RÉNYI FUNCTION FOR MULTIFRACTAL RANDOM FIELDS

NIKOLAI N. LEONENKO AND NARN-RUEIH SHIEH

Abstract. This paper presents the basic scheme and the log-normal, log-gamma and log-negative-inverted-gamma scenarios to establish the Rényi function for infinite products of homogeneous isotropic random fields on R^n; in particular for random fields on the on the sphere in R^3. The motivation of this paper is the test of (non-)Gaussianity on the Cosmic Microwave Background Radiation (CMBR) in Cosmology. In the presentation, we need to employ spherical harmonics for some concrete computations.

1. Introduction

Multifractal models have been used in many applications in hydrodynamic turbulence, finance, genomics, computer network traffic, etc; see Kolmogorov (1941,1962), Kahane (1985,1987), Gupta and Waymire (1993), Novikov (1994), Frisch (1995), Mandelbrot (1974), and Falconer (1997). Harte (2001) and Riedi (2003) contain an extensive bibliography of the subject. There are many ways to construct random multifractal measures such as via the binomial cascade, branching processes and other stochastic processes; see Kahane (1985,1987), Gupta and Waymire (1993), Frisch (1995), Taylor (1995), Molchan (1996), Falconer (1997), Jaffard (1999), Schmitt and Marsan (2001), Barral and Mandelbrot (2002), Shieh and Taylor (2002), Riedi (2003), Bacry and Muzy (2003), Schmitt (2003), Barndorf-Nilsen and Shmigel (2004), Mörters and Shieh (2002,2004,2008). In these works of such *Multifracal Analysis*, the *Rényi function*, which is also termed as the *deterministic partition function* or the *moment-scaling function*, plays a central role, as we may see in the seminal works of Mandelbrot (1974) and Kahane (1987). In a recent work, Mannersalo *et al* (2002) discuss Rényi functions for random measures induced from infinite products of stationary processes. Their work has been much examined for several classes of exponential (geometric) processes by Anh, Leonenko, and Shieh (2008a,b, 2009a,b, 2010); see also the results related to the topics in Shieh (2009) and Matsui and Shieh (2009,2012), and a simulation paper by Anh, Leonenko, Shieh and Taufer (2010).

The purpose of this work is to present a basic scheme and some important scenarios for the Rényi function of the infinite products of measurable homogeneous and isotropic random fields, which may show the multifractality of such fields. Our scheme is novel in two aspects; the first one is that the process (one-parameter) case is not easily seen to have the field (multi-parameter) corresponding, and the second one is that our scheme may include the model needed for study on the problems in Cosmic Microwave Background Radiation (CMBR); see a review paper by Marinucci (2004), a recent monograph by Marinucci and Peccati (2011), and the references therein. Moreover, our scenarios include several ones which are of intrinsic interest in previous mathematics and physics literatures.

The paper is organized as follows. In Section 2, we list some preliminaries and present the basic scheme. In Section 3, we provide some important scenarios. All proofs are given in the Section 4. In the concluding Section 5 we summarize our results and remark the possible statistical calibration of our model based on the dataset released by The Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP).

Key words and phrases. Rényi function; infinite products; random fields; multifractality; singularity spectrum; log-normal scenario; log-gamma scenario; log-negative-inverted- gamma scenario.

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2. Infinite products of random fields: multifractal schemes

In this paper we consider a measurable homogeneous and isotropic random field (HIRF, for brevity) on the n-dimensional Euclidean space $\mathbb{R}^n, n \geq 2$ (the case $n = 1$, i.e. the process case, is already in literatures), that is

$$\Lambda = \{\Lambda(x) = \Lambda_\omega(x), x \in \mathbb{R}^n, \omega \in \Omega\},$$

is a measurable separable mean-square continuous random field on a complete probability space $(\Omega, \mathcal{F}, P)$, such that

$$\mathrm{E}\Lambda(x) = m = const, \quad \mathrm{Var}\Lambda(x) < \infty, \quad \mathrm{Cov}(\Lambda(x), \Lambda(y)) = R_\Lambda(\|x - y\|),$$

where $R_\Lambda(\|x - y\|), (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, is a continuous positive definite kernel, which depends on the Euclidean distance $r = r_{xy} = \|x - y\|$ only.

