Geometric Aspects on Minimum Tetrahedralizations of Glued Polyhedrons

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1 Abstract and Introduction

三角錐化(Tetrahedralization) 在計算幾何裡是一塊很活躍的領域. 他在多種領域都有其應用, 如有限元分析、計算機輔助設計(CAD/CAM)、多邊形網路(mesh generation)、代數拓普、複雜度理論、計算機圖形. 一個多面體的三角錐化是一個分割, 使得其中所有元素皆為三角錐, 且倆倆交集為空集合或共同面、邊. 並不是所有多面體都有三角錐化, 但是如果允許其他的頂點存在(steiner points)則所有的多面體都能被三角錐化.

Ruppert 和 Seidel(1992)證明出決定一個多面體需不需要使用多餘的點來三角錐化是 NP 完全的. 其他有趣的成果如下, Rothschild 和 Straus(1985) 證明出任意一個凸多面體都能切成 $O(n^2)$ 個三角錐、Chazelle和Palios(1990) 對一些特殊的多面體給出了允許 steiner points 的三角錐化的算法、Chazelle 和 Shouraboura(1995)給出了切割兩個多面體之間的區域的切法, 如果允許 steiner points, 能切出 $O(n)$、Palios(1996) 對此想法做了個推廣, 給出切割一多面體和一多邊形間區域的算法. 三角錐化中的最佳化成果則比較少.

我們在這篇主要是探討其中一種很重要的最佳化, 當三角錐化的大小(裡面的三角錐的個數)為最少時, 我們叫此三角錐化為完美切法(minimum tetrahedralization). 這領域裡面有一些結果特別引人注意, Below(2004)等人證明出找出一個完美切法為 NP-難, 而決定一個多面體的大小的上界為 NP-完全. 其他的結果大多是給出分解一些特殊的多面體的算法, 如 Wang and Yang 給出 BP-多面體、雙層 BP-多面體(2004)和 k-vet 多面體(2000) 能在多項式時間內被分解的算法. 裡面我覺得最重要的成果由 Chin 等人(2001)做出, 他們給出了找出一個完美切法的近似算法. 更令人驚訝的是, 他們證明出如果只考慮多面體的組合結構, 那麼他們的結果是最佳上界.

而在我這篇論文中, 我探討一個問題, ”把兩個多面體沿著一個面黏起來, 那麼此面是否屬於這個新的多面體的完美切法?” 很明顯, 他不一定是, Chin 等人在他們的論文中也給出了反例. 這點正好是限制住只考慮組合結構的方法的巨大障礙, 因如果只考慮組合結構, 此面一定要切, 但是根據 Below 等人(2000)的論文中給出的例子的應用, 如果切了此面, 其誤差會呈線性成長, 因此給不出更好的結果. 我在這篇論文中給出了此敘述為正確的幾何限制條件, 並使用穿越多面體 $(A+B)(\mathrm{tran}(A+B))$ 的想法, 使得裡邊(inner edge)與頂點串聯起來, 進而能夠進行估計而導出矛盾, 是非常創新而且簡潔的想法.

我在這篇研究作出的結果有下列幾點

1. 引進了穿越多面體的想法, 並證明其關聯裡邊和頂點的性質
2. 舉出了命題成立的幾何限制條件
3. 舉出在沒有限制條件下的反例並以其映證此反例的確違反了我舉出的幾何限制條件
4. 證明如果多面體符合我給的幾何限制條件, 那麼命題成立
5. 以同樣的想法舉出 glued face 是兩個三角形的類似定理並解釋如果 glued face 再多, 則類似定理不成立的理由

我在這篇研究的方法我簡述一下, 首先先藉由畫廊問題的想法來找出一個對滿足某些條件的多面體的很自然的三角錐化(Lemma 1), 因其條件不容易進行操作, 我給出了決定其等價關係的方法(Lemma 2), 即是以點面之間的相對關係來表達, 再者我介紹了穿越多面體的概念並歸納地利用 Van Kampen 定理來證明此立體的確是多面體, 再者利用不能被三角錐化的多面體和 Below 的 Lemma(2000)我們可以很輕易地建構出對命題的反例. 從反例中我們可以清楚的看到問題所在, 就是穿越多面體並不符合 Lemma 1 的敘述, 所以我們對命題多加了一條幾何限制條件讓每一種穿越多面體都能符合 Lemma 1 的條件.

利用穿越多面體本身的特性加上從 Lemma 1 延伸出的不等式, 我們能輕易地對裡邊進行估計從而導出矛盾的不等式在兩多面體其中有一個為三角錐的情況下. 而當穿越多面體兩邊的頂點都多於 1 個的情況, 其特性更得到加强, 對於任何三角錐化, 裡邊的數量必定大於穿越多面體所有的頂點數減三(Lemma 4). 運用同樣的方法我們就能在更一般的情形導出結果. 雖然此結果極為漂亮, 但是我給出的幾何條件只是充分條件, 離充要條件的距離還有很大一段.

我在這裡簡述下未來可能的發展, 第一個自然能想到的問題就是找出此命題為真的充要條件, 我猜測能以理解更多穿越多面體的性質來達成. 第二個就是探討當我們考慮進幾何條件時, 如何給出比 Chin 等人的結果更佳的上界, 在這篇論文中, 因我們的充分條件太强, 並沒辦法使用在很一般的情形, 因此給不出確切的上界. 第三就是探討判斷此面是否屬於完美切法的複雜度, 值得注意的是, 如果使用 Below 等人作出的邏輯凸多面體, 那麼如果其中有一面屬於完美切法, 那麼他所代表的波爾代數式就不滿足, 但就算所有都不屬於完美切法, 我們也沒法斷定他的波爾代數式滿不滿足. 就我猜測, 判斷此面是否屬於完美切法的複雜度極有可能為 NP Hard.

Finding a tetrahedralization is an active area in computational geometry. It has applications in various topics, such as finite elements analysis, CAD/CAM, mesh generation, complexity, computer graphics and algebraic topology. Unlike triangulation in 2 dimensional cases, tetrahedralization has several interesting properties. Not all polyhedrons can be tetrahedralized, but if we allow steiner points, all polyhedron can be tetrahedralized. Also, the size of polyhedron can be varied while geometry realization changes (Below et al. 2000).

Chazelle and Palios (1990) proved that a polyhedron having n vertexes and r reflex edges can be tetrahedralized into $O(n + r^2)$ tetrahedron with $O(n + r^2)$ steiner points. Chazelle and Shouraboura (1995) developed an algorithm to tetrahedralize the region between two polyhedrons into $O(n)$ tetrahedrons. Ruppert and Seidel proved that it is NP-complete to determine if a polyhedron can be tetrahedralized without steiner points. For minimum tetrahedralization, fewer results were studied. Below et al. (2004) proved that finding minimum tetrahedralization is NP-hard and determining if a number is larger than the size of polyhedron is NP-complete. Thus, it becomes important to study how to find a minimum tetrahedralization.

