挑動心弦–探討圓內弦長分布情形

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Abstract

This study's aim is to build a new view of investigating the distribution of strings in a circle. Different from most people's first impression that strings are uniform distribution. First, we define a function, which contains degree the strings rotate on the point and distance between the point and center as input and length as output. Then, we invert the function and use the inverse function to get Cumulative Distribution Function and Probability Density Function of strings length. Under this view, the distribution of strings depend on their length and the distance between the point and the center. From that, we can get expected value and standard deviation of string length passed through every points and we can get the distribution of all string length by joint probability function. At the end, we compare the result with strings under uniform distribution.

中文摘要

本文所探討的是建立一種新的幾何觀點去探討圓內弦長的分布. 不同於一般人的第一印象：圓內每條弦長出現機率相同, 我們利用過圓內一點之弦長函數的反函數求出弦長累積分配函數與弦長機率密度函數, 在此種模型下, 每條弦長出現的機率都有所不同, 再求得子樣本的條件期望值與條件標準差, 並以聯合機率密度函數求出母體的分佈情形, 最後再與弦長為一致分配情形下的弦長分布相比較.

1 簡介

1.1 研究動機

圓是所有幾何圖形中, 最完美的一個對稱圖形, 在這樣一個圖形中, 具有無限多組的弦長.“這些弦長之間是否存在什麼有趣的分布情形呢? ”在好奇心的驅使之下.

1.2 研究目的

1. 找出過圓內一點的弦長集合的分佈情形.
2. 比較在不同觀點下, 圓內的弦長集合的分佈情形.

2 研究內容

2.1 過圓內一點的弦長集合分佈

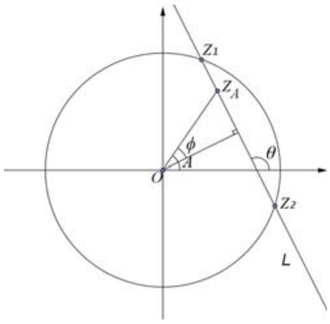
透過圓這個最對稱的圖形, 觀察過圓內一點的弦長, 看看這些弦長的集合有何特性. 我們先求出過圓內一點的弦長函數, 將弦長函數轉換成弦長的累積分配函數及機率密度函數, 再求出弦長的期望值、標準差和其分佈.

1. 過圓內一點的弦長函數

   證明.

(a) 定義符號和圖形

在複數平面上以 O 為原點做一個半徑為 R 的圓, 在圓內取一點 Z_A, Z_A 到圓心距離為 r, 過 Z_A 做一直線 L 交圓於 Z_1, Z_2, 直線 L 與實軸的水平夾角為 θ, O 點與 Z_A 的連線與實軸的夾角和過 O 點做一垂直於直線 L 的線段的夾角分別為 A 和 ϕ.



(b) 計算

令複數平面上一點 $Z_A(a+bi)$, $\sqrt{a^2+b^2}=r$, 其中 $a,b\in\mathbb{R}$, 過 Z_A 的直線 L 方程式為 $Z=Z_A+tZ_\lambda,\ t\in\mathbb{R},\ Z_\lambda=\cos\theta+i\sin\theta$, 而圓方程式為 $|Z|=R$

我們先求直線與圓的交點 Z_1, Z_2

$$\begin{cases} Z = Z_A + tZ_\lambda \\ |Z| = R \end{cases}$$
$$\Rightarrow |Z_A+tZ_\lambda| = R$$
$$\Rightarrow |Z_A+tZ_\lambda|^2 = R^2$$
$$\Rightarrow (Z_A+tZ_\lambda)(\overline{Z_A+tZ_\lambda}) = R^2$$
$$\Rightarrow Z_\lambda\overline{Z_\lambda}t^2 + (Z_\lambda\cdot\overline{Z_A}+\overline{Z_\lambda}\cdot Z_A)t + (Z_A\cdot\overline{Z_A}-R^2) = 0$$
$$\Rightarrow t = \frac{-(Z_\lambda\overline{Z_A}+\overline{Z_\lambda}Z_A)\pm\sqrt{D}}{2Z_\lambda\overline{Z_\lambda}},$$

其中 $D=(Z_\lambda\overline{Z_A}+\overline{Z_\lambda}Z_A)^2-4(Z_\lambda\overline{Z_\lambda})(Z_A\overline{Z_A}-R^2)$, t 代入$Z=Z_A+tZ_\lambda$, 解得交點

$$Z = Z_A + \frac{-(Z_\lambda\overline{Z_A}+\overline{Z_\lambda}Z_A)\pm\sqrt{D}}{2Z_\lambda\overline{Z_\lambda}}\cdot Z_\lambda = \frac{Z_A\overline{Z_\lambda}-\overline{Z_A}Z_\lambda\pm\sqrt{D}}{2\overline{Z_\lambda}}.$$

得直線與圓之兩交點分別為 $Z_1=\dfrac{Z_A\overline{Z_\lambda}-\overline{Z_A}Z_\lambda+\sqrt{D}}{2\overline{Z_\lambda}}$, $Z_2=\dfrac{Z_A\overline{Z_\lambda}-\overline{Z_A}Z_\lambda-\sqrt{D}}{2\overline{Z_\lambda}}$.