The Schoenberg theorem implies that a function $R_\Lambda(\|x - y\|)$ is the covariance function of a mean-square continuous homogeneous isotropic random field $\Lambda(x)$, $x \in \mathbb{R}^n$, if and only if there exists a finite measure G on the measurable space $(\mathbb{R}^1_+, \mathfrak{B}(\mathbb{R}^1_+))$ such that

$$R_\Lambda(r) = \int_0^\infty Y_n(ur)\ G(du), \tag{2.1}$$

To define Y_n, let, for $\nu > -\frac{1}{2}$,

$$J_\nu(z) = \sum_{m=0}^\infty (-1)^m (\frac{z}{2})^{2m+\nu} [m!\ \Gamma(m + \nu + 1)]^{-1}, \quad z > 0$$

be the Bessel function of the first kind of order ν, and we define

$$Y_n(z) = 2^{(n-2)/2} \Gamma(\frac{n}{2})\ J_{(n-2)/2}(z)\ z^{(2-n)/2}, \quad z \geq 0, \quad n \geq 2.$$

See, for example, Leonenko (1999), for more details and proofs on the above facts.

The 2-dimensional and the 3-dimensional fields are physically important and the above displays are then more explicit; thus, we list them in the below.

For $n = 2$ formula (2.1) is

$$R_\Lambda(r) = \int_0^\infty J_0(ur)\ G(du).$$

For $n = 3$ formula (2.1) is

$$R_\Lambda(r) = \int_0^\infty \frac{\sin\ ur}{ur}\ G(du).$$

We begin with the following conditions:

$\mathbf{A}'$. Let a given random field $\Lambda = \{\Lambda(x), x \in \mathbb{R}^n\}$ be a HIRF such that

$$\mathrm{E}\Lambda(x) = 1, \mathrm{Var}\Lambda(x) = \sigma^2_\Lambda < \infty, \Lambda(x) > 0, x \in \mathbb{R}^n,$$

$$\mathrm{Cov}(\Lambda(x), \Lambda(y)) = R_\Lambda(\|x - y\|) = \sigma^2_\Lambda \rho_\Lambda(\|x - y\|), \quad \rho(0) = 1,$$

and $\Lambda^{(i)}(x), x \in \mathbb{R}^n$, $i = 0, 1, 2, \ldots$ be a sequence of independent fields re-scaled from Λ by

$$\Lambda^{(i)}(x) \stackrel{d}{=} \Lambda^{(i)}(b^i x),$$

where $b > 1$ is a scaling factor, and $\stackrel{d}{=}$ denotes equality in finite-dimensional distributions. That $b^i x$ denotes the product of a vector x by a scalar b^i.

For the convenience, we will call the field Λ in the above condition $\mathbf{A}'$ to be a *mother field*; referring the sense that Λ generates our scheme presented in the main Theorem 2.1 in the below.

We denote the n-dimensional ($n \geq 2$) unit ball as

$$B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\},$$

and we will use the spherical coordinates for x: $x = (r, e) = re, \quad r > 0, \quad e \in S^{n-1}$; the latter one denotes the unit spherical surface in $\mathbb{R}^n$.

Now, we define a sequence of finite-product fields on B^n, $k = 1, 2, \ldots$, by

$$\Lambda_k(x) = \prod_{i=0}^{k} \Lambda^{(i)}(xb^i), \tag{2.2}$$

and the random measure on Borel σ-algebra $\mathcal{B}$ of a unit ball B^n

$$\mu_k(B) = \int_{y \in B} \Lambda_k(y) dy, \quad k = 0, 1, 2, \ldots, \quad B \in \mathcal{B};$$

using the spherical coordinates, we may write the above as,

$$\mu_k(B) = \int \int_{y=(r,e) \in B} r^{n-1} \Lambda_k(re) dr de, \quad k = 1, 2, \ldots.$$

In the above, we have used the normalized Lebesque measure dy on the ball B^n and the normalized uniform spherical measure de on S^{n-1} (normalized so that the total measure is 1).

We denote

$$\mu_k \to^D \mu, k \to \infty,$$

the weak convergence of the measures μ_k to some measure μ, that is

$$\int_{B^n} f(y) \mu_k(dy) \to \int_{B^n} f(y) \mu(dy), k \to \infty,$$

for all continuous functions $f(y), y \in B^n$.

For a random measure μ, the Rényi function of μ is a *deterministic* function defined as

$$T(q) = \liminf_{m \to \infty} \frac{\log_2 \mathrm{E} \sum_l \mu \left(B_l^{(m)} \right)^q}{\log_2 \left| B_l^{(m)} \right|}$$

where $\{B_l^{(m)}, l, m\}$, $l = 0, 1, \ldots, 2^m - 1$ and $m = 1, 2, \ldots$, denotes the mesh formed by the m-th level dyadic decomposition of the unit ball B^n based on the spherical coordinates (note that this mesh is different from the rectangle-type decomposition; we thanks to the referee to remind this). The notation $|\cdot|$ denotes the normalized Lebesgue measure on $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$; in the above and henceforth, the $\log_a$ denotes the logarithm in the base $a > 0$.

