The 4-choosability of planar graphs and cycle adjacency

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Abstract

This article studies list coloring of planar graphs. Suppose G is a plane graph and H is a subgraph of G which is induced by the edges of some 4-, 5-faces of G. Let D_H be a graph constructed from H with the vertex set $V(D_H) = \{f : f$ is a 4-, 5-face of H and $G\}$ and the edge set $E(D_H) = \{ff' :$ f and f' are adjacent in $G\}$. An edge of H is called an inner-edge if it is a common edge of two 4-, 5-faces of G, otherwise it is called an outer-edge. If D_H is a tree and all outer-edges of H are adjacent to 3-cycles in G, then H is called a star-sun in G. It is proved that a planar graph without adjacent 3-cycles, adjacent 4-cycles and star-suns is 4-choosable.

中文摘要

本篇研究平面圖的列表著色問題。令圖 G 為一個簡單平版圖，並令圖 H 是一個由圖 G 中的 4-面與 5-面的邊生成的連通子圖。令圖 D_H 為一個由圖 H 中建構出來的圖，其點集 $V(D_H) = \{f : f$ 是 H 和 G 的一個 4-面或 5-面$\}$、邊集 $E(D_H) = \{ff' :$ f 和 f' 在 G 中相鄰$\}$。在圖 H 中的邊 e，若 e 為兩個 4-面或 5-面的共同邊，則稱 e 為內部邊，反之則稱為外圍邊。若圖 D_H 是樹型圖且圖 H 中所有的外圍邊皆與 3-圈相鄰，則稱圖 H 為在圖 G 中的星日圖。對於平面圖，針對長度較小的圈進行限制，我們設計充分條件『不包含相鄰 3-圈、相鄰 4-圈與星日圖』，使得滿足條件的平面圖為可四元列表著色。

1 Introduction and Preliminaries

In 1852, Francis Guthrie, while trying to color the map of counties of England, noticed that only four different colors were sufficient in order that two countries sharing a boundary receiving different colors. The Four Color Map Problem then spread out through Augustus De Morgan of University College London. Figure 1.1 shows that four colors are sufficient for the maps of Taiwan and the United States.

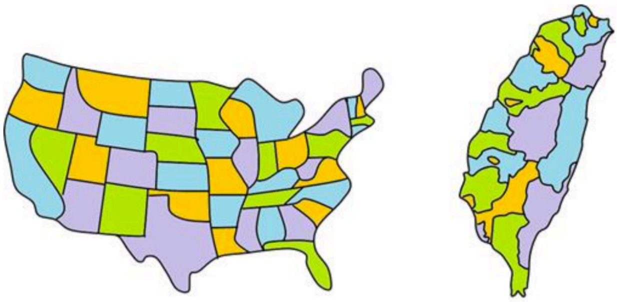


Figure 1.1 Examples for The Four Color Map Problem.

The problem was solved by Appel, Haken and Koch with the aids of computer in 1977, and known as The Four Color Theorem. The theorem now is often stated in term of the dual graph of a planar graph as: every planar graph is 4-colorable, namely the vertices of a planar graph can be colored by using 4 colors such that adjacent vertices receive different colors. Soon after then, Vizing [7] and Erdős et al. [2] generalized the concept of coloring to list coloring. This article studies list coloring for planar graphs.

Let G be a graph. A *proper vertex coloring* of G is a mapping $\varphi : V(G) \to \mathbb{N}$ such that $\varphi(v) \neq \varphi(u)$ whenever v and u are adjacent. A *list-assignment* of G is a mapping $L : V(G) \to 2^{\mathbb{N}}$ and is called a *k-list-assignment* if each $L(v)$ is of size k. The graph G is *L-colorable* if there exists an *L-coloring* which is a proper vertex coloring φ of G such that the color $\varphi(v) \in L(v)$ for each $v \in V(G)$. Figure 1.2 demonstrates a list-assignment L, using real colors, and an L-coloring of the map of Taiwan. If G is L-colorable for any k-list-assignment L, then G is said to be *k-choosable*. The *choosability* of a graph G is the minimum integer k such that G is k-choosable.

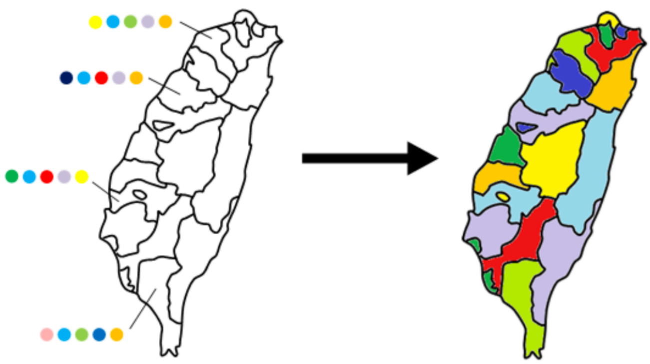


Figure 1.2 The example of an L-coloring of the map of Taiwan.