所以得弦長$=|Z_1-Z_2|=\left|\dfrac{\sqrt{D}}{\overline{Z_\lambda}}\right|=\dfrac{|\sqrt{D}|}{|\overline{Z_\lambda}|}$.

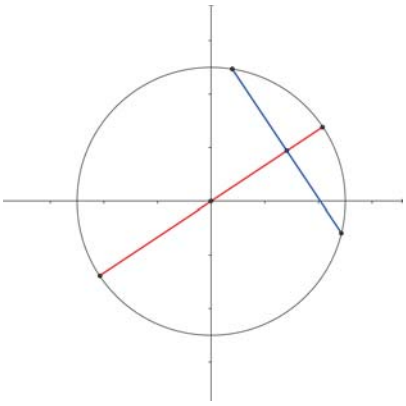
$$\because |\overline{Z_\lambda}| = \sqrt{(\cos\theta)^2+(-\sin\theta)^2} = 1,\ \therefore \frac{|\sqrt{D}|}{|\overline{Z_\lambda}|} = \sqrt{D}.$$

$$\begin{aligned} \text{其中 } D &= (Z_\lambda\overline{Z_A}+Z_A\overline{Z_\lambda})^2 - 4|Z_\lambda|^2(|Z_A|^2-R^2) \\ &= (Z_\lambda\overline{Z_A})^2+(Z_A\overline{Z_\lambda})^2+2|\overline{Z_\lambda}|^2|\overline{Z_A}|^2-4|\overline{Z_\lambda}|^2|\overline{Z_A}|^2+4|\overline{Z_\lambda}|^2R^2 \\ &= (Z_\lambda\overline{Z_A}-Z_A\overline{Z_\lambda})^2+4R^2. \end{aligned}$$

$Z_\lambda = \cos\theta + i\sin\theta, Z_A = a + bi$ 代入上式

$$
\begin{aligned}
\text{得 } D &= [(\cos\theta + i\sin\theta)(a - bi) - (a + bi)(\cos\theta - i\sin\theta)]^2 + 4R^2 \\
&= [(a\cos\theta + ai\sin\theta - bi\cos\theta + b\sin\theta) - (a\cos\theta - ai\sin\theta + bi\cos\theta + b\sin\theta)]^2 + 4R^2 \\
&= 4R^2 - 4(a\sin\theta - b\cos\theta)^2 \\
&= 4R^2 - 4(a^2 + b^2)\left(\sin\theta \cdot \frac{a}{\sqrt{a^2 + b^2}} - \cos\theta \cdot \frac{b}{\sqrt{a^2 + b^2}}\right)^2 \\
&= 4R^2 - 4(a^2 + b^2)[\cos(90^\circ - \theta)\cos A - \sin(90^\circ - \theta)\sin A]^2 \\
&= 4R^2 - 4r^2\cos^2(A + 90^\circ - \theta) \\
&= 4R^2 - 4r^2\cos^2\phi.
\end{aligned}
$$

得弦長函數 $s(r, \phi) = \sqrt{4R^2 - 4r^2\cos^2\phi}$ 為二元變數函數, 自變數為距離 r 和夾角 ϕ. 其中 $0 \le r < R, 0 \le \phi < \frac{\pi}{2}$.
如下圖, 當 $\phi = 0$ 時, 所對應到的弦長為藍色的那條弦長, 也是過該定點所取到的最短弦長; 當 $\phi = \frac{\pi}{2}$ 時, 所對應到的弦長為紅色的那條弦長, 為過該定點所取到的最短弦長.



□

2. 過圓內一點弦長的累積分配函數及機率密度函數

證明.

(a) 想法與定義符號
令過圓內一點弦長函數為 $s(r, \phi) = \sqrt{4R^2 - 4r^2\cos^2\phi}$, 符號定義如 1. 所述.
而在圓內取一距圓心 r 的定點, 過該點的弦長隨機變數定義為 S_r.
因為直接討論二元變數比較複雜, 所以我們在圓內取一個固定點, 也就是在 r 固定的情況下, 作為子樣本. 先推導出以弦長為隨機變數的累積分配函數 $F_{S_r}(s)$ 與機率密度函數 $f_{S_r}(s)$, 再運用機率密度函數求過圓內一點的弦長期望值 $E(S_r)$ 與變異數 $V(S_r)$.

