Mathematics: CALCULUS

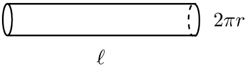
Principia Mathematica 自然哲學的數學原理

更一般的曲線理論



Q: 長度 ℓ, 半徑 r 的水管表面積

及容積 $=?$

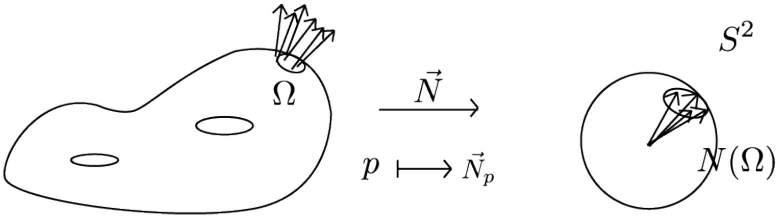


$\text{Area} = 2\pi r\ell \pm (\cdots)?$

曲率的概念 (Curvature)

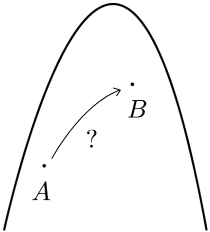
Gauss 高斯 1777-1855

1818 地形測量 $\rightarrow$ 微分幾何誕生

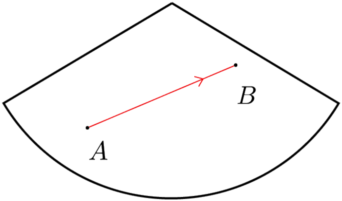


曲率 $K = \pm \lim_{\Omega \to p} \frac{|N(\Omega)|}{|\Omega|}$

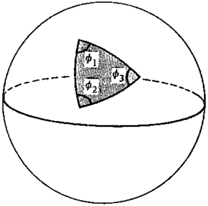
這是一個 不變量 (invariant)



測地線

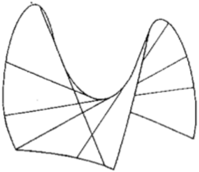


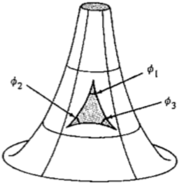
$k > 0$



半徑 R 的球, $K = \dfrac{1}{R^2}$

$k < 0$





Gauss 的偉大發現

三角形之內角和 $\neq 180^\circ$ $\left(=\pi+\int_\Omega KdA\right)$

半徑 r 的圓周長

$$L=2\pi r-\frac{\pi}{3}r^3K+o(r^3)$$

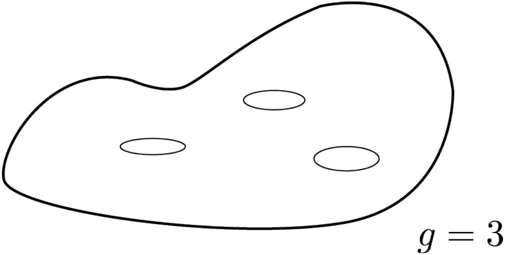
$\Rightarrow$ 在真實的世界中, 畢氏定理不一定正確！

Gauss-Bonnet 定理 (陳省身, 高維度)

$$\int_S K dA = 2\pi\chi(S)$$

其中, $\chi(S) \triangleq$ 點 $-$ 線 $+$ 面 $= 2 - 2g$

Euler number



我們如何知道我們所在的世界是否是彎曲的?

- Riemann (黎曼, 1855)

$$ds^2 = \sum g_{ij} dx^i \otimes dx^j$$

- Einstein (愛因斯坦, 1907-1915)

$$R_{ij} - \frac{1}{2} g_{ij} = T_{ij}$$

- Nash 1951
- YAU (丘成桐, 1976 宇宙的內在模型)

STRING THEORY (Witten $\cdots$)

END