Wang and Yang proved that we can tetrahedralize k-vet polyhedron (2000) and BP-polyhedron (2004) in polynomial time. I think the most important result was given by Chin et al. (2001). They gave an approximation algorithm and, even more surprisingly, proved that their upper bound is tight if we only consider the computational structure. In this paper, I raise an intuitively true question, "Given two tetrahedralizable polyhedron with their minimum tetrahedralization and we glue them together along one common face to form a new polyhedron. It is obvious that there exist a tetrahedralization for new formed polyhedron, the union of two minimum tetrahedralizations, but is it a minimum tetrahedralization? In the other word, should we cut the gluing face?"

The answer is no. Chin et al. (2001) gave a counter example. This is also the biggest obstacle to improve the upper bound given by Chin et al. If we only consider computational structure, the gluing face must be cut. However, Below et al. (2000) showed that the error will grow linearly in a specific class of polyhedron if we cut that face. Hence, Chin et al. cannot give a better upper bound. In order to improve the upper bound, it is necessary to study the geometric condition of this question. In this paper, I give a geometric sufficient condition when the answer of the question is yes by using tran $(A+B)$ to connect two separate idea, vertexes and inner edges, leading to a contradictory estimation.

2 Notation and some terms

Convention: in this paper, I only discuss polyhedrons whose any 4 vertexes are not coplanar Tetrahedralization of a polyhedron A (here refers to polyhedron in general sense including whose fundamental group is not simple) is a partition of A into several tetrahedrons that only meet on shared face such that each tetrahedron's vertexes meet on polyhedron's vertexes.

The size of tetrahedralization represents the number of tetrahedrons in tetrahedralization Minimum tetrahedralization of A is a tetrahedralization whose size is the smallest. The size of polyhedron represents the size of minimum tetrahedralization.

Given a polyhedron A (in general sense), $MT(A)$ denotes the set that consists of all tetrahedrons in minimum tetrahedralizaion.

Given n points $a_1, a_2, \ldots, a_n$ in R^3, let $conv(a_1, a_2, \ldots, a_n)$ denote its convex hull.

For convenience, denote both the point set of $(n-1)$-simplex and itself by $(a_1, a_2, \ldots, a_n)$

Given a polyhedron P, using $P_{[n]}$ to denote n dimensional data set of P and $P_{(n)}$ be their union. For example, $P_{[2]}$ is a set consists of all surfaces triangle while $P_{(2)}$ is the union of these triangles.

Given a polyhedron A with its vertexes $a_1, a_2, \ldots, a_n$, let $v(a_i)$ denote the number of edge that join to a_i.

For convenience, Given a set A of the set of geometric objects, let PA denote point set of these geometric objects. For example, $PMT(A) = A$.

Given 3 points x, y, z, let $Pla(x, y, z)$ denote unbounded plane pass through x, y, z.

3 Main body

In whole paper, I use a trivial fact that for any two tetrahedralizations on one polyhedron, the one which has less inner edges will have smaller size. It is a trivial consequence of following formula derived from Euler formula. $S = n + i - 3$ where S denotes the size of tetrahedralization, n denotes the number of vertexes, i denotes the number of inner edges, the edges wholly contained in the interior of polyhedron.

Also in this paper, I only discuss polyhedrons whose any 4 vertexes are not coplanar since polyhedrons not in general position can be perturbed. I will implicitly use this property in various places. This condition is important since if we allow 4 points to be coplanar, there will be lots of awkward phenomena such as the size of minimum dissection might be smaller than the size of minimum tetrahedralization (Below et al. 2000).

Although for convenience I only discuss convex polyhedron in theorem 1 and 2, it is worth to notice that all results in this paper do not depend on convexity of A, B and $A + B$. Hence, the general results when A and B are both tetrahedralizable (if not, the case is trivial or meaningless) follow easily.

Lemma 1. *Given a polyhedron P and vertex a_1 such that for any $x \in int(P)$, $\overleftrightarrow{a_1 x} \cap P_{(2)}$ is a two elements set. Prove that, there is a natural tetrahedralization N_P of P,*

$$N_P = \left\{ (a_1, a_x, a_y, a_z) \middle| (a_1, a_x, a_y, a_z) \in P_{[2]} \right\} \text{ where } x, y, z \neq 1.$$

Proof. For any $x, y \in P_{(2)}$, $x \neq y$, $\overleftrightarrow{a_1 x} \neq \overleftrightarrow{a_1 y}$ since $\overleftrightarrow{a_1 x} \cap P_{(2)}$ is a two element set. This implies that $\overleftrightarrow{a_1 x} \cap \overleftrightarrow{a_1 y} = a_1$. Since

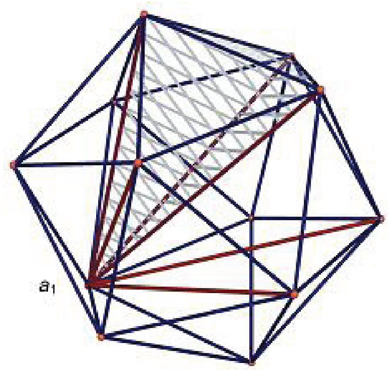
$$(a_1, a_x, a_y, a_z) = \cup (a_1, x)\Big|_{x \in (a_x, a_y, a_z)},$$

$$(a_1, a_x, a_y, a_z) \cap (a_1, a_i, a_j, a_k) = \cup (a_1, x)\Big|_{x \in (a_x, a_y, a_z) \cap (a_i, a_j, a_k)}$$

for which $(a_i, a_j, a_k), (a_x, a_y, a_z) \in P_{[2]}$. This shows that any intersection of two tetrahedrons is still a simplex based on the vertexes of P.

To prove their union is P, assume there is $x \notin PN_P$ in P. It is obvious to see $x \notin P_{(2)}/(a_1, a_b, a_c)$ which $(a_1, a_b, a_c) \in P_{[2]}$ by definition of N_P. If $x \in (a_1, a_b, a_c)$, it must belongs to (a_1, a_b, a_c, a_k) for $(a_b, a_c, a_k) \in P_{[2]}$. Hence $x \in \text{int}(P)$, but now since $\overleftrightarrow{a_1 x} \cap P_{(2)}$ are two point sets, x must contain in one of tetrahedron due to the fact $\overleftrightarrow{a_1 x} \cap P_{(2)}/a_1 \in (a_x, a_y, a_z)\Big|_{(a_x, a_y, a_z) \in P_{(2)}}$. Hence N_P is a tetrahedralization of P. $\square$

Ex 1: The below is a natural tetrahedralization of regular isocahedron.