(b) 求累積分配函數與機率密度函數
如果固定 r,
則由 ϕ 決定的弦長集合 $S = \left\{\sqrt{4R^2 - 4r^2\cos^2\phi} \mid 0 \le r < R, 0 \le \phi < \frac{\pi}{2}\right\}$, 此為一連續且無限元素的集合, 因為此函數在 $\left[0, \frac{\pi}{2}\right)$與$\left[\frac{\pi}{2}, \pi\right)$ 對稱, 所以弦長樣本空間取定義域為 $\left[0, \frac{\pi}{2}\right)$ 即可.

因為函數 $s(\phi)=\sqrt{4R^2-4r^2\cos^2\phi}$ 在 $\left[0,\dfrac{\pi}{2}\right)$ 之間為單調遞增函數, 我們可以利用 s 跟 ϕ 的一對一映射關係求累積分配函數, 即取到某個區間的弦長集合的累積機率等於其對應到角度區間的幾何機率. 故把弦長函數轉換成反函數, 再除以整個定義域區間的長度 $\dfrac{\pi}{2}$:

$$\text{令 } s=\sqrt{4R^2-4r^2\cos^2\phi}\Rightarrow\phi=\cos^{-1}\sqrt{\frac{4R^2-s^2}{4r^2}}\Rightarrow\frac{\phi}{\frac{\pi}{2}}=\frac{2}{\pi}\cos^{-1}\sqrt{\frac{4R^2-s^2}{4r^2}}$$

求得累積分配函數 $F_{S_r}(s)=\dfrac{2}{\pi}\cos^{-1}\sqrt{\dfrac{4R^2-s^2}{4r^2}}$, 嚴格定義為

$$\begin{cases} F_{S_r}(s)=0, & s<\sqrt{4R^2-4r^2};\\ F_{S_r}(s)=\dfrac{2}{\pi}\cos^{-1}\sqrt{\dfrac{4R^2-s^2}{4r^2}}, & \sqrt{4R^2-4r^2}\le s\le 2R;\\ F_{S_r}(s)=1, & 2R<s.\end{cases}$$

然而, 光是累積分配函數不足以我們進行運算, 我們還需要進一步得到機率密度函數 $f_{S_r}(s)$, 就是 $F_{S_r}(s)$ 的導數, 將 $F_{S_r}(s)$ 微分求得機率密度函數為

$$\begin{cases} f_{S_r}(s)=0, & s<\sqrt{4R^2-4r^2};\\ f_{S_r}(s)=\dfrac{2}{\pi}\dfrac{s}{\sqrt{(4R^2-s^2)(4r^2)}\sin\left(\cos^{-1}\sqrt{\frac{4R^2-s^2}{4r^2}}\right)}, & \sqrt{4R^2-4r^2}\le s\le 2R;\\ f_{S_r}(s)=0, & 2R<s.\end{cases}$$

於是, 機率密度函數圖所占的面積等於隨機變數在該區域的累積機率.
我們求出機率密度函數後, 就可順利求出子樣本的期望值與變異數.

(c) 求條件期望值與條件變異數
期望值 $E(S_r)$:

$$\begin{aligned}E(S_r)&=\int_{\sqrt{4R^2-4r^2}}^{2R}s\cdot f_{S_r}(s)ds\\&=\int_{\sqrt{4R^2-4r^2}}^{2R}s\cdot\frac{2}{\pi}\frac{s}{\sqrt{(4R^2-s^2)(4r^2)}\sin\left(\cos^{-1}\sqrt{\frac{4R^2-s^2}{4r^2}}\right)}ds.\end{aligned}$$

再來看看變異數:

$$\begin{aligned}V(S_r)&=E(S_r^2)-[E(S_r)]^2\\&=\frac{2}{\pi}\int_{\sqrt{4R^2-4r^2}}^{2R}\frac{s^3}{\sqrt{(4R^2-s^2)(4r^2)}\sin\left(\cos^{-1}\sqrt{\frac{4R^2-s^2}{4r^2}}\right)}ds-[E(S_r)]^2\end{aligned}$$

□

3. 母體

證明.

