A symmetry problem of elliptic differential operators in potential theory

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Abstract

This paper is a study of the equation $(-\Delta_E)^{\frac{\alpha}{2}}u(x) = f(x)$, where $(-\Delta_E)^{\frac{\alpha}{2}}$ is an (elliptic pseudo-differential) operator defined by

$$(-\Delta_E)^{-\frac{\alpha}{2}}f = \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty t^{\frac{\alpha}{2}-1}(H_t * f)(x)dt,$$

$$H_t(x) \equiv H(x,t) = \frac{1}{\sqrt{(4\pi t)^n \eta_1\eta_2\cdots\eta_n}}\exp\left(-\sum_i \frac{x_i^2}{4\eta_i t}\right),$$

where $\eta_1, \eta_2, \cdots, \eta_n$ are a set of non-negative numbers that specify the operator. Note that it is an extension of the fractional Laplacian operator $(-\Delta)^{\frac{\alpha}{2}}$.

In this paper, we construct a solution, noted as $J_\alpha f$, by

$$J_\alpha f(x) \equiv \frac{1}{\beta(\alpha)}\int_{\mathbb{R}^n} \frac{f(y)}{|\eta^{-1}\cdot(x-y)|^{n-a}}dy,$$

where $|\eta^{-1}\cdot(x-y)|$ is $\sqrt{\sum_i^n \eta_i^{-1}(x_i-y_i)}$, and $\beta(\alpha)^{-1}$ equals

$$\beta(\alpha)^{-1} = \frac{1}{\sqrt{\eta_1\eta_2\cdots\eta_n}}\cdot\frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}}2^\alpha\Gamma(\frac{\alpha}{2})}.$$

Then if we set $f = \chi_\Omega$ where χ_Ω is the indicator function and Ω is some bounded domain in $\mathbb{R}^n$, then for all bounded domain Ω that is invariant under reflection transformation P_m, namely $P_m\Omega = \Omega$ for all $m = 1,\ldots,n$, $J_\alpha f \equiv J_\alpha(x)$ satisfies

$$J_\alpha(x) = J_\alpha(P_m x).$$

The reflection transformation is defined as

$$P_m x = P_m(x_1,\cdots,x_m,\cdots,x_n) = (x_1,\cdots,-x_m,\cdots,x_n),$$

where $m = 1,2,\ldots,n$.

摘要: 在這篇報告中, 我們要探討一個方程式 $(-\Delta_E)^{\frac{\alpha}{2}}u = f$, 其中 $(-\Delta_E)^{\frac{\alpha}{2}}$ 是一個分數次的橢圓形微分算子, 其定義為

$$(-\Delta_E)^{-\frac{\alpha}{2}}f = \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty t^{\frac{\alpha}{2}-1}(H_t * f)(x)dt,$$

$$H_t(x) \equiv H(x,t) = \frac{1}{\sqrt{(4\pi t)^n \eta_1\eta_2\cdots\eta_n}} \exp\left(-\sum_i \frac{x_i^2}{4\eta_i t}\right),$$

其中 $\eta_1, \eta_2, \cdots, \eta_n$ 是一群決定其算子特性的參數. 而它是從一般的分數次拉普拉斯算子延伸而得到的.

在報告中, 我們也將找出其一個解, 記為 $J_\alpha f$, 為

$$J_\alpha f(x) \equiv \frac{1}{\beta(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|\eta \cdot (x-y)|^{n-a}} dy,$$

其中 $|\eta^{-1} \cdot (x-y)|$ 代表 $\sqrt{\sum_i^n \eta_i^{-1}(x_i - y_i)}$, 而 $\beta(\alpha)^{-1}$ 等於

$$\beta(\alpha)^{-1} = \frac{1}{\sqrt{\eta_1\eta_2\cdots\eta_n}} \cdot \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}.$$

如果在 $J_\alpha f$ 中令 $f = \chi_\Omega$, 其中 χ_Ω 是指示函數, 而 Ω 是一個在 $\mathbb{R}^n$ 中的有界區域, 則對於所有滿足鏡射變換 P_m 的 Ω, 更精確的說, 對於 $m = 1, \ldots, n$, 都有 $P_m\Omega = \Omega$, $J_\alpha f \equiv J(x)$ 滿足

$$J_\alpha(x) = J_\alpha(P_m x).$$

鏡射變換定義為

$$P_m x = P_m(x_1, \cdots, x_m, \cdots, x_n) = (x_1, \cdots, -x_m, \cdots, x_n)$$

其中 $m = 1, 2, \ldots, n$.

1 Introdution

The basic idea of this paper is derived from an important concept in potential theory, the Riesz potential $I_\alpha f$. It is known that Riesz potential is closely related to the fractional Laplacian operator. It is actually the inverse operator of $(-\Delta)^{\frac{\alpha}{2}}$, namely, $u(x) = I_\alpha f$ if $(-\Delta)^{\frac{\alpha}{2}} u = f$ [1]. Now we let $f \equiv \chi_\Omega$, where χ_Ω is the indicator function. Then this function denoted as $I_\alpha(x)$ in some bounded domain Ω has an interesting property. $I_\alpha(x)$ is radially symmetric to a center of a ball. In other words, $u(x)|_{\partial\Omega} = \text{const.}$ if and only if Ω is a ball [4].

In this paper, we will extend the fractional Laplacian to an elliptic operator

$$(-\Delta_E)^{\frac{\alpha}{2}} u = \left(-\sum_j^n \eta_j \frac{\partial^2}{\partial x_j^2}\right)^{\frac{\alpha}{2}} u,$$

where $\eta_1, \eta_2, \cdots, \eta_n > 0$ and they are independent of the variables. The fractional exponent will be defined in the article. We hope to achieve the following things in the paper:

1. Find the solution of $(-\Delta_E)^{\frac{\alpha}{2}} u = f$, which is denoted by $J_\alpha f(x)$. Then $u(x) = J_\alpha f$ if $(-\Delta_E)^{\frac{\alpha}{2}} u(x) = f(x)$.

2. Discuss the integrability of $J_\alpha f$.

3. Discuss the symmetry property of the solution of $(-\Delta_E)^{\frac{\alpha}{2}} u = \chi_\Omega$ where Ω is an n-dimensional ellipsoid centered at origin point and axis parallel to the axis $(x_1, x_2, \cdots, x_n)$ of some cartesian coordinate system.

4. Consider symmetry property of the solution of another equation $(-\Delta_E)^{\frac{\alpha}{2}} u = \chi_\Omega x_i$, where $i = 1, 2, \cdots, n$. (The antisymmetric property)

But before doing all this, we will first define some concepts.

1.1 Fractional Laplacian

Now we turn to an important concept of this paper: the fractional Laplacian operator $(-\Delta)^{-\frac{\alpha}{2}}$. Only the fractional exponent of a positive definite operator can be defined, so we need to take a minus sign in front of the ordinary Laplacian Δ.

One way to define $(-\Delta)^{-\frac{\alpha}{2}}$ is to use the Gamma function $\Gamma(\alpha)$. We can start from the fact that for any number A [1, 3]:

$$A^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tA} dt. \tag{1}$$

If we exchange A to a Laplacian, $A \mapsto -\Delta, s \to \frac{\alpha}{2}$, then we get the definition.

Definition 1. The fractional Laplacian $(-\Delta)^{-\frac{\alpha}{2}}$ is defined by

$$(-\Delta)^{-\frac{\alpha}{2}} f = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}-1} e^{t\Delta} f dt, \tag{2}$$

where

$$e^{\Delta t} f(x) = G_t * f(x) = \int_{\mathbb{R}^n} G_t(x-y) f(y) dy \tag{3}$$

and

$$G(x,t) = G_t(x) = (4\pi t)^{-n} \exp(-\frac{|x|^2}{4t}) \geq 0. \tag{4}$$

$G_t(x)$ is called the Gauss-Weierstrass kernel [1]. It is the fundamental solution of heat equation, and it is not difficult to see why we use it to define $e^{t\Delta}$

$$\frac{\partial G_t(x)}{\partial t} = \Delta G_t(x) \Longleftrightarrow G_t(x) = e^{\Delta t}, \quad t > 0. \tag{5}$$

However, there is a problem in this definition. When $\alpha = -2n$, where n is a positive integer, then the $\frac{1}{\Gamma(\frac{\alpha}{2})} = \frac{1}{\Gamma(-n)}$ part will be zero, and the integral part diverges. We fix this problem by taking the limit

$$\lim_{\alpha \to 2n} \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}-1} e^{tA} f dt \tag{6}$$

where A could be any number, and we find this limit to be A^n by using the equation

$$\frac{\Gamma(s+1)}{A^{-s+1}} = \int_0^\infty t^s e^{-At} dt.$$

So it is reasonable to redefine the fractional Laplacian by taking limits in the definition of it. Now we can define the fractional Laplacian with a positive integer exponent by

$$(-\Delta)^n = \lim_{\alpha \to 2n} (-\Delta)^{-\frac{\alpha}{2}}. \tag{7}$$

1.2 Riesz potential

Riesz potential is closely related to the fractional Laplacian, for it can be seen as an inverse of the fractional Laplacian [1].