Remark. *It is worth to notice that convex polyhedrons satisfy the statement of lemma1. In this case, can be any vertexes of convex polyhedron. Also, it is worth to notice that the condition can be revised as "for any $x \in P_{(2)}$, $(x, a_1) \subset P$ ". A good analogy associating with art of gallery problem is " a_1 is enough to be a guardian of the gallery P ".*

However, the condition of Lemma 1 is not a property that can be easily manipulated. Thus, I raise lemma 2 which allow us to determine the equivalence class of a_1.

First, let me define "located at the same side". Given a plane P and two points x_1, x_2. We say that x_1, x_2 are at the same side of P if and only if there is $\triangle ABC$ on P, $(A, B, C, x_1) \cap (A, B, C, x_2) \neq (A, B, C)$. Also, if one of x_i is on P, then we say x_1, x_2 are at the same side of P in general sense.

We can easily verify that this definition is well-defined, so we omit it here.

Lemma 2. *Let polyhedron P and a_1 satisfy the condition of Lemma 1. Then the $(P, x) x \in P$ also satisfy the condition of Lemma 1 if and only if for any $B \in P_{[2]}$, x and a_1 are located at the same side of B in general sense.*

Proof. First prove the forward direction. x and a_1 are located at the same side of (a_1, a_j, a_k) in general sense by definition. Now, for any $(a_i, a_j, a_k) \in P_{[2]}$ which $i, j, k \neq 1$, choose $y \in (a_i, a_j, a_k)$. By condition of Lemma 1, we know that (y, a_1) is contained in P and (a_i, a_j, a_k, a_1). Pick z on (y, a_1), such that $\overrightarrow{xz} \cap pla(a_i, a_j, a_k) \neq \emptyset$, let it be w. Now, consider a big(A,B,C) on $pla(a_i,a_j,a_k)$ such that (A,B,C) contains $conv(a_i,a_j,a_k,w)$. Then, $(z,y)\subset(A,B,C,a_1)\cap(A,B,C,x)$. Since z doesn't belong to $Pla(a_i,a_j,a_k)$,

$$(A,B,C,a_1)\cap(A,B,C,x)\neq(A,B,C).$$

For the backward direction, for any $(a_i,a_j,a_k)\in P_{[2]}$ which $i,j,k\neq 1$, assume there is $y\in(a_i,a_j,a_k)$, such that $(x,y)\not\subset P$. So, from x to y, there exists a point z that is the transition from outside to inside. Now, since $(a_i,a_j,a_k,a_1)\subset P$ and x and a_1 are at the same side of (a_i,a_j,a_k), there is a small neighborhood B of z, such that

$$(z,x)\cap B\subset(a_i,a_j,a_k,x)\cap(a_i,a_j,a_k,a_1).$$

But it is impossible since LHS contains points that don't belong to P while RHS is wholly contained in P. $\square$

Corollary 1. *Given a polyhedron P and a_1 satisfying the condition of Lemma 1. Let N be the number of vertexes and I be the number of inner edges in minimum tetrahedralization. Then the following inequality holds. $N-1-v(a_1)\geq I$.*

I will prove this corollary by finding a triangulation of polyhedron P which has $N-1-v(a_1)$ inner edges. Since I is the number of inner edges in minimum tetrahedralization, it must be smaller or equal than $N-1-v(a_1)$. It is obvious that N_P in Lemma 1 satisfied our requirement.

Definition $(tran(A+B))$**.**
Given a $conv(a_1,a_2,\dots,a_n,x,y,z)$. Consider $conv(a_1,a_2,\dots,a_n,x,y,z,b_1,b_2,\dots,b_m)$, such that

$$\begin{aligned}&conv(a_1,a_2,\dots,a_n,x,y,z,b_1,b_2,\dots,b_m)/conv(a_1,a_2,\dots,a_n,x,y,z)\\&=conv(b_1,b_2,\dots,b_m,x,y,z).\end{aligned}$$

For convenience denote $conv(a_1, a_2, \dots, a_n, x, y, z)$ as A, $conv(b_1, b_2, \dots, b_m, x, y, z)$ as B, $conv(a_1,a_2,\dots,a_n,x,y,z,b_1,b_2,\dots,b_m)$ as $A+B$. Given a tetrahedralization λ of $A+B$, such that for any $\varphi\in\lambda$, $(x,y,z)\not\subset P\varphi$. Consider a collection δ of tetrahedron from λ consisting all tetrahedrons $T\in\delta$, such that the vertexes of T consist of at least one a_i and one b_j. We define $P\delta$ as $tran(A+B)$.

Lemma 3. *Prove that δ is a tetrahedralization of a solid $tran(A+B)$ that wholly contains (x,y,z). Furthermore, int($tran(A+B)$) is connected and $tran(A+B)_{(2)}$ is simply connected.*

Proof. It is obvious that δ is a tetrahedralization since it is a subset of tetrahedralization λ. Assume $\text{int}(P\delta)$ is disconnected. It means that $\text{int}(P\delta)=U\cup V$ for some open U, V such that $U\cap V=\emptyset$.

First, observe that the only kind of tetrahedrons whose intersection with (x,y,z) is nonempty is the kind of tetrahedrons whose vertexes consist of at least one a_i and one b_j. Then, $P\delta$ contains (x,y,z) because for any $\varphi\in\lambda\Big|_{(x,y,z)\not\subset P\varphi}$, and $P=P\lambda\supset(x,y,z)$.

Now, I claim that for any $\alpha\in\text{int}((x,y,z))$ in subspace topology on (x,y,z), $\alpha\in\text{int}(P\delta)$. Consider $T\in\delta$ and $T\cap(x,y,z)$. If $\alpha\in\text{int}(T\cap(x,y,z))$ on (x,y,z), $\alpha\in\text{int}(P\delta)$ since it is even an interior point of T.

If $\alpha\in\partial T$ but not x, y, or z, it belongs to one of the surface ϑ of T. Let $\vartheta\cap(x,y,z)=(a,b)$. I claim that there are exactly two tetrahedron contains ϑ. Now consider the unbounded planes that do not intersect with T formed by any three vertexes of polyhedron. Consider the distance from each plane to α, it is trivial to see that the distances are non-zero. Since it is a finite set, I can choose ε smaller than every element. Now consider $B_\varepsilon(\alpha) \cap (x, y, z)$ and pick y not belonged to T from it. It is obvious that this y cannot belong to tetrahedron doesn't contain ϑ as desired.

If $\alpha \notin \{a, b\}$, it is trivial to see α is an interior point of $P\delta$ since it is interior point of union of two tetrahedrons that contain ϑ. Hence $\alpha \in \{a, b\}$.