(a) 想法與定義符號
定義 S 為過圓內任一點的弦長隨機變數. 接下來, 我們討論母體的情況, 也就是 r 和 ϕ 皆會變動的情況下, 母體由

1. 圓內任取一點 Z_A,

2. 過點 Z_A 任取弦長, 兩個獨立變數一起構成, 其映射情形如下：
子樣本（固定 r）令為 r_k 的映射情形

$$\begin{array}{rccccc}
S_k = \{ & S_1, & S_2, & S_3, & \cdots, & S_n, \cdots\} \\
 & \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\
\phi = \{ & \phi_1, & \phi_2, & \phi_3, & \cdots, & \phi_n, \cdots\}
\end{array}$$

母體的映射情形

$$\begin{array}{c}
\\
r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_k \\ \vdots
\end{array}
\begin{array}{c}
\begin{array}{cccccc}
\phi_1 & \phi_2 & \phi_3 & \cdots & \phi_n & \cdots
\end{array} \\
\left\{\begin{array}{cccccc}
S_{11} & S_{12} & S_{13} & \cdots & S_{1n} & \cdots \\
S_{21} & S_{22} & S_{23} & \cdots & S_{2n} & \cdots \\
S_{31} & S_{32} & S_{33} & \cdots & S_{3n} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \\
S_{k1} & S_{k2} & S_{k3} & \cdots & S_{kn} & \cdots \\
\vdots & \vdots & \vdots & & \vdots &
\end{array}\right\}
\end{array}$$

構成母體的變數有 r, ϕ 兩個, 所以當我們要計算母體的弦長期望值時, 需要用到聯合機率密度函數, 因此, 除了$f_{S_r}(s)$, 我們還需要另外求出 $f(r)$, 為在圓內取到距離圓心小於等於 r 的點的機率密度函數.

(b) 計算
令一圓圓心為 O, 半徑為 R, 若任意在圓內取一點, 該點距離圓心小於等於 r 的機率為 $\frac{r^2}{R^2}$, 故圓內取一點距離圓心小於等於 r 的累積分配函數 $F(r) = \frac{r^2}{R^2}$, 得機率密度函數 $f(r) = F'(r) = \frac{2r}{R^2}$. 將子樣本的弦長期望值 $E(S_r)$ 對 r 積分, 即可求得母體期望值 $E(S)$ 為

$$\begin{aligned}
E(S) &= \int_0^R \left(\int_{\sqrt{4R^2-4r^2}}^{2R} s \cdot f_{S_r}(s) ds \right) \cdot f(r) dr \\
&= \int_0^R \left(\frac{2}{\pi} \int_{\sqrt{4R^2-4r^2}}^{2R} \frac{s^2}{\sqrt{(4R^2-s^2)(4r^2)} \sin\left(\cos^{-1}\sqrt{\frac{4R^2-s^2}{4r^2}}\right)} ds \right) \cdot \frac{2r}{R^2} dr.
\end{aligned}$$

而母體變異數 $V(S)$ 為

$$\begin{aligned}
V(S) &= E(S^2) - (E(S))^2 \\
&= \int_0^R \left(\int_{\sqrt{4R^2-4r^2}}^{2R} s^2 \cdot f_{S_r}(s) ds \right) \cdot f(r) dr - (E(S))^2 \\
&= \int_0^R \left(\frac{2}{\pi} \int_{\sqrt{4R^2-4r^2}}^{2R} \frac{s^3}{\sqrt{(4R^2-s^2)(4r^2)} \sin\left(\cos^{-1}\sqrt{\frac{4R^2-s^2}{4r^2}}\right)} ds \right) \cdot \frac{2r}{R^2} dr - (E(S))^2 .
\end{aligned}$$

□

4. 比較不同觀點下的弦長分佈
分別以兩種不同的觀點:

1. 把每條弦長出現的機率視為一樣的一致分配
2. 每條弦長出現的機率為機率分配函數, 我們將其稱為“反三角弦長分配函數”來看弦長的期望值和標準差的趨勢.

(a) 子樣本期望值和標準差的比較

做一個半徑為 100 公分的圓, 分別在 10 公分, 20 公分, 30 公分, 40 公分, 50 公分, 60 公分, 70 公分, 80 公分, 90 公分處取點(即 $r = 10, 20, 30, \cdots, 90$), 過此九點每 1 度取 1 條弦長, 各取 90 條弦, 如下表格(一)：