1.1 Survey and Motivations

In 1994, Thomassen [6] proved that every planar graph is 5-choosable. Voigt [8] showed that not all planar graphs are 4-choosable. Hence, over the past decades, determining whether a planar graph is 4-choosable or not has received significant attention. Some notable research studies of choosability on planar graphs are as follows:

1. Thomassen [6] proved that every planar graph is 5-choosable.
2. Voigt [8] presented an example of planar graph which is not 4-choosable.
3. Every planar graph G without 3-cycles is 4-choosable because $\delta(G) \leq 3$.
4. Lam et al. [4] proved the 4-choosability of planar graphs without 4-cycles.
5. Wang and Lih proved the 4-choosability of planar graphs without 5-cycles, or without 6-cycles, or without intersecting 3-cycles in [10, 9, 11].
6. Farzad [3] proved the 4-choosability of planar graphs without 7-cycles.
7. Chen et al. [1] proved the 4-choosability of planar graphs without 4-cycles adjacent to 3-cycles.
8. Lin [13] proved that the 4-choosability of planar graphs without adjacent 3-cycles, adjacent 4-cycles and k-suns.

This article provides a new sufficient condition for a planar graph to be 4-choosable. Different from the above research, in this study, each short cycle is allowed to exist. This article uses the discharging method, the proof by contradiction, the Euler's formula and the mathematical induction to prove the following theorem.

Theorem: *Every planar graph without adjacent* 3*-cycles, adjacent* 4*-cycles and star-suns is* 4*-choosable.*

1.2 Definitions

All graphs considered in this article are finite, simple and planar. A *plane graph* is a drawing of a planar graph in which edges only interest at their end vertices. Let G be a plane graph with $V(G)$, $E(G)$, $F(G)$ as its vertex set, edge set and face set, respectively. The *degree* of a vertex v in G, written as $d(v)$, is the number of edges incident to it. The *degree* of a face f in G, denoted by $d(f)$, is the number of edges incident to it. The minimum degree of G, $\min\{d(v) \mid v \in V(G)\}$, is denoted by $\delta(G)$. A vertex v is a *k-vertex* (*k^+-vertex* or *k^--vertex*, respectively) if $d(v) = k$

($d(v) \geq k$ or $d(v) \leq k$, respectively). A face f is a k-*face* (k^{+}-*face* or k^{-}-*face*, respectively) if $d(f) = k$ ($d(f) \geq k$ or $d(f) \leq k$ respectively).

Let G be a plane graph. Two cycles are *adjacent* if they have at least one edge in common. Two cycles are *intersecting* if they have at least one vertex in common. A cycle of length k is called a k-*cycle*. A k-*sun* in a plane graph G is a k-cycle whose each edge is adjacent to a 3-cycle, see Figure 1.3.

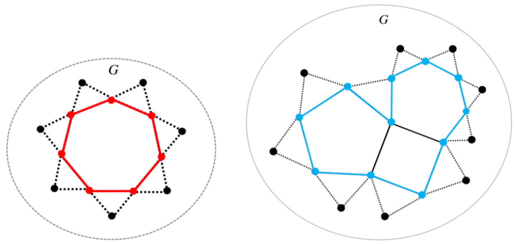
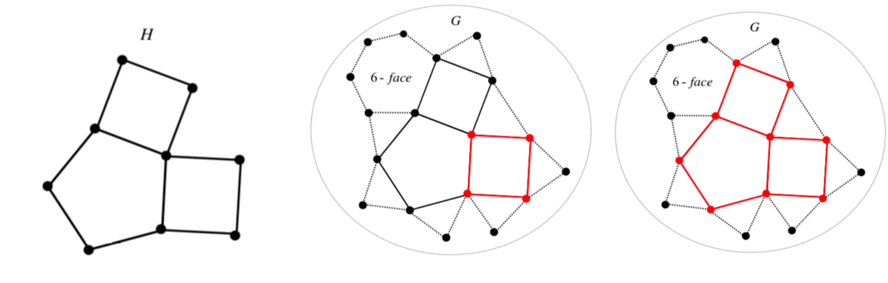


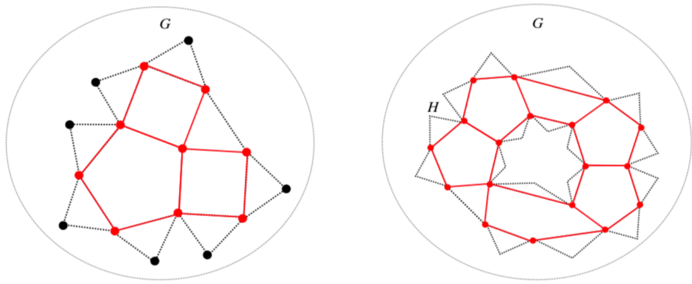
Figure 1.3 In G, the red 7-cycle is a 7-sun and the blue 11-cycle is an 11-sun.