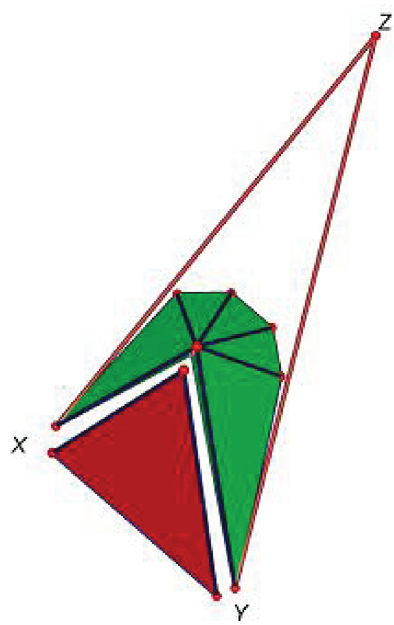
WLOG, assume $\alpha = a$. α belongs to a unique segment (a_i, b_j) due to the fact that any 4 vertexes of polyhedron are not co-planar. Since the segment based on vertexes is finite, I can find an open disk B of α on (x, y, z), such that for any $z \in B$, z only belongs to (a_i, b_j, f, g) for $(a_i, b_j, f, g) \in \delta$. Now it is trivial to see that α is an interior point of the smallest subcollection of δ that contains B (the set containing all tetrahedron that contains (a_i, b_j)). Hence, it belongs to $\text{int}(P\delta)$. □

Note: Notice that the construction of B also shows that $(a_i, b_j)/\{a_i, b_j\} \subset \text{int}(P\delta)$ by considering two cones formed by connecting disk B with a_i, b_j.

So, for any $\alpha \in \text{int}((x, y, z))$, I can choose an open ball B_α contained in $\text{int}(P\delta)$. This implies $\cup B_\alpha$ contained in $\text{int}(P\delta)$, too. Hence, $\cup B_\alpha$ is contained in one of U and V. WLOG, assume $\cup B_\alpha$ is contained in V which implies that U contained in $\text{int}(conv(a_1, a_2, \ldots, a_n, x, y, z))$. But it is not true since for any $s \in U$, s belongs to some $R \in \delta$ which implies that there is a path connecting U and V since R is convex and contains at least one b_k. Hence contradict the fact that $\text{int}(P\delta)$ is disconnected.

To prove it is simple, we prove it by induction. Consider the tetrahedron that contains (x, y), (x, y, a_i, b_j). By previous argument of choosing small enough neighborhood of the intersection of a segments and $\triangle xyz$, we can get the stacked polyhedron (green part) that we need to glue as picture shown. Both boundary of red part and green part are homeomorphic to a sphere.

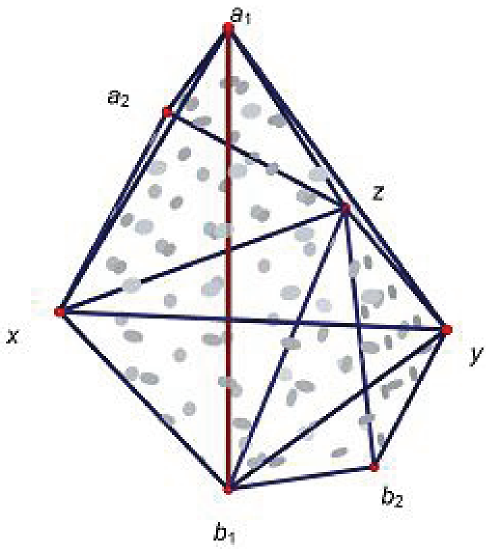
And since (a_i, b_j) will become inner edge after gluing (by note), the resulting polyhedron is homeomorphic to two spheres without a disk with an equivalence relationship on circle. By Van Kampen's theorem, fundamental group of resulting polyhedron is still trivial.



Van Kampen's theorem
If X, Y, $V \cap Y$, $X \cup Y$ is open and connected, then $\pi_1(X \cup Y)$ is isomorphic to free group of $\pi_1(X)$, $\pi_1(Y)$ with amalgamation of $\pi_1(X \cap Y)$.

Now, since we know (a_i, b_j) will become inner edge after gluing, we can apply Van Kampen's theorem to two balls without a disk with equivalence relationship on disk's boundary. Since now $\pi_1(X)$, $\pi_1(Y)$ are all trivial, the resulting free group with amalgamation is still trivial as desired.

We can inductively grow the polyhedron by considering new emerging inner edge that pierces through with same argument. Since there are only finite steps and $tran(A+B)$ is connected, it will stop and implies that $tran(A+B)_{(2)}$ is simply connected.
EQ: On below, $\delta = \{(r, s, a_1, b_1)\}$ which $r, s \in \{x, y, z\}$ and $tran(A+B) = conv(x, y, z, a_1, b_1)$.



One might guess
Given $conv(a_1, a_2, \ldots, a_n)$ whose size is m. Consider $conv(a_1, a_2, \ldots, a_n, x)$, such that $conv(a_1, a_2, \ldots, a_n, x)/conv(a_1, a_2, \ldots, a_n)$ is (a_i, a_j, a_k, x). Prove that $(a_i, a_j, a_k, x) \in MT(conv(a_1, a_2, \ldots, a_n, x))$, which implies that the size of $conv(a_1, a_2, \ldots, a_n, x)$ is $m+1$. For convenience, denote $conv(a_1, a_2, \ldots, a_n)$ as A, (a_i, a_j, a_k, x) as B and $conv(a_1, a_2, \ldots, a_n, x)$ as $A+B$.

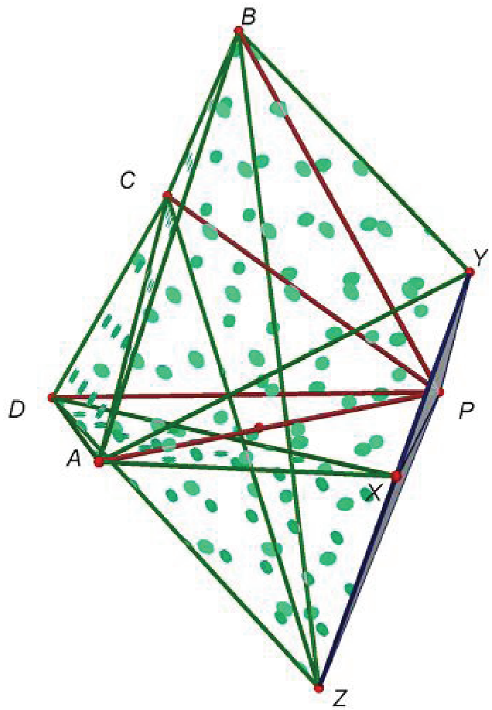
Intuitively, one might expect Guess is true, but in general it is wrong. It is true if $(tran(A+B) \cap A, a_q)\Big|_{q \in \{i,j,k\}}$ satisfy the condition of Lemma 1. Let us see the counterexample first.