r	10	20	30	40	50	60	70	80	90
s1	199.9996954	199.9987817	199.9972587	199.9951266	199.9923852	199.9890346	199.9850747	199.9805055	199.975327
s2	199.998782	199.995128	199.9890379	199.9805115	199.9695483	199.9561481	199.9403103	199.9220344	199.9013197
s3	199.9972609	199.9890435	199.975347	199.9561704	199.931512	199.9013698	199.8657414	199.8246238	199.7780136
s4	199.995134	199.9805352	199.9562015	199.9221294	199.8783138	199.8247485	199.7614254	199.6883354	199.6054676
s5	199.9924037	199.9696132	199.9316232	199.8784251	199.8100067	199.7263523	199.627443	199.5132558	199.3837646
s6	199.9890735	199.9562904	199.90164	199.8251043	199.7266582	199.6062693	199.4638977	199.2994965	199.113011
s7	199.9851473	199.9405826	199.8662861	199.7622245	199.6283513	199.4646065	199.2709164	199.0471936	198.7933368
s8	199.9806299	199.9225084	199.8256016	199.6898531	199.5151836	199.3014907	199.0486489	198.7565086	198.4248963
s9	199.9755268	199.9020891	199.7796329	199.6080681	199.3872679	199.1170684	198.7972682	198.4276276	198.0078674
s10	199.969844	199.8793488	199.7284324	199.5169576	199.2447317	198.9115051	198.5169708	198.0607623	197.5424523
s11	199.9635886	199.8543146	199.6720585	199.41662	199.0877175	198.6849862	198.2079761	197.6561493	197.0288772
s12	199.9567681	199.8270161	199.6105754	199.3071636	198.9163827	198.4377164	197.8705271	197.2140509	196.4673927
s13	199.9493906	199.7974856	199.5440535	199.1887069	198.730899	198.1699199	197.5048902	196.7347549	195.8582741
s14	199.9414652	199.765758	199.4725687	199.0613782	198.5314533	197.8818402	197.1113554	196.218575	195.2018212
s15	199.9330015	199.7318711	199.3962029	198.9253159	198.3182468	197.5737404	196.690236	195.6658508	194.4983587
s16	199.9240096	199.6958649	199.3150435	198.7806678	198.0914952	197.2459031	196.2418695	195.076948	193.7482365
s17	199.9145005	199.6577824	199.2291837	198.6275918	197.8514288	196.8986301	195.7666167	194.4522587	192.9518299
s18	199.9044857	199.6176685	199.138722	198.466255	197.5982919	196.5322425	195.2648625	193.7922019	192.1095399
s19	199.8939773	199.5755712	199.0437624	198.2968341	197.3323434	196.1470811	194.737016	193.0972233	191.2217932
s20	199.882988	199.5315402	198.9444143	198.119515	197.0538561	195.7435056	194.18351	192.3677958	190.2890432
s21	199.8715312	199.4856282	198.8407923	197.9344928	196.7631168	195.3218952	193.6048018	191.60442	189.3117698
s22	199.8596206	199.4378897	198.733016	197.7419717	196.460426	194.8826482	193.0013732	190.8076242	188.2904798
s23	199.8472709	199.3883815	198.62121	197.5421646	196.1460982	194.4261821	192.3737301	189.9779649	187.2257076
s24	199.8344968	199.3371628	198.5055039	197.3352932	195.8204612	193.9529334	191.7224033	189.1160272	186.1180158
s25	199.821314	199.2842946	198.386032	197.1215877	195.4838562	193.4633577	191.0479484	188.2224253	184.9679953
s26	199.8077383	199.2298401	198.2629331	196.9012867	195.1366377	192.9579297	190.3509455	187.2978027	183.7762659
s27	199.7937863	199.1738643	198.1363507	196.6746369	194.7791731	192.4371425	189.6320001	186.3428325	182.5434773
s28	199.7794749	199.1164341	198.0064323	196.4418929	194.4118425	191.9015083	188.8917427	185.3582185	181.2703093
s29	199.7648214	199.0576183	197.8733299	196.2033171	194.0350389	191.3515579	188.1308289	184.344695	179.957473
s30	199.7498436	198.9974874	197.7371993	195.9591794	193.6491673	190.7878403	187.34994	183.3030278	178.605711
s31	199.7345596	198.9361135	197.5982004	195.709757	193.254645	190.2109231	186.5497824	182.2340144	177.2157987
s32	199.7189882	198.8735702	197.4564966	195.4553342	192.851901	189.6213919	185.7310885	181.1384848	175.7885451
s33	199.703148	198.8099326	197.312255	195.196202	192.441376	189.0198504	184.8946162	180.017302	174.3247935
s34	199.6870585	198.7452774	197.1656458	194.9326579	192.0235219	188.4069199	184.0411492	178.8713627	172.8254229
s35	199.670739	198.6796822	197.0168426	194.6650057	191.5988014	187.7832395	183.1714972	177.7015977	171.2913492
s36	199.6542096	198.6132261	196.8660219	194.3935554	191.1676881	187.1494653	182.2864958	176.5089729	169.7235261
s37	199.6374901	198.5459893	196.7133631	194.1186223	190.7306655	186.5062706	181.3870064	175.2944898	168.1229466
s38	199.6206011	198.478053	196.5590482	193.8405274	190.2882274	185.8543453	180.4739166	174.0591861	166.4906445
s39	199.6035629	198.4094991	196.4032613	193.5595965	189.8408767	185.1943956	179.54814	172.8041366	164.8276961
s40	199.5863964	198.3404108	196.2461891	193.2761604	189.3891256	184.527144	178.610616	171.5304541	163.1352215