Assume H is a subgraph of a plane graph G which is induced by the edges of some 4-, 5-faces of G. Notice that H is also a plane graph. A 4-, 5-face of H is called a *real-face* if it is also a 4-, 5-face of G. An edge of H is called an *inner-edge* if it is a common edge of two real 4-, 5-faces of H, otherwise it is called an *outer-edge*. Let D_H be the graph with $V(D_H) = \{f : f \text{ is a real 4-, 5-face of } H\}$ and $E(D_H) = \{ff' : f \text{ and } f' \text{ are adjacent in } H\}$. If D_H is a tree and all outer-edges of H are adjacent to 3-cycles in G, then H is called a *star-sun* in G. Note that a star-sun is a special k-sun. If D_H is a tree and there is only one outer-edge of H not adjacent to 3-cycle in G, then H is called a *flower* in G. A *maximal flower* in G is a flower not a subgraph of another flower.

Example (in Figure 1.4)**.** In (a), let H be a subgraph which is induced by the edges of some 4-, 5-faces in G. The graph D_H is a tree. In (b), the red 4-cycle is a flower but not a maximal flower in G. The red part of (c) is a maximal flower in G since it is not a subgraph of another flower. The red part of (d) are star-suns.



(a) A special subgraph of H. (b) A flower (red). (c) A maximal flower (red).



(d) Two star-suns (red).

Figure 1.4

2 Proof of the Theorem

The idea of the proof follows a common routine as is in the articles mentioned in Section 1.1. First, we assume that there are counterexamples to the Theorem. Let G be a minimal counterexample with fewest vertices. We derive several forbidden configurations in G and discuss some properties of G.

We then apply the discharging method to show that the minimal counterexample does not exist. We design the initial charges $w : V(G) \to F(G)$ for all vertices and faces in the minimal counterexample G and the discharging rules. We modify w to w' by applying the discharging rules. Since the discharging rules only transfer charges in G, the initial total charge of w and w' should be the same. We shall show that the total charges of w and w' are different, which is a contradiction in total charge. Therefore, the minimal counterexample does not exist and the Theorem is proved.

2.1 Properties of a Minimal Counterexample

Let G be a minimal counterexample of the Theorem. Assume L is a 4-list-assignment of G such that G is not L-colorable. We derive some properties of G.

Lemma 1: *The minimum degree of vertex in G is at least 4, in other words,* $\delta(G) \geq 4$.

Proof. Suppose to the contrary that there is a vertex u of degree less than 4. Deleting u from G results a graph G' with fewer vertices than G. As G' satisfying the conditions of the Theorem, by the induction hypothesis, G' has an L-coloring. Because the list of u has 4 permissible colors and the number of the neighbors of u is less than 4, we can color u with a color in $L(u)$, and hence extending an L-coloring of G' to an L-coloring of G, see Figure 2.1. This is a contradiction. □

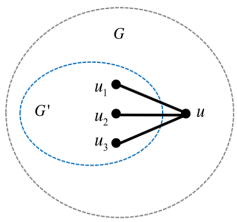


Figure 2.1 The vertex u has at most three neighbors u_1, u_2, u_3 in G.

2.1.1 Forbidden subgraphs of G : $\wp_1$-subgraph and $\wp_2$-subgraph

In the graph G, a $\wp_1$-subgraph is a 4-cycle in which every vertex is of degree 4 in G, and a $\wp_2$-subgraph is the union of two intersecting 4-cycles in which every vertex is of degree 4 except that the intersecting vertex is of degree 5 in G, see Figure 2.2.

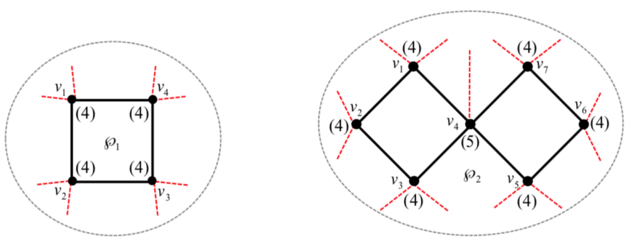


Figure 2.2 A $\wp_1$-subgraph and a $\wp_2$-subgraph.