Definition 2. For any $n \geq 2$, $0 < \alpha < n$, and $x \in \mathbb{R}^n$ the Riesz potential is

$$I_\alpha f(x) = (K_\alpha * f)(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-a}} dy, \tag{8}$$

where $\gamma(\alpha)$ is

$$\gamma(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}$$

and

$$K_\alpha = \frac{1}{\gamma(\alpha)} |x|^{\alpha - n} \tag{9}$$

is called the Risz kernel.

We are going to focus on Riesz potential in a compact domain Ω or

$$\frac{1}{\gamma(\alpha)} \int_\Omega \frac{f(y)}{|x-y|^{n-a}} dy = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-a}} \chi_\Omega dy, \tag{10}$$

where χ_Ω is the indicator function. The Riesz potential is a singular integral operator, so the concept of integrability is important. In other words, the question will be for $f \in L^p(\Omega)$, and $I_\alpha f \in L^q(\Omega)$, that p, q satisfy some condition which makes $I_\alpha : L^p(\Omega) \to L^q(\Omega)$ a bounded operator.

This property can be seen by the Hardy-Littlewood-Sobolev inequality [2]:

Theorem 1. For $0 < \alpha < n$, $1 \leq p, q \leq \infty$, $I_\alpha : L^p(\Omega) \to L^q(\Omega)$ is a bounded operator:

$$\|I_\alpha f\|_q \leq C\|f\|_p, \quad \text{if} \quad \frac{n}{p} \leq \frac{n}{q} + \alpha. \tag{11}$$

Proof. See [2]. This theorem says that if $f \in L^p(\Omega)$, then for $x \in \Omega$, $I_\alpha f(x)$ converges absolutely. □

We are going to see the relationship between fractional Laplacian and the Riesz potential.

Theorem 2. For any $0 < \alpha < n$, if $u(x)$ satisfies the equation

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \rho(x), \quad x \in \mathbb{R}^n, \tag{12}$$

then $u(\mathbf{x})$ can be written as the convolution of K_α and f:

$$u(x) = I_\alpha \rho(x) = (K_\alpha * \rho)(x). \tag{13}$$

Proof. The proof is standard [1]. For convenience, we will recall it in the appendix. □

2 Derive J_α

2.1 Extending the fractional Laplacian

Before extending the fractional Laplacian, we will start by looking at the normal Lalpcian first:

$$\Delta \equiv \sum_i \frac{\partial^2}{\partial x_i^2}. \tag{14}$$

We will extend this to

$$-\Delta_E = -\sum_i \eta_i \frac{\partial^2}{\partial \xi_i^2}, \quad \text{where } (\eta_1, \eta_2, \cdots \eta_n > 0), \tag{15}$$

because it is positive definite, $\eta_1, \eta_2, \eta_3 \cdots > 0$. For the specified case $\eta_1 = \eta_2 = \cdots = \eta_n = 1$, it reduces to the ordinary Laplacian.

The question is how to define this operator with a fractional exponent $(-\Delta_E)^{\frac{\alpha}{2}}$. We can do the same as the original fractional Laplacian:

Definition 3. The fractional exponent for the elliptical operator can be written as

$$(-\Delta_E f)^{-\frac{\alpha}{2}} = \frac{1}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}-1} e^{\Delta_E t} f dt, \tag{16}$$

where $e^{\Delta_E t} f = (H * f)(\xi)$, $e^{\Delta_E t}\delta(\xi) \equiv H(t, \xi)$ is the fundamental solution for $\partial_t u = \Delta_E u$, and

$$H(\xi, t) = \frac{1}{\sqrt{(4\pi t)^n \eta_1 \eta_2 \cdots \eta_n}} \exp\left(-\sum_i \frac{\xi_i^2}{4\eta_i t}\right) \geq 0. \tag{17}$$

(17) can be easily calculated,

$$\frac{\partial H(\xi, t)}{\partial t} - \Delta_E H(\xi, t) = 0 \; (t > 0, \; \lim_{t \to 0} = \delta(x)) \tag{18}$$

and we apply (18) to the Fourier transformation

$$\frac{\partial \widehat{H}(\xi,t)}{\partial t}+\left(\sum_i \eta_i\xi_i^2\right)\widehat{H}(\xi,t)=0, \tag{19}$$

$$\widehat{H}(\xi,t)=\exp\left(-\sum_i \eta_i\xi_i^2\right)t=0, \tag{20}$$

$$H(\xi,t)=\prod_i \frac{1}{2\sqrt{\pi t\eta_i}}\exp\left(-\frac{\xi_i^2}{4t\eta_i}\right)=\frac{1}{\sqrt{(4\pi t)^n\eta_1\eta_2\cdots\eta_n}}\exp\left(-\sum_i\frac{\xi_i^2}{4\eta_i t}\right). \tag{21}$$

2.2 The solution for Δ_E

With all these definitions, we can start to derive the solution for fractional elliptic operator associated to Δ_E.

Theorem 3. The solution for fractional elliptic operator

$$(-\Delta_E)^{\frac{\alpha}{2}}u(x)=\rho(x) \tag{22}$$

can be taken as $u(x)=J_\alpha\rho(x)$, where

$$J_\alpha u(x)\equiv\frac{1}{\beta(\alpha)}\int_{\mathbb{R}^n}\frac{f(y)}{|\eta^{-1}\cdot(x-y)|^{n-a}}dy, \tag{23}$$

and $|\eta^{-1}(x-y)|$ stands for $\sqrt{\sum_i^n \eta_i^{-1}(x_i-y_i)}$, and $\beta(\alpha)^{-1}$ equals

$$\beta(\alpha)^{-1}=\frac{1}{\sqrt{\eta_1\eta_2\cdots\eta_n}}\cdot\frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}}2^\alpha\Gamma(\frac{\alpha}{2})}. \tag{24}$$

Proof. This theorem can be proved by some simple transformation of the variables.

For the equation

$$(-\Delta_E)^{\frac{\alpha}{2}}u(x)=\rho(x), \tag{25}$$

consider a transformation:

$$x_i\mapsto\frac{\xi_i}{\sqrt{\eta_i}}. \tag{26}$$

Then (25) is transformed to

$$(-\Delta)^{\frac{\alpha}{2}}\tilde{u}(\xi)=\tilde{\rho}(\xi). \tag{27}$$

This is just the ordinary fractional Laplacian, so its solution is just the Riesz potential:

$$\tilde{u}(\xi)=\frac{1}{\gamma(\alpha)}\int_{\mathbb{R}^n}\frac{\tilde{\rho}(\xi)}{|\xi-\zeta|^{n-a}}d\zeta, \tag{28}$$

and we can transform it back to x_i variable, so the solution will be

$$u(x)=\frac{1}{\sqrt{\eta_1\eta_2\cdots\eta_n}}\cdot\frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}}2^\alpha\Gamma(\frac{\alpha}{2})}\int_{\mathbb{R}^n}\frac{f(y)}{|\eta^{-1}\cdot(x-y)|^{n-a}}dy. \tag{29}$$

□

It is also easy to define the solution for some compact domain Ω; simply set

$$J_\alpha f \equiv \frac{1}{\beta(\alpha)} \int_\Omega \frac{f(y)}{|\eta^{-1}(x-y)|^{n-\alpha}} dy = \frac{1}{\beta(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)\chi_\Omega}{|\eta^{-1}(x-y)|^{n-\alpha}} dy \tag{30}$$

where χ_Ω is the indicator function.

3 Integrability

We have to discuss the integrability of $J_\alpha f$. Because $J_\alpha f$ can be turned to $I_\alpha f$ by changing variables, they should satisfy the same inequality. This has been proven to be true, so we can apply everything in the same way.

Theorem 4. Let $0 \leq q \leq \infty$, $0 < \alpha < n$. Then $J_\alpha : L^p(\Omega) \to L^q(\Omega)$ is an continuous operator

$$\|J_\alpha f\|_{L^q(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \quad \text{for any } \frac{1}{p} \leq \frac{1}{q} + \frac{\alpha}{n}. \tag{31}$$

Proof. Before proofing this theorem we need some lemmas.