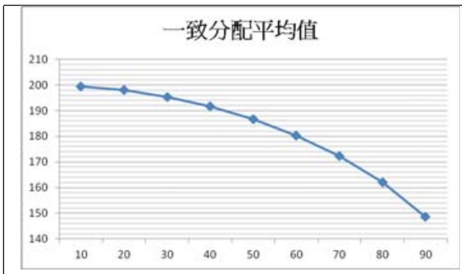
s41	199.5691224	198.2708715	196.08802	192.990554	188.9334949	183.8533283	177.66231	170.2392895	161.4143867
s42	199.5517619	198.2009656	195.9289443	192.7031164	188.4745137	183.1737015	176.7042131	168.9318334	159.6664057
s43	199.5343361	198.130778	195.769154	192.4141905	188.0127186	182.4890315	175.7373422	167.6093162	157.8925422
s44	199.5168662	198.0603938	195.6088421	192.1241224	187.5486536	181.8001	174.7627394	166.2730091	156.0941121
s45	199.4993734	197.9898987	195.4482029	191.8332609	187.0828693	181.1077028	173.781472	164.924225	154.2724862
s46	199.4818791	197.9193785	195.2874315	191.5419578	186.6159225	180.4126482	172.7946322	163.564319	152.4290922
s47	199.4644046	197.8489192	195.1267238	191.2505668	186.1483753	179.7157572	171.8033369	162.194689	150.5654181
s48	199.4469712	197.7786066	194.966276	190.9594431	185.6807951	179.0178624	170.8087265	160.8167767	148.6830148
s49	199.4296001	197.7085267	194.8062843	190.6689437	185.2137535	178.3198073	169.8119655	159.432068	146.7834996
s50	199.4123125	197.6387651	194.6469452	190.379426	184.7478257	177.6224454	168.8142407	158.0420935	144.8685595
s51	199.3951295	197.5694072	194.4884546	190.0912481	184.28359	176.9266397	167.8167615	156.6484292	142.9399546
s52	199.3780721	197.5005379	194.3310078	189.8047679	183.8216269	176.2332612	166.8207584	155.2526964	140.9995223
s53	199.361161	197.4322418	194.1747995	189.5203431	183.3625186	175.5431885	165.8274824	153.8565626	139.0491814
s54	199.344417	197.3646027	194.0200232	189.2383302	182.906848	174.8573065	164.8382039	152.461741	137.0909358
s55	199.3278605	197.2977037	193.8668712	188.9590843	182.455198	174.1765052	163.8542114	151.0699909	135.1268799
s56	199.3115117	197.2316271	193.715534	188.6829587	182.008151	173.5016787	162.8768105	149.6831173	133.1592024
s57	199.2953905	197.1664543	193.5662007	188.4103042	181.5662876	172.8337241	161.9073219	148.3029702	131.1901916
s58	199.2795167	197.1022655	193.4190578	188.1414689	181.130186	172.1735396	160.9470806	146.9314443	129.2222404
s59	199.2639096	197.0391401	193.27429	187.8767974	180.7004211	171.5220241	159.9974334	145.5704778	127.2578512
s60	199.2485885	196.977156	193.1320792	187.6166304	180.2775638	170.8800749	159.0597372	144.222051	125.2996409
s61	199.2335718	196.9163898	192.9926043	187.3613043	179.8621797	170.2485868	158.1353572	142.888185	123.3503462
s62	199.2188782	196.8569168	192.8560416	187.1111507	179.4548285	169.6284501	157.2256644	141.5709392	121.4128287
s63	199.2045254	196.7988105	192.7225637	186.8664956	179.0560631	169.0205496	156.332033	140.2724092	119.4900787
s64	199.1905312	196.742143	192.5923398	186.6276595	178.6664284	168.4257622	155.4558379	138.9947233	117.5852206
s65	199.1769125	196.6869846	192.4655354	186.3949561	178.2864604	167.8449559	154.5984522	137.7400399	115.7015156
s66	199.1636861	196.6334039	192.3423118	186.1686925	177.9166855	167.2789874	153.7612437	136.5105426	113.8423655
s67	199.1508683	196.5814673	192.2228263	185.9491684	177.5576192	166.728701	152.9455719	135.308436	112.0113137
s68	199.1384745	196.5312397	192.1072314	185.7366755	177.2097655	166.194926	152.1527849	134.1359406	110.2120467
s69	199.1265202	196.4827833	191.9956752	185.5314975	176.8736155	165.6784755	151.3842155	132.9952865	108.4483925
s70	199.1150198	196.4361587	191.8883008	185.3339089	176.5496468	165.1801441	150.6411778	131.8887074	106.7243179
s71	199.1039875	196.3914239	191.7852462	185.1441751	176.2383223	164.7007061	149.9249633	130.8184328	105.0439232
s72	199.0934369	196.3486348	191.6866438	184.9625519	175.9400893	164.2409134	149.2368368	129.7866806	103.411434
s73	199.0833807	196.3078448	191.5926209	184.7892848	175.6553789	163.8014941	148.5780327	128.7956485	101.8311903
s74	199.0738315	196.2691049	191.5032987	184.6246086	175.3846046	163.38315	147.9497504	127.8475044	100.307631
s75	199.0648008	196.2324634	191.4187929	184.4687472	175.1281616	162.9865549	147.3531508	126.9443769	98.84527535
s76	199.0562998	196.1979662	191.3392127	184.3219132	174.8864261	162.6123529	146.7893511	126.0883453	97.4486993
s77	199.0483388	196.1656564	191.2646614	184.1843073	174.6597543	162.2611564	146.2594214	125.281428	96.12250751
s78	199.0409277	196.1355746	191.1952358	184.0561179	174.4484815	161.9335441	145.7643801	124.5255722	94.87130012
s79	199.0340755	196.1077584	191.1310259	183.9375211	174.2529217	161.6300599	145.3051899	123.822642	93.69963477
s80	199.0277907	196.0822427	191.0721154	183.8286801	174.0733664	161.3512105	144.8827537	123.1744067	92.6119838
s81	199.0220809	196.0590594	191.018581	183.7297449	173.9100843	161.0974645	144.4979105	122.5825297	91.61268709
s82	199.0169532	196.0382377	190.9704923	183.6408521	173.7633204	160.8692506	144.1514321	122.0485571	90.70590127
s83	199.0124138	196.0198036	190.9279123	183.5621248	173.6332957	160.6669561	143.8440193	121.5739063	89.89554624
s84	199.0084684	196.0037804	190.8908964	183.493672	173.5202063	160.4909258	143.5762986	121.159856	89.18525028
s85	199.0051217	195.990188	190.8594929	183.4355887	173.4242233	160.3414612	143.3488194	120.807536	88.57829532
s86	199.0023778	195.9790436	190.833743	183.3879554	173.3454922	160.2188188	143.1620513	120.5179187	88.07756404
s87	199.0002403	195.9703612	190.8136803	183.3508384	173.2841323	160.1232099	143.0163817	120.2918108	87.68549079
s88	198.9987115	195.9641517	190.799331	183.3242893	173.2402371	160.0547995	142.9121139	120.1298471	87.40401813
s89	198.9977935	195.9604229	190.7907139	183.308345	173.2138732	160.0137058	142.8494659	120.0324848	87.23456082
s90	198.9974874	195.9591794	190.7878403	183.3030278	173.2050808	160	142.8285686	120	87.17797887