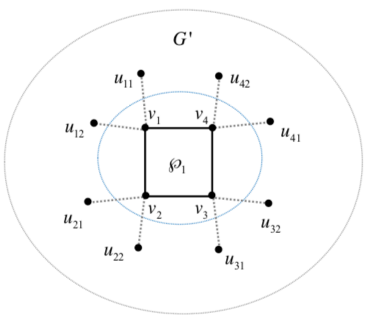
Lemma 2: *The graph G contains neither a $\wp_1$-subgraph nor a $\wp_2$-subgraph.*

Proof. Suppose G contains a $\wp_1$-subgraph with vertex set $V(\wp_1) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(\wp_1) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Delete $V(\wp_1)$ from G, and then the remaining graph $G' = G - V(\wp_1)$ admits an L-coloring φ' by the induction hypothesis. For all $v \in V(\wp_1)$, let $L_1(v) = L(v) \setminus \{\varphi'(u) \mid u \in V(G') \text{ and } uv \in E(G)\}$. Then, $|L_1(v_i)| \geq 2$ for $i \in \{1,2,3,4\}$, see Figure 2.3 (a).

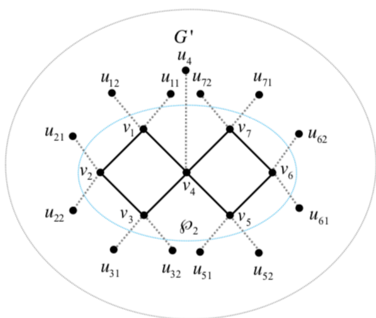
First, suppose that $L_1(v_1) = L_1(v_2) = L_1(v_3) = L_1(v_4) = \{a, b\}$. In this case, we color v_1 and v_3 by a, color v_2 and v_4 by b. Second, suppose that not all $L_1(v_i)$ being the same, say $L_1(v_1) \neq L_1(v_4)$. In this case, choose some $a \in L_1(v_1) \setminus L_1(v_4)$. Then color v_1 by a, color v_2 by a color $b \in L_1(v_2) \setminus \{a\}$, color v_3 by a color $c \in L_1(v_3) \setminus \{b\}$ color v_4 by a color $d \in L_1(v_4) \setminus \{c\}$ which is not equal to a. Then, $\wp_1$ is L_1-colorable. Hence, we can extend φ' to an L-coloring of G.

Suppose G contains a $\wp_2$-subgraph with $V(\wp_2) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $E(\wp_2) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_4v_5, v_5v_6, v_6v_7, v_7v_4\}$. Then $G' = G - V(\wp_2)$ admits an L-coloring φ'. For all $v \in V(\wp_2)$, let $L_2(v) = L(v) \setminus \{\varphi'(u) \mid u \in V(G') \text{ and } uv \in E(G)\}$. Then, $|L_2(v_i)| \geq 2$ for $i \in \{1,2,3,5,6,7\}$ and $|L_2(v_4)| \geq 3$, see Figure 2.3 (b).

If $L_2(v_5) \cap L_2(v_7) \neq \emptyset$, then we choose a color $a \in L_2(v_5) \cap L_2(v_7)$ and a color $b \in L_2(v_6) \setminus \{a\}$. Let $\varphi'(v_5) = \varphi'(v7) = a$ and $\varphi'(v_6) = b$. For the remaining 4-cycle $\{v_1, v_2, v_3, v_4\}$, each vertex still has at least 2 permissible colors that are not used by any of its colored neighbors. Therefore, we can extend φ' to an L-coloring of G. Assume $L_2(v_5) \cap L_2(v_7) = \emptyset$. According to the proof of $\wp_1$-subgraph, there is an L-coloring φ' to the 4-cycle $\{v_1, v_2, v_3, v_4\}$. Without loss of generality, assume $\varphi'(v_4) \notin L_2(v_7)$. We choose a color $a \in L_2(v_5) \setminus \{\varphi'(v_4)\}$, $b \in L_2(v_6) \setminus \{a\}$ and $c \in L_2(v_7) \setminus \{b\}$. Let $\varphi'(v_5) = a$, $\varphi'(v_6) = b$ and $\varphi'(v_7) = c$. Therefore, we can extend φ' to an L-coloring of G. This completes the proof of Lemma 2. □



(a) A $\wp_1$-subgraph.



(b) A $\wp_2$-subgraph.

Figure 2.3

2.1.2 The distribution of all 4-, 5-faces in G

Consider the special graph D_G, it presents the distribution of all 4-, 5-faces in the minimal counterexample G. Assume H is a subgraph of G which is induced by the edges of some 4-, 5-faces of G and D_H is a connected component of D_G. Let α be the number of real-5-faces in H, and β be the number of real-4-faces in H. The number of outer-edges and inner-edges of H are denoted by $\mathsf{Outer}(H)$ and $\mathsf{Inner}(H)$. Note that the sum of $\mathsf{Outer}(H)$ and $\mathsf{Inner}(H)$ is $|E(H)|$.