Consider polyhedron A as below consists of

$$\{\triangle XYZ, \triangle ADX, \triangle DXZ, \triangle ADC, \triangle ACB, \triangle ABY, \triangle AYX, \triangle BYZ, \triangle BCZ\}.$$

It is a twisted version of a polyhedron that cannot be tetrahedralized given by Lennes (1911). Now, glue a tetrahedron (X, Y, Z, P) such that $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, $\overline{PD} \subset A+B$. We can do that by twisting polyhedron a little bit as picture shown.

By Lemma 1, there is a natural tetrahedralization of $A+B$ as picture shown. Now,



consider the operation $N_m : (\triangle adc, \triangle bdc) \to conv(a,\, b,\, c,\, d,\, x_1,\, x_2,\, \ldots,\, x_m)$ such that

$$surf(conv(a, b, c, d, x_1, x_2, \ldots, x_n)) = \{\triangle adc, \triangle bdc, \triangle x_i x_j c, \triangle x_i, x_j d\}$$

which $0 \le i, j \le m+1$ and $x_0 = a$, $x_{m+1} = b$. Now we use this operation on A to form a new convex polyhedron A' which is a desired counterexample. Let

$$A' = A \cup N_m(\triangle ABC, \triangle ACD) \cup N_m(\triangle ADX, \triangle DXZ) \cup N_m(\triangle AYX, \triangle BAY) \\ \cup N_m(\triangle ZBC, \triangle BZY)$$

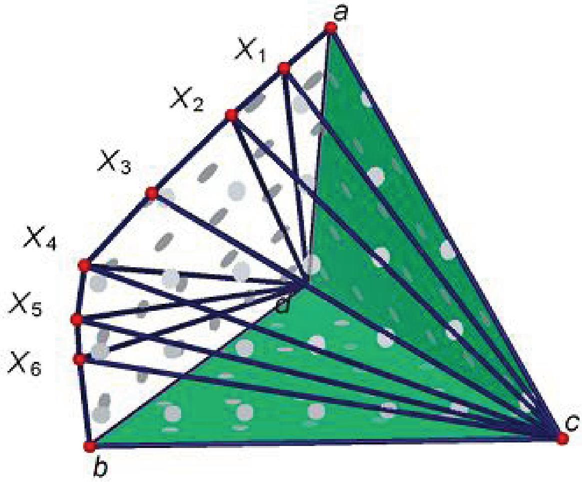
and twist A' a little bit such that it is a convex polyhedron. (We can always do that since in the operation of N_m, we can choose x_i that is extremely closed to $\overline{ab}$. Hence we can choose those new added point such that they are extremely closed to $conv(A, B, C, D, X, Y, Z)$) and let X_i close to one vertexes sufficiently enough such that (c, d, X, Y) which $Y \notin \{a, b, X_j\}$ will always have improper intersection with one of $\{\overline{AC}, \overline{AY}, \overline{DX}, \overline{BZ}\}$ (Above, I abuse the notation x_i, a, b, c, and d to every operation N_m. What I mean is that the outcomes of 4 operations should all satisfy the constraints I mentioned above.)

I claim that when m is big enough, $B \notin MT(A' + B)$ which $B = (X, Y, Z, P)$. First, there is a trivial tetrahedralization T such that $B \notin T$ whose inner edge set is

$$\{\overline{AC}, \overline{AY}, \overline{DX}, \overline{BZ}, \overline{PA}, \overline{PB}, \overline{PC}, \overline{PD}\}.$$

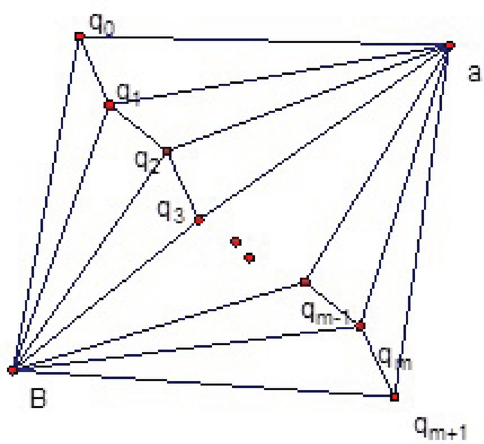
On the other hand, I will claim that for any tetrahedralization R that contains B, its cardinality of its inner edge set will be more than m. Hence, when $m \ge 9$, the tetrahedralization containing B cannot be minimum tetrahedralization.

First, this tetrahedralization should contain $\{\overline{AC}, \overline{AY}, \overline{DX}, \overline{BZ}\}$, otherwise by lemma proved by Below et al. (2000), we can get the cardinality of its inner edge is more than m.



If one of $\{\triangle ADX,\ \triangle DXZ,\ \triangle ADC,\ \triangle ACB,\ \triangle ABY,\ \triangle AYX,\ \triangle BYZ,\ \triangle BCZ\}$ doesn't belong to minimum tetrahedralization. WLOG, assume it is $\triangle ADX$. Consider the tetrahedron that contains $\overline{DX}$ and transverse two side of $\triangle ADX$. This tetrahedron will have improper intersection with one of $\{\overline{AC}, \overline{AY}, \overline{DX}, \overline{BZ}\}$ by construction. Now, if this tetrahedralization contains $\{\triangle XYZ, \triangle ADX, \triangle DXZ, \triangle ADC, \triangle ACB, \triangle ABY, \triangle AYX, \triangle BYZ, \triangle BCZ\}$, they form a tetrahedron A that cannot be tetrahedralized. We get a contradiction, hence for any tetrahedralization R that contains B, its cardinality of its inner edge set must be more than m as we desired.

Lemma of Below at el.(2000) (The proof of this lemma is in the attachment)
"Let P be a convex 3-polytope on n vertices, that contains the following collection of



triangular facets: $[a, q_i, q_{i+1}]$ and $[b, q_i, q_{i+1}]$ for $i = 0, 1, \ldots, m$ (see below), with the additional restrictions that $conv(a, b)$ does not intersect $conv(q_0, q_1, \ldots, q_{m+1})$. Then for each triangulation of P that does not use the edge $\overline{ab}$, the number of interior edges is at least m."

Remark. *The problem of this counterexample occurs due to the fact that* $(S,i)\Big|_{i\in\{x,y,z\}}$ *doesn't satisfy the condition of Lemma 1 where* $S = tran(A'+B)\cap A'$.