計算可得各點所取出之弦長的一致分配平均值和反三角分配期望值, 如下表格(二)：

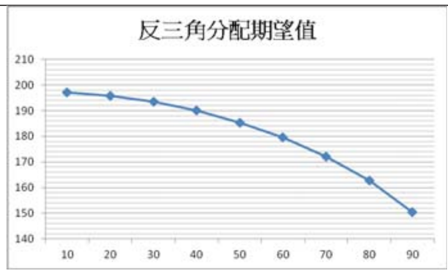
r	10	20	30	40	50	60	70	80	90
平均值	199.494	197.962	195.37	191.65	186.694	180.334	172.291	162.066	148.558
期望值	197.268	195.849	193.448	190.004	185.421	179.547	172.137	162.751	150.442

畫成圖形比較如下：

由表格(二)來看, 我們可以知道：子樣本的弦長一致分配平均值隨著 r 的增加下降的幅度較反三角分配期望值明顯, 且兩者差值隨著 r 的增加, 先縮小後放大, 我們觀察弦長機率密度函數將會發現：隨著 ϕ 逐漸增加, 弦長機率密度函數會先遞減再遞增, 且弦長較短的一側出現機率較高, 因此弦長反三角分配期望值小於弦長一致分配平均值；由比較圖一和圖二來看, 可以得知: 不論以哪種觀點來分析圓內弦長分布, 弦長期望值與 r 皆呈現負相關, 因為過圓內任意一點所能取到的最長弦長必為直徑, 但隨著 r 的增加, 取到的最短弦長的長度



(圖一)