Lemma 1. If a function $f(x)$ depends only on $|\eta^{-1}x| \equiv r$ (where the norm stands for $(\sum_i^n \eta_i^{-1} x_i)^{1/2}$), then we have the integral equality

$$\int_{\mathbb{R}^n} f(x)dx = \omega_n \int_0^\infty f(r) r^{n-1} dr, \tag{32}$$

where ω_n is

$$\omega_n = \sqrt{\eta_1 \cdots \eta_n} \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{33}$$

Proof. We can start from the fact that [3]

$$\begin{aligned} &\int_{\mathbb{R}^n_+} f(x_1^{b_1} + x_2^{b_2} + \cdots + x_n^{b_n}) x_1^{a_1-1} x_1^{a_2-1} \cdots x_n^{a_n-1} dx \\ &= \frac{\Gamma(\frac{a_1}{b_1})\Gamma(\frac{a_2}{b_2}) \cdots \Gamma(\frac{a_n}{b_n})}{b_1 \cdots b_n \Gamma(\frac{a_1}{b_1} + \frac{a_2}{b_2} \cdots + \frac{a_n}{b_n})} \int_0^\infty f(t) t^{\frac{a_1}{b_1} + \frac{a_2}{b_2} \cdots + \frac{a_n}{b_n} - 1} dt. \end{aligned} \tag{34}$$

$\mathbb{R}^n_+$ is defined as

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_1, \cdots, x_n > 0\}. \tag{35}$$

By setting $b_1 = b_2 = \cdots = b_n = 2$, and $a_1 = a_2 = \cdots = a_n = 1$, and a transformation,

$$x_i \mapsto \sqrt{\eta_i} x_i, \quad i = 1, 2, \cdots n, \tag{36}$$

we get

$$\int_{\mathbb{R}^n_+} f(|\eta^{-1}x|^2) dx = \sqrt{\eta_1 \cdots \eta_n} \frac{\pi^{n/2}}{2^n \Gamma(\frac{n}{2})} \int_0^\infty f(t) t^{\frac{n}{2}-1} dt. \tag{37}$$

Last, consider a change of variable $t = r^2$:

$$\int_{\mathbb{R}^n_+} f(|\eta^{-1}x|)dx = \sqrt{\eta_1 \cdots \eta_n} \frac{\pi^{n/2}}{2^n \Gamma(\frac{n}{2}+1)} \int_0^\infty f(r) r^{n-1} dr. \tag{38}$$

By the symmetry of $f(|\eta^{-1}x|)$, it is easy to check

$$2^n \int_{\mathbb{R}^n_+} f(|\eta^{-1}x|)dx = \int_{\mathbb{R}^n} f(|\eta^{-1}x|)dx, \tag{39}$$

then the lemma is proven. □

Lemma 2. For some $1 \le p, q, r \le \infty$, if they satisfy

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}, \tag{40}$$

then

$$\beta(\alpha) \| J_\alpha f \|_r \le \| f \|_p \| h \|_q, \tag{41}$$

where

$$h(x,y) \equiv h(\eta^{-1}(x-y)) = \frac{1}{|\eta^{-1}(x-y)|^{n-\alpha}}. \tag{42}$$

Proof. First, we set

$$\begin{aligned} |J_\alpha f| &= \frac{1}{\beta(\alpha)} \left| \int_{\mathbb{R}^n} f(y) h(x,y) dy \right| \le \frac{1}{\beta(\alpha)} \int_{\mathbb{R}^n} |h(x,y) f(y)| dy \\ &= \frac{1}{\beta(\alpha)} \int_{\mathbb{R}^n} |f(y)|^{\frac{p}{r}} |f(y)|^{1-\frac{p}{r}} |h(x,y)|^{\frac{q}{r}} |h(x,y)|^{1-\frac{q}{r}} dy. \end{aligned} \tag{43}$$

We can see that,

$$\frac{1}{r} + \left(\frac{1}{p} - \frac{1}{r}\right) + \left(\frac{1}{q} - \frac{1}{r}\right) = \frac{1}{r} + \frac{1}{pr/(p-r)} + \frac{1}{qr/(q-r)} = 1. \tag{44}$$

Then we can apply the Hölder inequality to it:

$$\begin{aligned} \beta(\alpha)|J_\alpha f| &\le \left(\int_{\mathbb{R}^n} |f(y)|^p |h(x,y)|^q dy \right)^{\frac{1}{r}} \cdot \left(\int_{\mathbb{R}^n} |f(y)|^p dy \right)^{\frac{1}{p}-\frac{1}{r}} \\ &\quad \cdot \left(\int_{\mathbb{R}^n} |h(x,y)|^q dy \right)^{\frac{1}{q}-\frac{1}{r}}. \end{aligned} \tag{45}$$

Take both sides to an exponent r, then integrate it by x, and we get

$$\beta(\alpha)^r \| J_\alpha f \|_r^r \le \left(\int_{\mathbb{R}^n} |f|^p dy \right) \left(\int_{\mathbb{R}^{2n}} |h|^q dx dy \right) \| f \|_p^{r-p} \| h \|_q^{r-q} = \| f \|_p^r \| h \|_q^r \tag{46}$$

and the lemma is proven. □

Lemma 3. For $n \geq 2, 0 < \alpha < n$, one has

$$\int_\Omega \frac{1}{|\eta^{-1}(x-y)|^\alpha} dy \leq \frac{n|E_n|}{n-\alpha} \left(\frac{|\Omega|}{|E_n|} \right)^{1-\frac{\alpha}{n}}, \tag{47}$$

where E_n is

$$E_n = \sqrt{\eta_1 \cdots \eta_n} \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}. \tag{48}$$

It is the volume of a n-dimensional ellipsoid with axes $\sqrt{\eta_1}, \sqrt{\eta_2}, \cdots, \sqrt{\eta_n}$.

Proof. First, we set $S \in \mathbb{R}^n$ an n-dimensional ellipsoid centered at x with axes $\sqrt{\eta_1}R$, $\sqrt{\eta_2}R, \cdots, \sqrt{\eta_n}R$ and each parallel to the axis $x_1, x_2, \cdots, x_n$ of coordinate, and $|S| = E_n R^n$ is the volume of S. Then we set $|\Omega| = |S|$, so that $R = (|\Omega|/|S|)^{1/n}$.

$$\int_S \frac{dy}{|\eta^{-1}(x-y)|^\alpha} = \int_{S\cap\Omega} \frac{dy}{|\eta^{-1}(x-y)|^\alpha} + \int_{S-(S\cap\Omega)} \frac{dy}{|\eta^{-1}(x-y)|^\alpha} \tag{49}$$

and

$$\int_\Omega \frac{dy}{|\eta^{-1}(x-y)|^\alpha} = \int_{S\cap\Omega} \frac{dy}{|\eta^{-1}(x-y)|^\alpha} + \int_{\Omega-(S\cap\Omega)} \frac{dy}{|\eta^{-1}(x-y)|^\alpha}. \tag{50}$$

Because $S - (S \cap \Omega)$ is inside S, $\frac{1}{|\eta^{-1}(x-y)|^\alpha} \leq R^{-\alpha}$; therefore

$$\int_{\Omega-(S\cap\Omega)} \frac{dy}{|\eta^{-1}(x-y)|^\alpha} \leq R^{-\alpha}(|\Omega| - |S\cap\Omega|). \tag{51}$$

Similarly, $\Omega - (S \cap \Omega)$ is outside S, so that

$$\int_{S-(S\cap\Omega)} \frac{dy}{|\eta^{-1}(x-y)|^\alpha} \geq R^{-\alpha}(|S| - |S\cap\Omega|) = R^{-\alpha}(|\Omega| - |S\cap\Omega|), \tag{52}$$

thus we get

$$\int_{S-(S\cap\Omega)} \frac{dy}{|\eta^{-1}(x-y)|^\alpha} \geq \int_{\Omega-(S\cap\Omega)} \frac{dy}{|\eta^{-1}(x-y)|^\alpha}, \tag{53}$$

or

$$\begin{aligned} \int_\Omega \frac{dy}{|\eta^{-1}(x-y)|^\alpha} &\leq \int_S \frac{dy}{|\eta^{-1}(x-y)|^\alpha} = \int_0^R r^{-\alpha} r^{n-1} |nE_n| dr \\ &= \frac{R^{n-\alpha}}{n-\alpha} nE_n = \frac{n|E_n|}{n-\alpha} \left(\frac{|\Omega|}{|E_n|} \right)^{1-\frac{\alpha}{n}}. \end{aligned} \tag{54}$$

Replace $\chi_\Omega f(x)$ by $f(x)$ in lemma one in (54), and the lemma is proven. □

The rest of the proof is obvious. For some r we can let

$$\frac{1}{r} + \frac{1}{p} = 1 + \frac{1}{q} \tag{55}$$

and if $1 \leq r \leq 1/(1-n/\alpha)$ then $n/p \leq n/q + n$ is satisfied. By Lemma 2

$$\|J_\alpha f\| \leq \beta(\alpha)^{-1}\|h\|_r\|f\|_p. \tag{56}$$

Note that we have replaced $f(x)\chi_\Omega$ by $f(x)$, and $h(x,y)\chi_\Omega$ by $h(x,y)$ because (57) only integrates over a bounded domain.