Example (in Figure 2.4)**.** Let H be a subgraph of G which is induced by the edges of some 4-, 5-faces of G and D_H be a connected component of D_G. In H, there are 6 real-5-faces, 3 real-4-faces, one 6-face (may be not real), one 3-face (may be not real), one 13-face (may be not real), 22 outer-edges and 10 inner-edges. Hence, $\alpha = 6$, $\beta = 3$, $\mathsf{Outer}(H) = 22$ and $\mathsf{Inner}(H) = 10$.

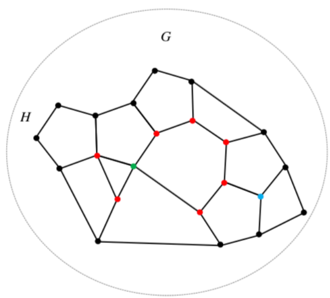
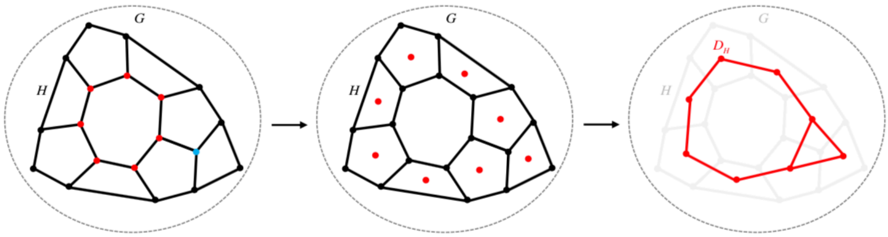


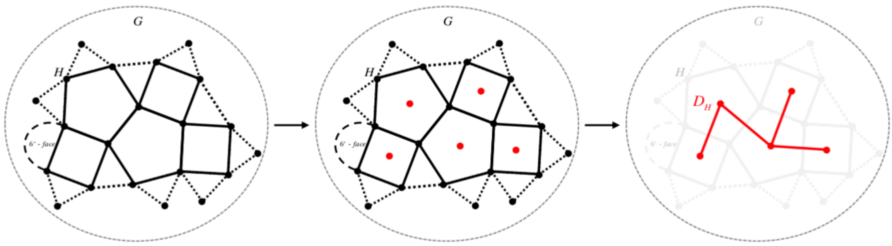
Figure 2.4 A subgraph H with D_H being a connected component of D_G.

2.1.3 A special graph D_H constructed by the real 4-, 5-faces in H

To describe the distribution of the 4-, 5-faces in the minimal counterexample G, let H be a subgraph of G which is induced by the edges of some 4-, 5-faces of G and D_H be a connected component of D_G. The graph D_H is constructed by real-4-faces and real-5-faces in H, see Figure 2.5(a). Note that if H is a maximal flower in G, then D_H is a tree, see Figure 2.5(b).



(a) The graph D_H has a cycle.



(b) The graph D_H is a tree.

Figure 2.5 The construction of D_H.

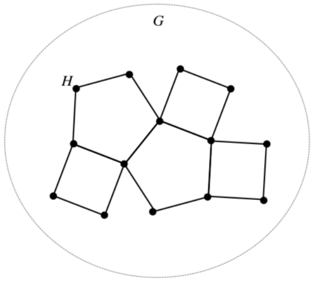
Lemma 3: *Let H be a subgraph of G which is induced by the edges of some 4-, 5- faces of G and D_H be a connected component of D_G.*

1. *If D_H is a tree, then* $\mathsf{Outer}(H) = 3\alpha + 2\beta + 2$.
2. *If D_H is not a tree, then* $\mathsf{Outer}(H) \leq 3\alpha + 2\beta$.

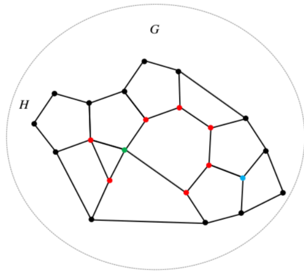
Proof. Note that the degree sum of all real-4-faces, real-5-faces of H counts outer-edges once and inner-edges twice. Hence, $5\alpha + 4\beta = \mathsf{Outer}(H) + 2\,\mathsf{Inner}(H)$. If D_H is a tree, then the number of the inner-edges of H is $\alpha + \beta - 1$. This means $\mathsf{Inner}(H) = \alpha + \beta - 1$. Therefore, we have $\mathsf{Outer}(H) = 3\alpha + 2\beta + 2$. If D_H is not a tree, then $\mathsf{Inner}(H) \geq \alpha + \beta$. This implies that $\mathsf{Outer}(H) \leq 3\alpha + 2\beta$. □

Example (for Lemma 3)**.** In Figure 2.6 (a), D_H is a tree with $\alpha = 2$, $\beta = 3$ and $\mathsf{Outer}(H) = 14$. Then, $\mathsf{Outer}(H) = 14 = 3 \cdot 2 + 2 \cdot 3 + 2 = 3\alpha + 2\beta + 2$.