Theorem 1. *Given a* $conv(a_1,a_2,\ldots,a_n)$ *whose size is* m. *Consider* $conv(a_1,\ a_2,\ \ldots,\ a_n,\ x)$, *such that* $conv(a_1,a_2,\ldots,a_n,x)/conv(a_1,a_2,\ldots,a_n)$ *is* $(a_i,\ a_j,\ a_k,\ x)$ *and there exist a point* z *from* $\{a_i,a_j,a_k\}$ *such that for any* $Y = Pla(a_w,a_e,a_r)$, x *and* z *are at the same side of* Y *in general sense. Prove that* $(a_i,a_j,a_k,x)\in MT(conv(a_1,a_2,\ldots,a_n,x))$, *which implies that the size of* $conv(a_1,a_2,\ldots,a_n,x)$ *is* $m+1$. *For convenience, denote* $conv(a_1,a_2,\ldots,a_n)$ *as* A, (a_i,a_j,a_k,x) *as* B *and* $conv(a_1,a_2,\ldots,a_n,x)$ *as* $A+B$.

Proof. The bold face statement assures that $(tran(A+B),z)$ satisfy the condition of Lemma 1 by Lemma 2 and construction of $tran(A+B)$.

If $(a_i,a_j,a_k,x)\notin MT(conv(a_1,a_2,\ldots,a_n,x))$, we can apply Lemma 3. There exists a simple $tran(A+B)$. So it is trivial to see that the number of inner edges of $tran(A+B)$ is at least the number of vertexes minus 4 by Lemma 1. Let this number be Q. By Corollary 1, $(Q+4)-1-4\geq Q$ (left hand side is the upper bound of inner edge when (a_i,a_j,a_k,x) belongs to a tetrahedron), a contradiction. Hence, $(a_i,a_j,a_k,x)\in MT(conv(a_1,\ a_2,\ \ldots,\ a_n,\ x))$ as desired. □

Now, by the same way, consider following statement.

Theorem 2. *Given a* $conv(a_1,a_2,\ldots,a_n)$ *whose size is* S_A. *Consider* $conv(a_1,\ a_2,\ \ldots,\ a_n,\ b_1,\ b_2,\ \ldots,\ b_m)$, *such that* $conv(a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_m)/conv(a_1,\ a_2,\ \ldots,\ a_n)$ *is* $conv(a_i,a_j,a_k,b_1,b_2,\ldots,b_m)$ *whose size is* S_B *and there exist a point* z *from* $\{a_i,a_j,a_k\}$ *such that for any* $Y = Pla(a_w,a_e,a_r)$, b_h *and* z *are at the same side of* Y *in general sense and there exist a point* y *from* $\{a_i,a_j,a_k\}$ *such that for any* $Z = Pla(b_t,b_y,b_u)$, a_s *and* z *are at the same side of* Z *in general sense. Then* (a_i,a_j,a_k) *is cut in minimum tetrahedralization which implies that the size of* $conv(a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_m)$ *is* S_A+S_B. *For convenience, denote* $conv(a_1,a_2,\ldots,a_n)$ *as* A, $conv(a_i,a_j,a_k,b_1,b_2,\ldots,b_m)$ *as* B *and* $conv(a_1,a_2,\ldots,a_n,x)$ *as* $A+B$.

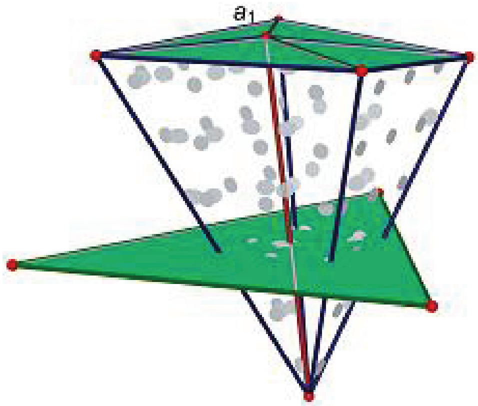
Before proving Theorem 2, let us analyze it first. We can use same way as Theorem 1 which only deals with $tran(A+B)$. However, there are some subtleties; Lemma 2 cannot precisely apply here.

However, if we scrutinize the proof of Lemma 2, we can see that what it assures is that there is no improper intersection of those faces and segments (there is no point from outside to inside). So, the upper part of polyhedron and the lower part still both satisfy the condition of Lemma 1 by construction of $tran(A+B)$, the bold face statement, and the remark in Lemma 1. Furthermore, observe that if we can prove that the inner edge in $tran(A+B)$ cannot be smaller than the number of all vertexes minus 3, denote it as Q. We can easily get contradiction by Corollary 1, $(Q+3+3)-2-6\geq Q$.

Lemma 4. *Consider* $tran(A+B)$ *and relabel the vertexes* a_i, a_j, a_k *as* x, y, z, $\{a_d\}$ *as* α_1, α_2, $\ldots$, α_u, $\{b_e\}$ *as* β_1, β_2, $\ldots$, β_r. *Prove that for any tetrahedralization of* $tran(A+B)$ *such that every tetrahedron intersects with both side, the cardinality of its inner edge set is larger or equal than* $u+r$ *when* $u,r\geq 2$.

Proof. First, observe that there are three kinds of tetrahedrons in $MT(tran(A+B))$,

$$\begin{aligned} &\text{three vertexes are belonged to } \{\alpha_i\} \text{ or } \{x,y,z\}, && (1)\\ \text{or}\quad &\text{two vertexes are belonged to } \{\alpha_i\} \text{ or } \{x,y,z\}, && (2)\\ \text{and}\quad &\text{three vertexes are belonged to } \beta_i \text{ or } \{x,y,z\}. && (3)\end{aligned}$$



Now, choosing any vertex α_i, consider the set consisting of every tetrahedron in $MT(tran(A+B))$ containing α_i, called this set as $adj(\alpha_i)$. If every tetrahedrons belonged to $adj(\alpha_i)$ is type(1) tetrahedron, we get one inner edge correspond to α_i(ex: in the picture, since $adj(\alpha_i)$ is all type(1) tetrahedrons, brown inner edge correspond to a_1.) Repeating this operation until it stops. (Do same thing with β_j and type(3) tetrahedron.)

Now, it is trivial to see that the vertexes we remain α_j, $adj(\alpha_i)$ contains at least one type(2) tetrahedron since finite points generating by type(3) cannot composite a segment (there is more subtleties caused by confusion between type(2) and type(3), see remark).

Same as β_j. Considering all type(2) tetrahedron, we can group them together by following way: beginning from any type(2) tetrahedron, if there is any other type(2) tetrahedron that has common vertexes with it, group them together. Keep doing until it stops. I claim that in each group, the number of inner edges is at least the number of vertexes.