Then by Lemma 3

$$\|h\|_r \leq \left(\frac{nE_n}{n-(n-\alpha)r}\right)^{\frac{1}{r}} \left(\frac{|\Omega|}{|E_n|}\right)^{\frac{1}{r}+\frac{\alpha}{n}-1}. \tag{57}$$

So the theorem is proven. □

4 The symmetry problem

We know that for I_α the solution of $(-\Delta)^{\alpha/2}u = \chi_\Omega$ has some very interesting property, such as its the volume on $\partial\Omega$ is a constant if any only if Ω is a ball [4].

$(-\Delta_E)^{\frac{\alpha}{2}}u = \chi_\Omega$ is invariant under some "elliptical rotation" that preserves $|\eta^{-1}x|$, just like the $(-\Delta)^{\alpha/2}u = \chi_\Omega$ is invariant under rotations that preserve $|x|$, but the same property cannot carry over; that is, $J_\alpha(x)|_\Omega$ will not be a constant where Ω is the ellipsoid with axis parallel to the coordinate. It is because not all the transformation that preserves $|\eta^{-1}x|$ preserves an infinitesimal volume dV in $\mathbb{R}^n$, (if we see this transformation as a coordinate transformation, then it means the Jacobian does not equal one) [3], and therefore $J_\alpha(x)|_{\partial\Omega}$ does not satisfy this property.

There is a transformation that preserves $|\eta^{-1}x|$ and infinitesimal volume. It is the reflection transformation P_m (See Definition 4). It is a discrete transformation, so instead of $J_\alpha(x)|_{\partial\Omega} = \text{const}$, we will get $J_\alpha(x)|_{\partial\Omega} = J_\alpha(P_m x)|_{\partial\Omega}$. (See Theorem 5.)

Definition 4. We are going to introduce the reflection transformation $P_m : \mathbb{R}^n \to \mathbb{R}^n$.

$$P_m x = P_m(x_1, \cdots, x_m, \cdots, x_n) = (x_1, \cdots, -x_m, \cdots x_n), \tag{58}$$

where $m = 1, 2, \ldots, n$.

For an n-dimensional ellipsoid with axis $\sqrt{\eta_1}, \sqrt{\eta_2}, \cdots, \sqrt{\eta_n}$, and each parallel to the axis of the coordinate $(x_1, x_2, \cdots, x_n)$, which will be noted as Ω is symmetric under reflection transformation. That is, for a $x \in \Omega$, then $P_m x \in \Omega$, and for some $x \in \partial\Omega$, then $P_m x \in \partial\Omega$.

Theorem 5. Let

$$u(x) \equiv \frac{1}{\beta(\alpha)} \int_\Omega \frac{dy}{|\eta^{-1}(x-y)|^{n-\alpha}} = J_\alpha, \tag{59}$$

then there is a property of $u(x)|_{\partial\Omega}$.

$$u(x)|_{\partial\Omega} = u(P_m x)|_{\partial\Omega}, \text{ for all } n = 1, 2, \cdots, n. \tag{60}$$

For convenience, we need to use a different kind of coordinate instead of the ordinary Cartesian coordinate.

Definition 5. We are going to define an elliptical coordinate $(\rho, \phi_1 \cdots \phi_{n-1})$ with the center at some point p.

$$\begin{aligned}
x_1 - p_1 &= \sqrt{\eta_1}\rho\cos\phi_1 \\
x_2 - p_2 &= \sqrt{\eta_2}\rho\sin\phi_1\cos\phi_2 \\
&\vdots \\
x_{n-1} - p_{n-1} &= \sqrt{\eta_{n-1}}\rho\sin\phi_1\cdots\sin\phi_{n-2}\cos\phi_{n-1} \\
x_n - p_n &= \sqrt{\eta_n}\rho\sin\phi_1\cdots\sin\phi_{n-2}\sin\phi_{n-1}
\end{aligned}$$

We can set that $p \in \partial\Omega$.

Then another coordinate $(r, \theta_1, \cdots \theta_{n-1})$ at a point p'.

$$\begin{aligned}
x_1 - p'_1 &= \sqrt{\eta_1}r\cos\theta_1 \\
x_2 - p'_2 &= \sqrt{\eta_2 r\sin\theta_1}\cos\theta_2 \\
&\vdots \\
x_{n-1} - p'_{n-1} &= \sqrt{\eta_{n-1}}r\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1} \\
x_n - p'_n &= \sqrt{\eta_n}r\sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1}
\end{aligned}$$

Then we can set that $p' = P_m p \in \partial\Omega$.

With this coordinate, we shall define a subset in $\mathbb{R}^n$ by

$$\tau_l^{(k)} = \{x \in \Omega \mid 2\frac{l-1}{k} \le \rho < 2\frac{l}{k}\},\ 1 \le l \le k. \tag{61}$$

It is easy to check out that

$$\bigcup_{1\le l\le k} \tau_l^{(k)} = \Omega. \tag{62}$$

We will do the same to coordinate $(r, \theta_1 \cdots \theta_{n-1})$.

$$\tau_l'^{(k)} = \{x \in \Omega \mid 2\frac{l-1}{k} \le r < 2\frac{l}{k}\},\ 1 \le l \le k \tag{63}$$

and

$$\bigcup_{1\le l\le k} \tau_l'^{(k)} = \Omega. \tag{64}$$

Lemma 4. For any k and any $1 \le l \le k$, it satisfies

$$|\tau_l^{(k)}| = |\tau_l'^{(k)}|. \tag{65}$$

Proof. For any $x \in \tau_l^{(k)}$ it satisfies the condition

$$\begin{aligned}\left(2\frac{l-1}{k}\right)^2 \le \frac{(x_1-p_1)^2}{\eta_1} + \frac{(x_2-p_2)^2}{\eta_2} + \cdots \frac{(x_m-p_m)^2}{\eta_m} \\ + \cdots + \frac{(x_{n-1}-p_{n-1})^2}{\eta_{n-1}} + \frac{(x_n-p_n)^2}{\eta_n} \le \left(2\frac{l}{k}\right)^2\end{aligned} \tag{66}$$

and

$$x \in \Omega. \tag{67}$$

If we transform any $x \in \tau_l^{(k)}$ with the reflection transformation P_m , then $P_m x \equiv x'$ satisfies

$$\begin{aligned}\left(2\frac{l-1}{k}\right)^2 \le \frac{(P_m x_1-p_1)^2}{\eta_1} + \frac{(P_m x_2-p_2)^2}{\eta_2} + \cdots \frac{(P_m x_m-p_m)^2}{\eta_m} \\ + \cdots + \frac{(P_m x_n-p_n)^2}{\eta_n} \le \left(2\frac{l}{k}\right)^2\end{aligned} \tag{68}$$

or

$$\begin{aligned}\left(2\frac{l-1}{k}\right)^2 \le \frac{(x_1-p_1)^2}{\eta_1} + \frac{(x_2-p_2)^2}{\eta_2} + \cdots \frac{(x_m+p_m)^2}{\eta_m} \\ + \cdots + \frac{(x_{n-1}-p_{n-1})^2}{\eta_{n-1}} + \frac{(x_n-p_n)^2}{\eta_n} \le \left(2\frac{l}{k}\right)^2,\end{aligned} \tag{69}$$

and $x' \in \Omega$. This is exactly the condition that satisfies for any $x' \in \tau_l^{\prime(k)}$. Thus,

$$P_m \tau_l^{(k)} = \tau_l^{\prime(k)}. \tag{70}$$

Since the reflection transformation preserves the volume, so that

$$|\tau_l^{(k)}| = |\tau_l^{\prime(k)}|. \tag{71}$$

□

We know that $\tau_l^{(k)}$ and $\tau_l^{\prime(k)}$ approach to zero as k approaches to infinity. But how exactly and how rapidly it approaches to zero, we can see it by Lemma 5.