In Figure 2.6 (b), D_H is not a tree with $\alpha = 6$, $\beta = 3$ and $\mathsf{Outer}(H) = 22$. Then, $\mathsf{Outer}(H) = 22 \leq 3 \cdot 6 + 2 \cdot 3 = 3\alpha + 2\beta$.



(a)



(b)

Figure 2.6

2.2 Discharging Method

2.2.1 Discharging rules

For $x \in V(G) \cup F(G)$, the initial charge of x is defined by the weight function $w(x) = 3d(x) - 12$. By using the Euler's formula of a connected planar graphs, we have $V(G) - E(G) + F(G) = 2$. Hence, the total charge of w is

$$\begin{aligned} w(G) &= \sum_{x \in V(G) \cup F(G)} w(x) = \sum_{v \in V(G)} (3d(v) - 12) + \sum_{f \in F(G)} (3d(f) - 12) \\ &= 3\left(\sum_{v \in V(G)} d(v) + \sum_{f \in F(G)} d(f)\right) - 12(|V(G)| + |F(G)|) \\ &= 3\,(2|E(G)| + 2|E(G)|) - 12\,(|E(G)| + 2) = -24. \end{aligned}$$

Now we will discharge the charges according to the following rules, see also Figure 2.7.

(R-1) If a 4-face f is incident to exactly one 5^+-vertex v, then v sends charge 2 to f.

(R-2) If a 4-face f is incident to more than one 5^+-vertex v, then each v sends charge 1 to f.

(R-3) Discharge 1 of the charge of a non-triangular face to each of its adjacent 3-faces.

(R-4) If a 6^+-face f' is adjacent to a maximal flower H, then f' sends charge 1 to the face f in H that is adjacent to f'.

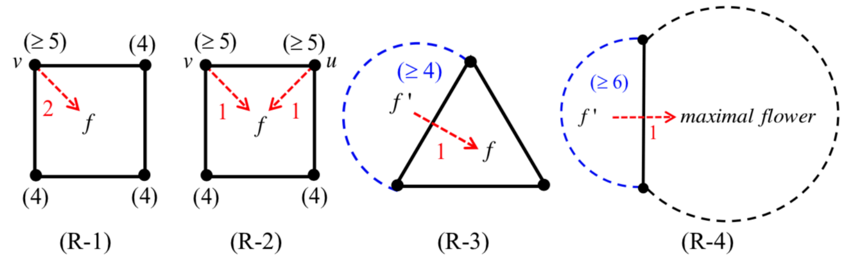


Figure 2.7 Discharging rules.

Let w and w' denote the weight functions for the initial charge and the one after discharging. Now we estimate the total charge of w'. Notice that the total charge of w' is supposed to be the same as that of w. The proof of the Theorem will be obtained by showing the total charge of w' is nonnegative.

2.2.2 New charges of the vertices and faces

Assume $v \in V(G)$. By Lemma 1, we may assume that v is of degree at least 4.

1. If $d(v) = 4$, then $w'(v) = w(v) = 0$ since v does not transfer any charges to others.
2. Let $d(v) = 5$. The vertex v is incident to at most two 4-faces, since G contains no adjacent 4-faces. According to R-1 and R-2, if v is incident to at most one 4-face which receives at most 2 from v, then $w'(v) \geq w(v) - 2 = 3 - 2 = 1 > 0$. If v is incident to 2 intersecting 4-faces, and one of them that only has an incident 5^+-vertex receives 2 from v, then the other must receive only 1 from v because G forbids $\wp_2$-subgraph. Hence, $w'(v) \geq w(v) - 1 - 2 = 3 - 1 - 2 = 0$, see Figure 2.8.

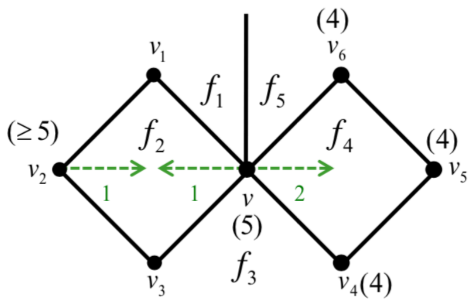


Figure 2.8 The situation that a 5-vertex sends 3 charges to 4-faces.