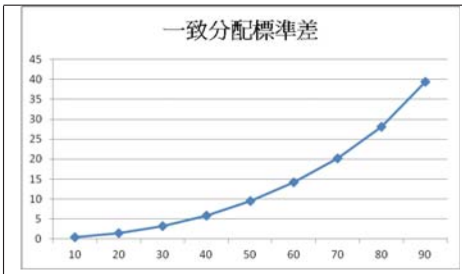
(圖二)

逐漸縮短, 而產生此種趨勢.
而以不同觀點所計算出來的各點的標準差, 如下表格(三)：

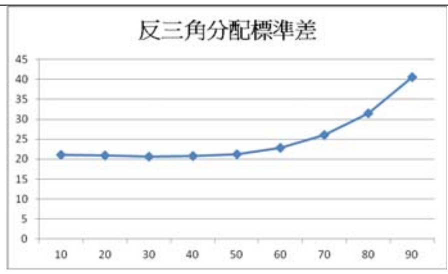
r	10	20	30	40	50	60	70	80	90
一致分配標準差	0.3544	1.4285	3.2568	5.9029	9.4715	14.1327	20.1764	28.1524	39.3755
反三角分配標準差	21.077	20.889	20.699	20.715	21.272	22.846	26.016	31.491	40.515

(表格三)

畫成圖形比較如下：



(圖三)



(圖四)

由表格(三)來看, 我們可以知道：隨著 r 增加, 弦長一致分配標準差呈現單調遞增的趨勢, 而弦長反三角分配標準差則會先略為下降後, 再呈現上升的趨勢；由比較圖三和圖四則可以明顯發現：當 r 越小時, 以不同觀點所得到的弦長標準差差異越大.

(b) 母體期望值的比較
做一個半徑為 100 公分的圓, 在 r 公分處取點, 過該點每隔 1 度取 1 條弦長, 共取 90 條弦, 分別以弦長為一致分配或弦長為反三角分配的方式將弦長積分, 再對 r 積分可得母體弦長一致分配平均值為 169.37, 以及母體弦長反三角分配期望值為 195.06.

3 結論

3.1 研究結果

1. 過圓內一點的弦長函數 $s(r,\phi) = \sqrt{4R^2 - 4r^2\cos^2\phi}$, 藉由反函數在角度上的累積機率可得弦長的累積分配函數

$$\begin{cases} F_{S_r}(s) = 0, & s < \sqrt{4R^2 - 4r^2}; \\ F_{S_r}(s) = \dfrac{2}{\pi}\cos^{-1}\sqrt{\dfrac{4R^2 - s^2}{4r^2}}, & \sqrt{4R^2 - 4r^2} \le s \le 2R; \\ F_{S_r}(s) = 1, & 2R < s. \end{cases}$$

和機率密度函數

$$\begin{cases} f_{S_r}(s) = 0, & s < \sqrt{4R^2 - 4r^2}; \\ f_{S_r}(s) = \dfrac{2}{\pi} \dfrac{s}{\sqrt{(4R^2 - s^2)(4r^2)} \sin\left(\cos^{-1}\sqrt{\frac{4R^2-s^2}{4r^2}}\right)}, & \sqrt{4R^2 - 4r^2} \le s \le 2R; \\ f_{S_r}(s) = 0, & 2R < s. \end{cases}$$

並求得期望值

$$E(S) = \int_0^R \left(\frac{2}{\pi} \int_{\sqrt{4R^2-4r^2}}^{2R} \frac{s^2}{\sqrt{(4R^2 - s^2)(4r^2)} \sin\left(\cos^{-1}\sqrt{\frac{4R^2-s^2}{4r^2}}\right)} ds \right) \cdot \frac{2r}{R^2} dr.$$

和變異數

$$V(S) = \int_0^R \left(\frac{2}{\pi} \int_{\sqrt{4R^2-4r^2}}^{2R} \frac{s^3}{\sqrt{(4R^2 - s^2)(4r^2)} \sin\left(\cos^{-1}\sqrt{\frac{4R^2-s^2}{4r^2}}\right)} ds \right) \cdot \frac{2r}{R^2} dr - (E(S))^2 .$$

2. 分別以兩種不同的觀點:
    1. 把每條弦長出現的機率視為一樣的一致分配
    2. 每條弦長出現的機率為機率分配函數 $F_{S_r}(s)$, 我們將其稱為“反三角弦長分配函數”來比較不同觀點之下, 弦長期望值之間的差異以及弦長標準差之間的差異.

3.2 未來展望

將這種每條弦長出現機率不一致的分析模型推廣至橢圓, 甚至是不規則凸圖形, 並且與弦長為一致分配時所得的結果互相比較.

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