Lemma 5. For any integer k, and some $1 \le l \le k$ the volume of $\tau_l^{(k)}$ and $\tau^{\prime(k)}$ satisfy

$$|\tau_l^{(k)}| \le C(l) k^{-n}, \tag{72}$$

$$|\tau_l^{\prime(k)}| \le C(l) k^{-n}, \tag{73}$$

where C is a constant independent of k but dependent to l.

Proof. For convenience, we define

$$\sigma_l^{(k)} = \{x \in \mathbb{R}^n \mid 2\tfrac{l-1}{k} \leq \rho < 2\tfrac{l}{k}\} \tag{74}$$

and

$$\sigma_l^{'(k)} = \{x \in \mathbb{R}^n \mid 2\tfrac{l-1}{k} \leq r < 2\tfrac{l}{k}\}. \tag{75}$$

where $1 \leq l \leq k$. By the definition of (74) and (75), we can see that $\tau_l^{(k)} \subseteq \sigma_l^{(k)}$, and $\tau^{'(k)} \subseteq \sigma_l^{'(k)}$; therefore, $|\tau_l^{(k)}| \leq |\sigma_l^{(k)}|$, and $|\tau_l^{'(k)}| \leq |\sigma_l^{'(k)}|$. Since the volume of $\sigma_l^{(k)}$ and $\sigma_l^{'(k)}$ can be computed

$$|\sigma_l^{(k)}| = \int_{\mathbb{R}^n} \chi_{\sigma_l^{(k)}} dx = \omega_n \int_{\frac{l-1}{k}}^{\frac{l}{k}} r^{n-1} dr = \frac{\omega_n}{n-1}\left[\left(\frac{l}{k}\right)^n - \left(\frac{l-1}{k}\right)^n\right]. \tag{76}$$

We have used Lemma 1 in Equation (76). The ω_n has been defined in (33). Therefore, we can see that

$$|\tau_l^{(k)}| \leq |\sigma_l^{(k)}| = \frac{\omega_n}{n-1}\left[\left(\frac{l}{k}\right)^n - \left(\frac{l-1}{k}\right)^n\right] \equiv C(l)k^{-n}, \tag{77}$$

where $C(l)$ equals

$$C(l) = \frac{\omega_n}{n-1}[l^n - (l-1)^n]. \tag{78}$$

The case for $\tau_l^{'(k)}$ can be proven in the same way. □

Now, we can divide the function $J_\alpha(x)$ into

$$\begin{aligned} J_\alpha(x) &= \frac{1}{\beta(\alpha)} \int_\Omega \frac{dy}{|\eta^{-1}(x-y)|^{n-\alpha}} \\ &= \sum_{1\leq l\leq k} \int_{\tau_l^{(k)}} \frac{dy}{|\eta^{-1}(x-y)|^{n-\alpha}} \equiv \sum_{1\leq l\leq k} j_\alpha^{(l)} \\ &= \sum_{1\leq l\leq k} \int_{\tau_l^{'(k)}} \frac{dy}{|\eta^{-1}(x-y)|^{n-\alpha}} \equiv \sum_{1\leq l\leq k} j_\alpha^{'(l)}, \end{aligned} \tag{79}$$

where

$$j_\alpha^{(l)}(x) \equiv \int_{\tau_l^{(k)}} \frac{dy}{|\eta^{-1}(x-y)|^{n-\alpha}}, \tag{80}$$

and

$$j_\alpha^{'(l)}(x) \equiv \int_{\tau_l^{'(k)}} \frac{dy}{|\eta^{-1}(x-y)|^{n-\alpha}}. \tag{81}$$

We are going to define an approximation of $j^{(l)}$

$$j_{appx.\alpha}^{(l)}(x) = |\tau_l^{(k)}| \frac{1}{|\eta^{-1}(x-y)|^{n-\alpha}}, \tag{82}$$

$$j_{appx.\alpha}^{'(l)}(x) = |\tau_l^{'(k)}| \frac{1}{|\eta^{-1}(x-y')|^{n-\alpha}}, \tag{83}$$

for some $y \in \tau_l^{(k)}$ and $y' \in \tau_l^{'(k)}$.

Lemma 6. For any integer k and $1 \leq l \leq k$ it satisfies

$$|j_{\alpha}^{(l)}(p) - j_{appx.\alpha}^{(k)}(p)| \leq C'(l)k^{-\alpha} \tag{84}$$

and

$$|j_{\alpha}^{'(l)}(p') - j_{appx.\alpha}^{'(k)}(p')| \leq C'(l)k^{-\alpha}, \tag{85}$$

where $C'(l)$ is independent of k.

Proof. First, it is obvious that

$$|\tau_l^{(k)}| \min_{x\in\tau_l^{(k)}} \frac{1}{|\eta^{-1}(p-y)|^{n-\alpha}} \leq j_{\alpha}^{(l)}(p), j_{appx.\alpha}^{(l)}(p) \leq |\tau_l^{(k)}| \max_{x\in\tau_l^{(k)}} \frac{1}{|\eta^{-1}(p-y)|^{n-\alpha}}, \tag{86}$$

where

$$\max_{y\in\tau_l^{(k)}} \frac{1}{|\eta^{-1}(p-y)|^{n-\alpha}} = \frac{1}{(2\frac{l-1}{k})^{n-\alpha}}, \tag{87}$$

and

$$\min_{y\in\tau_l^{(k)}} \frac{1}{|\eta^{-1}(p-y)|^{n-\alpha}} = \frac{1}{(2\frac{l}{k})^{n-\alpha}}. \tag{88}$$

So for a sufficiently large k, it can satisfy

$$\max_{y\in\tau_l^{(k)}} \frac{|\tau_l^{(k)}|}{|\eta^{-1}(p-y)|^{n-\alpha}} - \min_{x\in\tau_l^{(k)}} \frac{|\tau_l^{(k)}|}{|\eta^{-1}(p-y)|^{n-\alpha}} = \frac{|\tau_l^{(k)}|}{(2\frac{l-1}{k})^{n-\alpha}} - \frac{|\tau_l^{(k)}|}{(2\frac{l}{k})^{n-\alpha}}, \tag{89}$$

so that

$$|j_{\alpha}^{(l)}(p) - j_{appx.\alpha}^{(k)}(p)| \leq |\tau_l^{(k)}| \left(\frac{1}{(2\frac{l-1}{k})^{n-\alpha}} - \frac{1}{(2\frac{l}{k})^{n-\alpha}} \right) \equiv C'(l)k^{-\alpha}. \tag{90}$$

We have used Lemma 5 in this equation and $C'(l)$ equals

$$C'(l) = C(l)2^{\alpha-n} \cdot \left[(l-1)^{\alpha-n} - l^{\alpha-n}\right]. \tag{91}$$

Then the theorem is proven. □

Of course, we can basically do the same with $|\tau_l^{'(k)}|$. Then we can get

$$|j_{\alpha}^{'(l)}(p') - j_{appx.\alpha}^{'(k)}(p')| \leq C'(l)k^{-\alpha}. \tag{92}$$

Note that $C'(l)$ increases as l increases, and by (91) and (78), we can see that $C'(l)$ increases in the order of $k^{n-1} \cdot k^{\alpha-n-1} = k^{\alpha-2}$

Now, back to the main theorem, we can see that $j_{appx.\alpha}^{(k)}(p) = j_{appx.\alpha}^{'(k)}(p')$ because by Lemma 4 $|\tau_l^{(k)}| = |\tau_l^{'(k)}|$, and $\frac{1}{|\eta^{-1}(p-y)|^{n-\alpha}} = \frac{1}{|\eta^{-1}(p'-y')|^{n-\alpha}}$ for some $y \in \tau_l^{(k)}$ and $P_m y = y' \in \tau_l^{'(k)}$, therefore

$$|j_{\alpha}^{'(l)}(p') - j_{\alpha}^{(l)}(p)| \leq |j_{\alpha}^{'(l)}(p') - j_{appx.\alpha}^{'(k)}(p')| + |j_{\alpha}^{(l)}(p) - j_{appx.\alpha}^{(k)}(p)| \leq 2C'(l)k^{-\alpha}, \tag{93}$$

thus, for sufficiently large k

$$
\begin{aligned}
|J_\alpha(p) - J_\alpha(p')| &= \left| \sum_l^k j_\alpha^{(l)}(p) - j_\alpha^{'(l)}(p') \right| \leq \sum_l^k |j_\alpha^{'(l)}(p') - j_\alpha^{(l)}(p)| \\
&\leq \sum_l^k C'(l) k^{-\alpha} \leq kC'(k) \cdot k^{-\alpha}.
\end{aligned} \tag{94}
$$

Since $kC'(k)$ increases in the order of $k^{\alpha-1}$, so (94) will decrease in the order of k^{-1} as k approaches to infinity. So the theorem is proven. □

4.1 Generalization

In Theorem 5, we have assumed Ω to be an n-dimensional ellipsoid centered at the origin point and it has axis of $\sqrt{\eta_1}, \sqrt{\eta_2} \cdots \sqrt{\eta_n}$ each parallel to the coordinate $(x_1 \cdots x_n)$. But this assumption is superfluous, for all we need is the restriction for Ω is $P_m\Omega = \Omega$ and Ω is bounded. From (66) to (70) we can see that Lemma 4 still holds under this restriction, and therefore, so does in Theorem 5.