3. If $d(v) \geq 6$, then v is incident to at most $\left[\frac{d(v)}{2}\right]$ 4-faces because G contains

no adjacent 4-cycles and each 4-face receives 2 at most from v, where $[x]$ is the largest integer not exceeding x, see Figure 2.9. Therefore,

$$w'(v) \geq w(v) - 2\left[\frac{d(v)}{2}\right] = 3d(v) - 12 - 2\left[\frac{d(v)}{2}\right] \geq 2d(v) - 12 \geq 0.$$

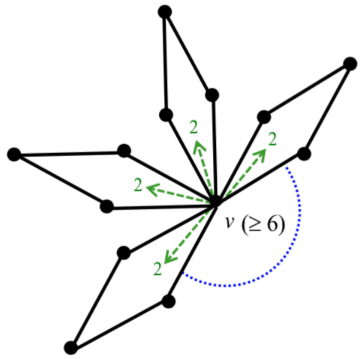


Figure 2.9 Each 6^+-vertex sends at most $\left[\frac{d(v)}{2}\right]$ charges to its incident 4-faces.

Next, assume $f \in F(G)$.

1. If $d(f) = 3$, then the faces adjacent to f are not 3-faces, since G contains no adjacent 3-faces. Therefore, $w'(f) = w(f) + 1 \cdot 3 = (-3) + 1 \cdot 3 = 0$, because f gets 3 charges from its adjacent 4^+-faces according to R-3, see Figure 2.10.

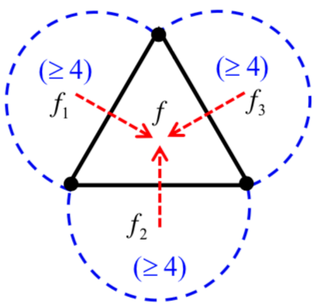


Figure 2.10 Every 3-face receives 3 charges from its adjacent 4^+-faces.

2. If $d(f) \in \{4, 5\}$, then we compute their charges together in group. Let H be a subgraph of G which is induced by the edges of some 4-, 5-faces of G and

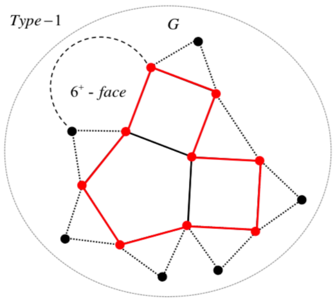
D_H be a connected component of D_G. Since G contains no star-suns, we can classify H into three types:

(1) H is called **Type-1** if D_H is a tree and H is a maximal flower of G, see Figure 2.11 (a).

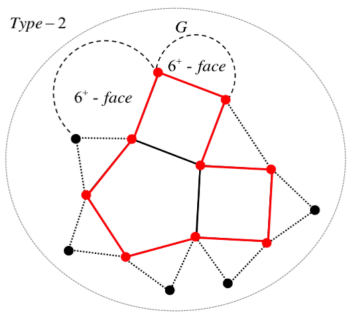
(2) H is called **Type-2** if D_H is a tree and H is not a maximal flower of G, see Figure 2.11 (b).

(3) H is called **Type-3** if D_H is not a tree, see Figure 2.11 (c).

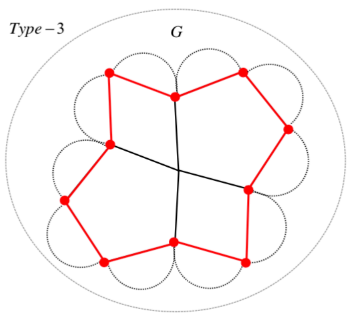
Let f be a real 4-, 5-face of H. We call f Type-i if H is Type-i for some $i \in \{1, 2, 3\}$.



(a) Type-1: D_H is a tree and H is a maximal flower in G.



(b) Type-2: D_H is a tree but H is not a maximal flower in G.



(c) Type-3: D_H is not a tree.

Figure 2.11

In the above three types, let the number of real-5-faces be α and the number of real-4-faces be β. Since G forbids $\wp_1$-subgraph, every 4-face of G is incident to one 5^+-vertex at least. As a result, after applying R-1 and R-2, each 4-face of G receives 2 charges from 5^+-vertices at least. The original charge of each 5-face is three, so the sum of the charges of the real 4-, 5-faces in H is at least $3\alpha + 2\beta$.