First, we consider a single type(2) tetrahedron. There are at least four inner edge and exactly four vertexes. Then use induction. If a group have already consisted of n type(2) tetrahedrons satisfy our requirement, we want to prove it is still true when we have $n+1$ type(2) tetrahedrons.

When we have $n+1$ type(2) tetrahedrons, we have following 4 cases,

Case 1 There is only one common vertex between original group and new tetrahedron.
In this case, we have 3 additional vertexes and at least 4 additional inner edges.

Case 2 There are two common vertexes between original group and new tetrahedron that located on same side.
In this case, we have 2 additional vertexes and at least 4 additional inner edges.

Case 3 There are two common vertexes between original group and new tetrahedron that located on different side.
In this case, we have 2 additional vertexes and at least 3 additional inner edges.

Case 4 There are three common vertexes between original group and new tetrahedron.
In this case, we have 1 additional vertex and at least 2 additional inner edges.

Hence by induction, in each group, the number of inner edge is at least the number of vertexes. By summing up all possibility, we get what we want. □

Remark. *When u or $r=1$, $adj(\alpha_j)$ might only contains both type(1) and type(3) polyhedron (since some are hybrid of type 2 and type 3, those tetrahedrons can compose segments), so the way I manipulate here will be invalid. But when $u, r \geq 2$, since the edge* *where tetrahedrons that can simultaneously belong to type 2 and 3 is only three and they don't share a common vertex, the awkward phenomena in previous case won't occur.*

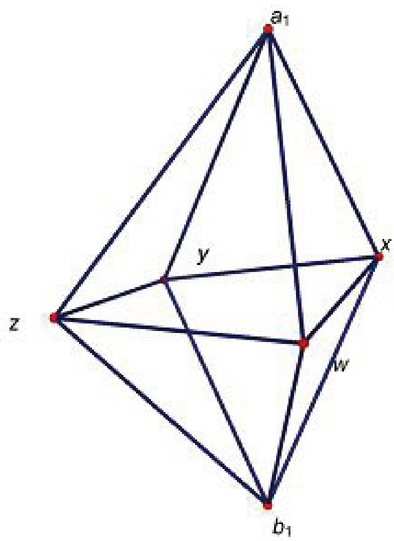
Now I am going to prove Theorem 2. Assume

$$\triangle a_i a_j a_k \notin MT\left(conv(a_1, a_2, \ldots, a_n, x)\right).$$

We can apply Lemma 3 to get $tran(A+B)$. If u or $r = 1$(as I defined in Lemma 4), by applying Theorem 1, we can immediately get contradiction. So now, we can apply Lemma 4 and Corollary 1. We also get a contradiction $(Q+3+3)-2-6 \geq Q$. Hence $\triangle a_i, a_j, a_k$ is cut in minimum tetrahedralization as desired.

Generalization

But notice that, the way I manipulate is invalid when the gluing face is not a single triangle since the contradicting inequality in Theorem 1 might become reasonable with same RHS and LHS. Hence, we cannot promise the way to cutting. There is an extremely easy counter example as RHS. If $conv(x, y, z, w, a_1, b_1)$ is $tran(A+B)$, then either $\{(x, z, y), (x, z, w)\}$ or $\{(y, w, x), (y, w, z)\}$ should belong to $MT(A+B)$ since the tetrahedralization formed by inner edge $\overline{a_1 b_1}$ is also minimum tetrahedralization.

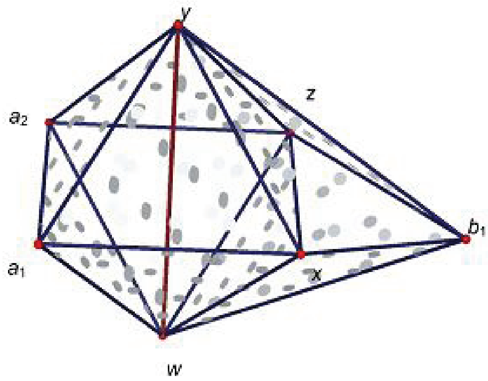


But there is indeed a weaker version of Theorem 1 and 2 when the gluing face is two triangles.

Theorem 3. *Given a $conv(a_1, a_2, \ldots, a_n, x, y, z, w)$ whose size is m.*
Consider $conv(a_1, a_2, \ldots, a_n, x, y, z, w, P)$, such that

$$conv(a_1, a_2, \ldots, a_n, x, y, z, w, P)/conv(a_1, a_2, \ldots, a_n, x, y, z, w)$$

is polyhedron(not necessarily convex) (x, y, z, w, P) and there exist 2 points t_1 from $\{x, z\}$ and t_2 from $\{y, w\}$ such that t_i, P are at the same side of $Pla(a_w, a_e, a_r)$. Consider 2 tetrahedralization, one is obvious tetrahedralization whose size is $m+2$. WLOG, assume in that case, the inner edge contain in $conv(x, y, z, w)$ is $\overline{xz}$. The other tetrahedralization is the smallest size of tetrahedralization which contains $\{(x, y, w), (z, y, w)\}$. Denote the size of this tetrahedrlization as M. Then, the size of $conv(a_1, a_2, \ldots, a_n, x, y, z, w, P)$ is $\min(m+2, M)$.



Why do I distinguish $m+2$ with M? Well, since they might not be equal. Now, consider following example as LHS which $conv(a_1, a_2, x, y, z, w)$ is regular octahedron whose size is 4, so in this case, $m+2=6$. However, the tetrahedranization formed by inner edge (y, w) having the smaller size M which is 5.

Theorem 4. *Given a $conv(a_1, a_2, \ldots, a_n, x, y, z, w)$ whose size is m and $conv(b_1, b_2, \ldots, b_n, x, y, z, w)$ whose size is n.*
Consider $conv(a_1, a_2, \ldots, a_n, x, y, z, w, b_1, b_2, \ldots, b_n)$, such that

$$conv(a_1, a_2, \ldots, a_n, x, y, z, w, b_1, b_2, \ldots, b_n)/conv(a_1, a_2, \ldots, a_n, x, y, z, w)$$

is polyhedron(not necessarily convex) $(b_1, b_2, \ldots, b_n, x, y, z, w)$ and there exists 2 points t_1 from $\{x, z\}$ and t_2 from $\{y, w\}$ such that t_i, b_y are at the same side of $Pla(a_w, a_e, a_r)$ and there exists 2 points w_1 from $\{x, z\}$ and w_2 from $\{y, w\}$ such that w_i, a_y are at the same side of $Pla(b_w, b_e, b_r)$. Consider 2 tetrahedralization, one is obvious tetrahedralization whose size is $m+n$. WLOG, assume in that case, the inner edge contain in $conv(x, y, z, w)$ is (x, z). The other tetrahedralization is the smallest size of tetrahedralization which contains $\{(x, y, w), (z, y, w)\}$. Denote the size of this tetrahedralization as M. Then, the size of $conv(a_1, a_2, \ldots, a_n, x, y, z, w, b_1, b_2, \ldots, b_n)$ is $\min(m+n, M)$.