Another assumption that is superfluous is that we only consider $p \in \partial\Omega$ and $P_m p \in \partial\Omega$. That is, we only consider $J_\alpha(x)$ under the restriction $J_\alpha(x)|_{\partial\Omega}$. We will extend it to any point $p \in \mathbb{R}^n$ and $p' = P_m p \in \mathbb{R}^n$.

We will redefine the coordinate $(\rho, \phi_1 \cdots \phi_{n-1})$ and $(r, \theta_1, \cdots \theta_{n-1})$ in Definition 5 basically in the same way but this time the coordinate will be centered at any point p and $P_m p$ which is not necessary on $\partial\Omega$. $|\tau_l^{(k)}|$ and $|\tau_l^{'(k)}|$ are now written as

$$
\tau_l^{(k)} = \{x \in \Omega \mid 2^{\frac{l-1}{k}} \leq \rho < 2^{\frac{l}{k}}\}, k_{min} \leq l \leq k_{max} \tag{95}
$$

and

$$
\tau_l^{'(k)} = \{x \in \Omega \mid 2^{\frac{l-1}{k}} \leq r < 2^{\frac{l}{k}}\}, k_{min} \leq l \leq k_{max}, \tag{96}
$$

where k_{max} is defined as $\forall l > k_{max}$, $\tau_l^{(k)} = \emptyset$. Since $P_m \tau_l^{(k)} = \tau_l^{'(k)}$, so that $\forall l > k_{max}$, $\tau_l^{'(k)} = \emptyset$. Such k_{max} exists because of the boundedness of Ω. Similarly, k_{min} is defined as $\forall l < k_{min}$, $\tau_l^{(k)} = \emptyset$. If such k_{min} does not exist, then set $k_{min} = 1$.

By this definition, we can get

$$
\Omega = \bigcup_{k_{min} \leq l \leq k_{max}} \tau_l^{(k)} = \bigcup_{k_{min} \leq l \leq k_{max}} \tau_l^{'(k)}, \tag{97}
$$

and therefore,

$$\begin{aligned} J_\alpha(x) &= \frac{1}{\beta(\alpha)}\int_\Omega \frac{dy}{|\eta^{-1}(x-y)|^{n-\alpha}} \\ &= \sum_{k_{min}\le l\le k_{max}} \int_{\tau_l^{(k)}} \frac{dy}{|\eta^{-1}(x-y)|^{n-\alpha}} \equiv \sum_{k_{min}\le l\le k_{max}} j_\alpha^{(l)} \\ &= \sum_{k_{min}\le l\le k_{max}} \int_{\tau_l'^{(k)}} \frac{dy}{|\eta^{-1}(x-y)|^{n-\alpha}} \equiv \sum_{k_{min}\le l\le k_{max}} j_\alpha'^{(l)}. \end{aligned} \tag{98}$$

The definition of $j_\alpha^{(l)}(x)$, $j_\alpha'^{(l)}(x)$, $j_{appx.\alpha}^{(l)}(x)$, and $j_{appx.\alpha}'^{(l)}(x)$ are still the same, so that we can see Theorem 5 still holds.

The generalization of Theorem 5 is:

Theorem 6 (generlization)**.** Let

$$u(x) \equiv \frac{1}{\beta(\alpha)}\int_\Omega \frac{dy}{|\eta^{-1}(x-y)|^{n-\alpha}} = J_\alpha. \tag{99}$$

For all bounded domain Ω that satisfies $\Omega = P_m\Omega$

$$u(x) = u(P_m x). \tag{100}$$

4.2 The antisymmetric property

In the equation of $(-\Delta_E)^{\frac{\alpha}{2}}u(x) = f(x)$, we have set $f(x) = \chi_\Omega$ and found out that the solution, noted as $J_\alpha(x)$, has the symmetric property. Now, if we replace the function $f(x)$ by another function $g(x) = \chi_\Omega x_i$, where $1 \le i \le n$, then the solution for $(-\Delta_E)^{\frac{\alpha}{2}}u(x) = g(x)$, noted as $J_\alpha g(x) = \mathfrak{J}_\alpha(x)$ will satisfy another property.

Theorem 7. Let

$$u(x) \equiv \frac{1}{\beta(\alpha)}\int_\Omega \frac{x_i dy}{|\eta^{-1}(x-y)|^{n-\alpha}} = \mathfrak{J}_\alpha(x), \tag{101}$$

then there is a property of $u(x)|_\Omega$.

$$u(x)|_{\partial\Omega} = u(P_m x)|_{\partial\Omega} \text{ ,for } m \neq i, \tag{102}$$

$$u(x)|_{\partial\Omega} = -u(P_m x)|_{\partial\Omega} \text{ ,for } m = i. \tag{103}$$

Where Ω is an n-dimensional ellipsoid centered at origin point and axis parallel to the coordinate $(x_1, x_2 \cdots x_n)$.

Proof. First, we will redefine, $j_\alpha^{(l)}$, $j_\alpha'^{(l)}$ as

$$j_\alpha^{(l)}(x) \equiv \int_{\tau_l^{(k)}} \frac{x_i dy}{|\eta^{-1}(x-y)|^{n-\alpha}} \tag{104}$$

and

$$j_{\alpha(x)}^{'(l)} \equiv \int_{\tau_l^{'(k)}} \frac{x_i dy}{|\eta^{-1}(x-y)|^{n-\alpha}}. \tag{105}$$

Since $\tau_l^{(k)}$ and $\tau_l^{'(k)}$ are the same as (61) and (63). so that

$$\begin{aligned} \mathfrak{J}_\alpha(x) &= \frac{1}{\beta(\alpha)} \int_\Omega \frac{x_i dy}{|\eta^{-1}(x-y)|^{n-\alpha}} \\ &= \sum_{1\le l\le k} \int_{\tau_l^{(k)}} \frac{x_i dy}{|\eta^{-1}(x-y)|^{n-\alpha}} \equiv \sum_{1\le l\le k} j_\alpha^{(l)} \\ &= \sum_{1\le l\le k} \int_{\tau_l^{'(k)}} \frac{x_i dy}{|\eta^{-1}(x-y)|^{n-\alpha}} \equiv \sum_{1\le l\le k} j_\alpha^{'(l)}. \end{aligned} \tag{106}$$

Now we are going to do something different. Define a subset in $\tau_l^{(k)}$,

$$\pi_{ml}^{(k)} = \{x \in \tau_l^{(k)} \mid \sqrt{\eta_i}\tfrac{m-1}{k} \le x_i < \sqrt{\eta_i}\tfrac{m}{k}\}. \tag{107}$$

Where m is an integer ranges from $-k+1$ to k. Similarly, we define

$$\pi_{ml}^{'(k)} = \{x \in \tau_l^{'(k)} \mid \sqrt{\eta_i}\tfrac{m-1}{k} \le x_i < \sqrt{\eta_i}\tfrac{m}{k}\}. \tag{108}$$

By (107) and (108), we can see that $P_m\pi_{ml}^{(k)} = \pi_{ml}^{'(k)}$ for $m \neq i$ and $P_m\pi_{ml}^{(k)} = \pi_{-m+1l}^{'(k)}$ for $m = i$, therefore, $|\pi_{ml}^{(k)}| = |\pi_{ml}^{'(k)}|$ for $m \neq i$, and $|\pi_{ml}^{(k)}| = |\pi_{-m+1l}^{'(k)}|$ for $m = i$. □

We can see that $|\pi_{ml}^{(k)}|$ decay to zero as k approach to infinity, and Lemma 7 told that who rapidly does it approaches to zero.