(1) If H is Type-1, then H sends at most $\mathsf{Outer}(H) - 1$ charges to its adjacent 3-faces according to R-3, and H receives charge 1 from a 4^+-face according to R-4. Note that D_H is a tree since H is a maximal flower. By Lemma 3 (1), $\mathsf{Outer}(H) = 3\alpha + 2\beta + 2$. Hence,

$$\sum_{\substack{f \in \{f \,:\, \text{a real-face of} \\ \text{Type-1 } H \text{ with } d(f) \in \{4,5\}\}}} w'(f) \geq 3\alpha + 2\beta - \big(\mathsf{Outer}(H) - 1\big) + 1$$
$$= 3\alpha + 2\beta - (3\alpha + 2\beta + 2 - 1) - 1 = 0.$$

(2) If H is Type-2, then H sends at most $\mathsf{Outer}(H) - 2$ charges to its adjacent 3-faces, or it is a maximal flower. By Lemma 3 (1), $\mathsf{Outer}(H) = 3\alpha + 2\beta + 2$. Hence,

$$\sum_{\substack{f \in \{f \,:\, \text{a real-face of} \\ \text{Type-2 } H \text{ with } d(f) \in \{4,5\}\}}} w'(f) \geq 3\alpha + 2\beta - \big(\mathsf{Outer}(H) - 2\big)$$
$$= 3\alpha + 2\beta - (3\alpha + 2\beta + 2 - 2) = 0.$$

(3) If H is Type-3, then H sends at most $\mathsf{Outer}(H)$ charges to its adjacent 3-faces. By Lemma 3 (2), $\mathsf{Outer}(H) \leq 3\alpha + 2\beta$. Hence,

$$\sum_{\substack{f \in \{f \,:\, \text{a real-face of} \\ \text{Type-3 } H \text{ with } d(f) \in \{4,5\}\}}} w'(f) \geq 3\alpha + 2\beta - \mathsf{Outer}(H)$$
$$\geq 3\alpha + 2\beta - (3\alpha + 2\beta) = 0.$$

3. If $d(f) \geq 6$, then $w'(f) \geq w(f) - 1 \cdot d(f) = 2d(f) - 12 \geq 0$ since f transfers at most charges $d(f)$ to its incident faces which are 3-cycles or maximal flowers according to R-3 and R-4, see Figure 2.12.

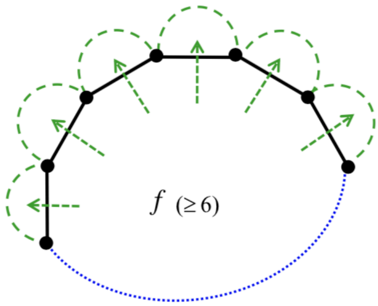


Figure 2.12 A 6^+-face sends at most $d(f)$ charges.

After discharging, the total charge of w' is

$$\sum_{x\in V(G)\cup F(G)} w'(x) = \sum_{v\in V(G)} w'(v) + \sum_{f\in\{\text{3-face, }6^+\text{-face in }G\}} w'(f) + \sum_{f\in\{\text{4-, 5-face is Type-1}\}} w'(f) + \sum_{f\in\{\text{4-, 5-face is Type-2}\}} w'(f) + \sum_{f\in\{\text{4-, 5-face is Type-3}\}} w'(f) \geq 0.$$

Since the original total charge is -24,

$$\sum_{x\in V(G)\cup F(G)} w'(x) \neq \sum_{x\in V(G)\cup F(G)} w(x),$$

a contradiction. In other words, the minimal counterexample G does not exist. Therefore, the proof of the Theorem is completed.

3 Summary and Conclusions

3.1 The Difference Between the Previous Result and the Theorem

The result in [13] shows that every planar graph without adjacent 3-cycles, adjacent 4-cycles and k-suns is 4-choosable. In the new Theorem, the k-suns can exist, and only a kind of particular suns, called star-suns, is forbidden. In other words, the one is a special case in light of the research. Hence, the Theorem is better than the previous result.

3.2 The Value of the Theorem

1. It is worth noticing in [5] that Montassier design a planar graph that satisfies the conditions of the Theorem but is not 3-choosable. Therefore, it is tight that every planar graph without adjacent 3-cycles, adjacent 4-cycles and star-suns is 4-choosable.

2. Construction of a Coloring Algorithm.

   For the planar graph G satisfying our conditions, we can color G according to the following steps:

   (1) Search for vertices with degrees less than 4, $\wp_1$-subgraphs and $\wp_2$-subgraphs and delete them recursively.
   (2) What remains is a graph G' with minimum degree at least 4 and with neither $\wp_1$-subgraph nor $\wp_2$-subgraph. But by the Theorem, G' cannot exist. This means graph G is deleted completely.
   (3) If we list what we have deleted in their order, then we can color the graph according to the backward order.

3.3 Conjectures

After completing the proof of the Theorem, based on the process of the discharging we make a conjecture as follows.

Conjecture: *Every planar graph without adjacent 3-cycles, adjacent 4-cycles is 4-choosable.*

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