For the gluing face is 5 or more-polygon, the way I manipulate won't work since the contradicting inequality in Theorem 1 can be true in this situation.

4 Conclusion

I have done the following things in this paper

1. Raise the notion of $tran(A+B)$ and assure that it is simple.
2. Use Lemma 4 to strengthen the properties of $tran(A+B)$ that is natural connection between the number of vertexes and the number of inner edges.
3. Give a counter example on "gluing a tetrahedron on an arbitrary convex polyhedron doesn't imply that the tetrahedron belongs to minimum tetrahedralization" and thus check it does counter the restricting condition I gave (In Chin et al's paper (2001), there is an alternative counterexample which includes Schonhardt polyhedron, twisted prism.)
4. Raise the sufficient condition of our desired statement.

In Chin et al's (2001) paper, their upper bound is tight if we only consider the computational structure. We call two polyhedrons is computational equivalent if the sets of all vertexes and edges of two polyhedron are homeomorphic while geometric equivalence requires an isometry between two polyhedrons.

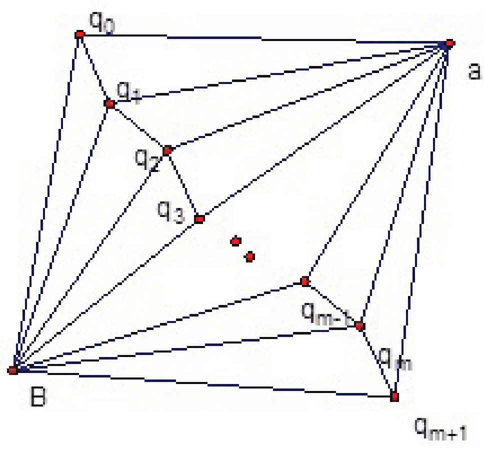
The biggest obstacle is that we cannot determine if the glued triangle belongs to minimum tetrahedralization or not, but if we only consider computational structure, we have no other choice but sever this triangle which will leading to linearly increasing error as examples posed in Below et al's paper shown(2000). So, few quiet interesting questions rises, how can we find a necessary and sufficient condition of cutting this triangle by consider polyhedron's geometry structure? If we find it, how much it can decrease the upper bound given by Chin et al? Also, what is the time complexity to check if all glued triangle should be cut or not?

If we use the logical polyhedron in Below et al's paper (2004), we can find out that if one of the glued triangles is cut, then the Boolean algebra it represents is unsatisfied. Yet, even all glued triangles are not cut; we cannot impetuously claim that the Boolean algebra is satisfied. If it is, then the complexity will be NP-Hard.

5 Appendix

For reader's convenience, here we reproduce Below et al.'s proof.

Lemma of Below et al. (2000) Let P be a convex 3-polytope on n vertices, that contains the followingcollection of triangular facets:
$[a, q_i, q_{i+1}]$ and $[b, q_i, q_{i+1}]$ for $i = 0, 1, \ldots, m$ (see Fig. below), with the additional restrictions that $conv(a, b)$ does not intersect $conv(q_0, q_1, \ldots, q_{m+1})$. Then for each triangulation of P that does not use the edge $\overline{ab}$, the number of interior edges is at least m.



Proof. Since $conv(q_0, q_{m+1})$ is in the interior of P, we obtain the following simple fact: for all $|i - j| \geq 2$, if q_iq_j is an edge of a triangulation, it will also be an interior edge. The proof of the lemma proceeds by induction on m. The lemma is clearly true for $m = 1$.

Call (*) the assumption that all vertices q_i, with $1 \leq i \leq m$, are incident to at least one interior edge of the triangulation T. We now show how to invoke induction in case (*) does not hold: A vertex q_i untouched by an interior edge belongs to the tetrahedra

$$\sigma_{i,a} = (a, q_{i-1}, q_i, q_{i+1}) \text{ and } \sigma_{i,b} = (b, q_{i-1}, q_i, q_{i+1}).$$

This is because the triangle (a, q_i, q_{i+1}) is in some simplex, and if the fourth point is some other vertex besides q_{i-1} or b we have an interior edge touching q_i. Furthermore, the fourth point cannot be b since in this case the edge ab would be present. By chopping off these two tetrahedra together with the vertex q_i (i.e., considering the convex hull of all of P's vertices except q_i) we can apply induction to guarantee that the remaining triangulation $T/\sigma_{i,a}, \sigma_{i,b}$ has at least $m-1$ interior edges. Together with the edge $q_{i-1}q_{i+1}$ they account for m interior edges in T. □

If (*) holds we will show the claim directly; we set up a one-to-one map from the set $\{q_1, \ldots, q_m\}$ to a subset of the interior edges that touch them: The vertices q_i come along a polygonal curve in a canonical order which is reflected by their indices.

We mark or orient the interior edges $q_i v$ that touch a vertex q_i as follows: If $v \notin \{q_0, \ldots, q_{m+1}\}$, we call the edge $q_i v$ special, otherwise we orient it from smaller to larger index.

For the vertices q_i with special edges incident to them, we map q_i to one of those. If a vertex q_i has no special edges, but has outgoing interior edges, we map it to the outgoing edge $q_i q_k$ with the smallest index k. We are left with the case of those vertices q_i that have only incoming interior edges incident to q_i.

Consider the triangle (a, q_i, q_{i+1}). It has to be in some tetrahedron of T whose fourth point is bound to be a q_{j_a} with $j_a < i$. Likewise (b, q_i, q_{i+1}) is in a tetrahedron with fourth point q_{j_b} with $j_b < i$. If both $j_a = j_b = i-1$, there can be no interior edges incident to q_i (see above), a contradiction to (*). Let j be any of j_a, j_b such that $j < i$. Map q_i to $q_j q_{i+1}$.

We claim that the given map is one-to-one. If some vertex q_i maps to the special edge $q_j v$, then necessarily $i = j$. There are potentially two vertices that can be mapped to an interior edge $q_j q_k$ with $j < k$: q_j when $q_j q_k$ is the chosen outgoing edge of q_j and q_{k-1}, in case q_{k-1} has only incoming edges.

In the latter case one of the tetrahedra (a, q_j, q_{k-1}, q_k) and (b, q_j, q_{k-1}, q_k) has to be in the triangulation, and q_j will be mapped to the smaller indexed edge $q_j q_{k-1}$. This is an interior edge since $j < k-2$, so q_j cannot also be mapped to $q_j q_k$. The injectivity of the map is proven.

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