Lemma 7. For any $-k+1 \le m \le k$, $|\pi_{ml}^{(k)}|$ and $|\pi_{ml}^{'(k)}|$ satisfy

$$|\pi_{ml}^{(k)}| \le Ck^{-n-1} \tag{109}$$

and

$$|\pi_{ml}^{'(k)}| \le Ck^{-n-1}, \tag{110}$$

where C is a constant independent of k.

Proof. Set

$$|\pi|_{max} \equiv \max\{|\pi_{-k+1l}^{(k)}|, |\pi_{-k+2l}^{(k)}| \cdots |\pi_{k-1l}^{(k)}|, |\pi_{kl}^{(k)}|\} \tag{111}$$

and

$$|\pi|_{min} \equiv \min\{|\pi_{-k+1l}^{(k)}|, |\pi_{-k+2l}^{(k)}| \cdots |\pi_{k-1l}^{(k)}|, |\pi_{kl}^{(k)}|\}. \tag{112}$$

Notice that,

$$2k|\pi|_{min} \le \sum_{m=-k+1}^{k} |\pi_{ml}^{(k)}| = |\tau_l^{(k)}|. \tag{113}$$

By Lemma 5 we can se that

$$|\pi|_{min} \leq \frac{1}{2}C(l)k^{-n-1} \tag{114}$$

or

$$|\pi_{ml}^{(k)}| \leq |\pi|_{max} \leq \frac{1}{2}\frac{|\pi|_{max}}{|\pi|_{min}}C(l)k^{-n-1}. \tag{115}$$

The same can be done with $|\pi_{ml}^{'(k)}|$, and the theorem is proven. □

Lemma 8. For any integer k and $1 \leq l \leq k$ it satisfies

$$|j_\alpha^{(l)}(p) + j_\alpha^{'(l)}(p')| \leq C'(l)k^{-\alpha}. \tag{116}$$

If $p \in \partial\Omega$, $p' = P_i p \in \partial\Omega$, and

$$|j_\alpha^{(l)}(p) - j_\alpha^{'(l)}(p')| \leq C'(l)k^{-\alpha}. \tag{117}$$

If $p \in \partial\Omega$, $p' = P_m p \in \partial\Omega$ where $m \neq i$.

The definition of $j_\alpha^{(l)}(x)$ and $j_\alpha^{'(l)}(x)$ are in (104) and (105), and $C'(l)$ is independent of k.

Proof. First, we define

$$\mathrm{j}_\alpha^{ml}(x) = \int_{\pi_{ml}^{(k)}} \frac{x_i dy}{|\eta^{-1}(x-y)|^{n-\alpha}} \tag{118}$$

and

$$\mathrm{j}_\alpha^{'ml}(x) = \int_{\pi_{ml}^{'(k)}} \frac{x_i dy}{|\eta^{-1}(x-y)|^{n-\alpha}}. \tag{119}$$

It is easy to see that

$$j_\alpha^{(l)}(x) = \sum_{m=-k+1}^{k} \mathrm{j}_\alpha^{ml}(x), \text{ and } j_\alpha^{'(l)}(x) = \sum_{m=-k+1}^{k} \mathrm{j}_\alpha^{'ml}(x). \tag{120}$$

Consider an approximation of $\mathrm{j}_\alpha^{ml}(x)$, and $\mathrm{j}_\alpha^{'ml}(x)$

$$\mathrm{j}_{appx.\alpha}^{ml}(x) = |\pi_{ml}^{(k)}| \frac{x_i}{|\eta^{-1}(x-y)|^{n-\alpha}} \tag{121}$$

and

$$\mathrm{j}_{appx.\alpha}^{'ml}(x) = |\pi_{ml}^{'(k)}| \frac{x_i}{|\eta^{-1}(x-y')|^{n-\alpha}}. \tag{122}$$

For some $y \in \pi_{ml}^{(k)}$, and $y' \in \pi_{ml}^{'(k)}$,

$$|\pi_{ml}^{(k)}| \min_{x\in\pi_{ml}^{(k)}} \frac{x_i}{|\eta^{-1}(p-y)|^{n-\alpha}} \leq \mathrm{j}_\alpha^{(ml)}(p), \mathrm{j}_{appx.\alpha}^{(ml)}(p) \leq |\pi_{ml}^{(k)}| \max_{x\in\pi_{ml}^{(k)}} \frac{x_i}{|\eta^{-1}(p-y)|^{n-\alpha}}. \tag{123}$$

By the difinition of $\pi_{ml}^{(k)}$ and $\pi_{ml}^{'(k)}$ we can see that

$$\max_{y\in\pi_{ml}^{(k)}} \frac{x_i}{|\eta^{-1}(p-y)|^{n-\alpha}} \leq \frac{m}{k}\frac{1}{(2\frac{l-1}{k})^{n-\alpha}} \tag{124}$$

and

$$\min_{y\in\pi_{ml}^{(k)}} \frac{x_i}{|\eta^{-1}(p-y)|^{n-\alpha}} \leq \frac{m-1}{k}\frac{1}{(2\frac{l}{k})^{n-\alpha}}. \tag{125}$$

Apply (124) and (125) and Lemma 7 to (123) we can get

$$|\mathrm{j}_{\alpha}^{(ml)}(p) - \mathrm{j}_{appx.\alpha}^{(ml)}(p)| \leq C(m,l)k^{-\alpha-1}. \tag{126}$$

Where $C(m,l)$ equals

$$C(m,l) = 2^{\alpha-n-1}\frac{|\pi|_{max}}{|\pi|_{min}}C(l)\left[\frac{m}{(l-1)^{n-\alpha}} - \frac{m-1}{l^{n-\alpha}}\right]. \tag{127}$$

The can similarly apply the same way to $\mathrm{j}_{\alpha}^{'(ml)}(p')$, and get

$$|\mathrm{j}_{\alpha}^{'(ml)}(p') - \mathrm{j}_{appx.\alpha}^{'(ml)}(p')| \leq C(m,l)k^{-\alpha-1}. \tag{128}$$

On the other hand, by (121) and (122), we can see that $\mathrm{j}_{appx.\alpha}^{ml}(p) = \mathrm{j}_{appx.\alpha}^{'ml}(p')$ for $P_m p = p'$ where $m \neq i$, and $\mathrm{j}_{appx.\alpha}^{ml}(p) = -\mathrm{j}_{appx.\alpha}^{'-m+1l}(p')$ for $P_i p = p'$. Therefore, we get

$$\sum_{m=-k+1}^{k} \mathrm{j}_{appx.\alpha}^{ml}(p) = \sum_{m=-k+1}^{k} \mathrm{j}_{appx.\alpha}^{'ml}(p') \tag{129}$$

for $P_m p = p'$, $m \neq i$.

$$\sum_{m=-k+1}^{k} \mathrm{j}_{appx.\alpha}^{ml}(p) = -\sum_{m=-k+1}^{k} \mathrm{j}_{appx.\alpha}^{'ml}(p') \tag{130}$$

for $P_i p = p'$.

So for the $m \neq i$ case, by (120) we can get,

$$\begin{aligned}
|j_{\alpha}^{(l)}(p) - j_{\alpha}^{'(l)}(p')| &= \left|\sum_{m=-k+1}^{k}(\mathrm{j}_{appx.\alpha}^{ml}(p) - \mathrm{j}_{\alpha}^{ml}(p)) + \sum_{m=-k+1}^{k}(\mathrm{j}_{\alpha}^{'ml}(p')) - \mathrm{j}_{appx.\alpha}^{'ml}(p'))\right| \\
&\leq \sum_{m=-k+1}^{k}|\mathrm{j}_{appx.\alpha}^{ml}(p) - \mathrm{j}_{\alpha}^{ml}(p)| + \sum_{m=-k+1}^{k}|\mathrm{j}_{appx.\alpha}^{'ml}(p') - \mathrm{j}_{\alpha}^{'ml}(p')| \\
&\leq 2\sum_{m=-k+1}^{k} C(l,m)k^{\alpha-1} = 2k\cdot C(l)k^{-\alpha-1} \equiv C(l)k^{-\alpha}
\end{aligned} \tag{131}$$

and for the $m = i$ case

$$
\begin{aligned}
|j_\alpha^{(l)}(p) + j_\alpha^{'(l)}(p')| &= \left| \sum_{m=-k+1}^{k} (\mathrm{j}_\alpha^{ml}(p) - \mathrm{j}_{appx.\alpha}^{ml}(p)) + \sum_{m=-k+1}^{k} (\mathrm{j}_\alpha^{'ml}(p')) - \mathrm{j}_{appx.\alpha}^{'ml}(p')) \right| \\
&\leq \sum_{m=-k+1}^{k} |\mathrm{j}_{appx.\alpha}^{ml}(p) - \mathrm{j}_\alpha^{ml}(p)| + \sum_{m=-k+1}^{k} |\mathrm{j}_{appx.\alpha}^{'ml}(p') - \mathrm{j}_\alpha^{'ml}(p')| \\
&\leq 2 \sum_{m=-k+1}^{k} C(l,m) k^{\alpha-1} = 2k \cdot C(l) k^{-\alpha-1} \equiv C(l) k^{-\alpha}.
\end{aligned}
\tag{132}
$$

Note that by (127), $\sum_{m=-k+1}^{k} C(l,m)k$ is independent of m. So we can see that the theorem is proven. □

We can see that the first equation in Lemma 7 is identical to (93), so by (94) and (106) we can get

$$\mathfrak{J}_\alpha(p)|_{\partial\Omega} = \mathfrak{J}_\alpha(P_m p)|_{\partial\Omega}, \text{ for } m \neq i. \tag{133}$$

By the second equation of Lemma 7, we can see that it is basically the same as (93) but $j_\alpha^{'(l)}(p')$ has been replaced by $-j_\alpha^{'(l)}(p')$, so by (94) we can see that $|\mathfrak{J}_\alpha(p) + \mathfrak{J}_\alpha(P_m p)|$ will approach to zero as k approaches to infinity, so that

$$\mathfrak{J}_\alpha(p)|_{\partial\Omega} = -\mathfrak{J}_\alpha(P_m p)|_{\partial\Omega}, \text{ for } m = i, \tag{134}$$

and therefore, the theorem has been proven. □

5 Appendix [1]

Theorem 8. For any $0 < \alpha < n$, if $u(x)$ satisfies the equation

$$-\Delta^{\frac{\alpha}{2}} u(x) = \rho(x), \quad \text{for any } x \in \mathbb{R}^n \tag{135}$$

then $u(\mathbf{x})$ can be written as the convolution between K_α and f

$$u(x) = I_\alpha \rho(x) = (K_\alpha * \rho)(x). \tag{136}$$

Proof. We can start by Fourier transformation.

$$
\begin{aligned}
\mathcal{F}(-\Delta^{\frac{\alpha}{2}} u(x)) &= -\widehat{\Delta^{\alpha/2} u}(\xi) \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{\Gamma(\frac{-\alpha}{2})} \int_0^\infty t^{-\frac{\alpha}{2}-1} e^{\Delta t} u(x) dt \right) e^{-ix\cdot\xi} dx \\
&= \frac{1}{\Gamma(\frac{-\alpha}{2})} \int_0^\infty t^{-\frac{\alpha}{2}-1} \left(\int_0^\infty G_t * u(x) e^{-ix\cdot\xi} dx \right) dt \\
&= \frac{1}{\Gamma(\frac{-\alpha}{2})} \int_0^\infty t^{-\frac{\alpha}{2}-1} \widehat{G_t} \cdot \widehat{u}(\xi) dt \\
&= \widehat{\rho}(\xi).
\end{aligned}
\tag{137}
$$

The Fourier transformation of the Gauss kernel was known as

$$\widehat{G_t}(\xi) = (4\pi t)^{-\frac{n}{2}} \int_0^\infty e^{-\frac{|x|^2}{4t}} \cdot e^{-ix\cdot\xi} d\mathbf{x} = e^{-t|\xi|^2}. \tag{138}$$

So,

$$-\widehat{\Delta^{\alpha/2} u}(\xi) = \frac{\widehat{u}(\xi)}{\Gamma(\frac{-\alpha}{2})} \int_0^\infty t^{-\frac{\alpha}{2}-1} e^{-t|\xi|^2} dt = |\xi|^\alpha \widehat{u}(\xi). \tag{139}$$

In this case, we have use the fact that

$$\int_0^\infty t^s e^{-tu^2} dt = \frac{\Gamma(s+1)}{u^{2(s+1)}}. \tag{140}$$

Therefore, we get

$$\widehat{u}(\xi) = |\xi|^{-\alpha} \widehat{\rho}(\xi), \;\; u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} |\xi|^{-\alpha} e^{i\xi\cdot x} d\xi. \tag{141}$$

The solution can be constructed to the convolution between two functions. $u(x) = (2\pi)^{-n} \phi * \rho(x)$. Where the ϕ is

$$\phi(x) = \int_{\mathbb{R}^n} |\xi|^{-\alpha} e^{i\xi\cdot x} d\xi, \tag{142}$$

$$\phi(\lambda x) = \int_{\mathbb{R}^n} |\xi|^{-\alpha} e^{i\xi\cdot(\lambda x)} d\xi = \int_{\mathbb{R}^n} |\xi|^{-\alpha} e^{i(\lambda\xi)\cdot x} d\xi. \tag{143}$$

Now consider a transformation:

$$\lambda\xi = \zeta, \; d\xi = \lambda^{-n} d\zeta.$$

Then,

$$\phi(\lambda\mathbf{x}) = \int_{\mathbb{R}^n} |\lambda^{-1}\zeta|^{-\alpha} e^{i\zeta\cdot x} \lambda^{-n} d\zeta = \lambda^{\alpha-n} \int_{\mathbb{R}^n} |\zeta|^{-\alpha} e^{i\zeta\cdot x} d\zeta = \lambda^{\alpha-n} \phi(x). \tag{144}$$

It is obvious that $\phi(x)$ is a homogeneous function, so we can express ϕ in $\phi(x) = C|x|^{\alpha-n}$, and now we are going to determine the constant C.

$$\begin{aligned}\int_{\mathbb{R}^n} \phi(x) \exp(-\frac{|x|^2}{2}) dx &= C \int_{\mathbb{R}^n} \exp(-\frac{|x|^2}{2}) |\xi|^{\alpha-1} dx \\ &= C|\mathbb{S}^{n-1}| \int_0^\infty r^{\alpha-1} \exp(-\frac{r^2}{2}) dr \\ &= C|\mathbb{S}^{n-1}| 2^{\frac{\alpha}{2}-1} \Gamma(\frac{\alpha}{2}).\end{aligned} \tag{145}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^n} \phi(x) \exp(-\frac{|x|^2}{2}) dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\xi|^{-\alpha} \exp(-\frac{|x|^2}{2}) e^{i\xi\cdot x} dx d\xi \\ &= (\sqrt{2\pi})^n \int_{\mathbb{R}^n} \exp(-\frac{|\xi|^2}{2}) |x|^{-\alpha} d\xi \\ &= (\sqrt{2\pi})^n |\mathbb{S}^{n-1}| \int_0^\infty \exp(-\frac{r^2}{2}) r^{n-\alpha-1} dr \\ &= (\sqrt{2\pi})^n |\mathbb{S}^{n-1}| 2^{\frac{n-\alpha}{2}-1} \Gamma(\frac{n-\alpha}{2}).\end{aligned} \tag{146}$$

Compare (145) and (146) with the results, we can see that

$$C = (\sqrt{2\pi})^n 2^{-\alpha+\frac{n}{2}} \frac{\Gamma(\frac{\alpha-n}{2})}{\Gamma(\frac{\alpha}{2})}, \tag{147}$$

$$\phi(x) = (\sqrt{2\pi})^n 2^{-\alpha+\frac{n}{2}} \frac{\Gamma(\frac{\alpha-n}{2})}{\Gamma(\frac{\alpha}{2})} |x|^{\alpha-n}. \tag{148}$$

And finally, we get our solution

$$u(x) = (2\pi)^{-n} \phi * \rho(x) = \pi^{-\frac{n}{2}} 2^{-\alpha} \frac{\Gamma(\frac{\alpha-n}{2})}{\Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}^n} \frac{\rho(y)}{|x-y|^{n-\alpha}} = I_\alpha \rho(x) dx. \tag{149}$$

If the exponent $\alpha = 2n$ (n is a natural number), then by definition:

$$-\widehat{\Delta^n u}(\xi) = \lim_{\alpha \to 2n} -\widehat{\Delta^{\alpha/2} u}(\xi) = \lim_{\alpha \to 2n} \frac{1}{\Gamma(\frac{-\alpha}{2})} \int_0^\infty t^{-\frac{\alpha}{2}-1} \widehat{G_t} \cdot \widehat{u}(\xi) dt. \tag{150}$$

By calculation, this limit will be

$$\lim_{\alpha \to 2n} -\widehat{\Delta^{\alpha/2} u}(\xi) = |\xi|^\alpha \widehat{u}(\xi). \tag{151}$$

□

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