Remark 2.1. *Rényi function plays a central role in multifractal analysis, since the multifractal formalism in the theory of random cascades can be understood in the sense that the Legendre transform of the Rényi function*

$$\mathcal{L}T(z) = \min_q (qz - T(q))$$

would be the dimension spectrum (in the sense of Hausdorff dimension) of the following sets (indexed by α)

$$F_\alpha = \left\{ x \in B^n : \lim_{m\to\infty} \log_2 \mu\left(B_l^{(m)}(x)\right) / \log_2 \left|B_l^{(m)}\right| = \alpha \right\},$$

where $B_l^{(m)}(x)$ is the sequence in the mesh $\{B_l^{(m)}, l, m\}$ that contain x; it shrinks to x as $m \to \infty$.

Our basic scheme can be summarized in the following statement.

Theorem 2.1. *Suppose that the condition $\boldsymbol{A}'$holds.*

(i) Assume that the correlation function $\rho_\Lambda(\|x-y\|) = \rho(r)$ of a mother field Λ satisfies the following condition:

$$|\rho_\Lambda(r)| \le Ce^{-\gamma r}, r > 0, \tag{2.3}$$

for some positive C and γ. Then, when the scaling factor

$$b : b > \sqrt[n]{1+\sigma_\Lambda^2},$$

on B^n the measures

$$\mu_k \to^D \mu, k \to \infty,$$

the random measure μ is non-degenerate, it has the finite second moment: $\mathrm{E}\mu^2(B^n) < \infty$, and it has the stochastic scaling-invariant (or say self-similar) property:

$$\mu(dy) = b^{-n}\Lambda(y)\hat{\mu}(bdy),$$

where the measure $\hat{\mu}(dy)$ is independent of Λ and has the same distribution as $\mu(dy)$.

(ii) Assume that for some range

$$q \in Q = [q_-, q_+],$$

both $\mathrm{E}^q\Lambda(0) < \infty$ and $\mathrm{E}\mu^q(B^n) < \infty$, $B \in \mathcal{B}$, then, the Rényi function $T(q)$ of μ is given by

$$T(q) = q - 1 - \frac{1}{n}\log_b \mathrm{E}\Lambda^q(x), q \in Q, \tag{2.5}$$

Remark 2.2. *Our scheme, suppose n being reduced to be 1, is complementary to that in Mannersalo et al (2002, p. 894); see the remark below the proof of Theorem 2.1. The scheme may be traced back to the "multiplicative chaos" theory of Kahane (1985, 1987), which lays a mathematical foundation on Mandelbrot's cascades; confer to the pioneering paper of Mandelbrot (1974) for the details of the physical theory.*

Remark 2.3. *The range Q in Theorem 2.1 at least contains [1,2]; while to determine the full range of validity for a given scenario is mathematically challenging, even in the classical cascades case; see Kahane (1985,1987).*

The proof of Theorem 2.1 will be given in the Section 4.

Motivated by CMBR problems (see Section 1), we consider the case when the the underlying random field is a 3-dimensional spherical one. Let the spherical surface in $\mathbb{R}^3$ (as a 2-dimensional manifold) with a given radius $r > 0$ be

$$s_2(r) = s(r) = \left\{x \in \mathbb{R}^3 : \|x\| = r\right\} \subset \mathbb{R}^3$$

while the Lebesgue measure (the area element on the sphere)

$$\sigma_r(du) = \sigma_r(d\theta.d\varphi) = r^2 \sin\theta d\theta d\varphi \ , \ (\theta,\varphi) \in s(1) \ , \ r = \|x\| > 0 \ .$$

A spherical random field, denoted by

$$T = \{T(r,\theta,\varphi) : 0 \le \theta \le \pi, \ 0 \le \varphi \le 2\pi, \ r > 0\} \text{ or } T = \left\{\widetilde{T}(x) \ , \ x \in s(r)\right\},$$

is a stochastic function defined on the sphere $s(r)$. We consider a real-valued spherical field T with the mean m, and finite second moments such that it is continuous in the mean-square sense and its (continuous) covariance function depends on the geodesic (or angular) distance between two points on the sphere. Under these conditions, the the isotropic random field on the sphere $s_2(r)$ can be expanded in mean-square sense as a Laplace series

$$T(r,\theta,\varphi) = m + \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^m(\theta,\varphi)a_{lm}(r) \ , \tag{2.1}$$

where $Y_l^m(\theta,\varphi)$ represent the spherical harmonics, i.e.

$$Y_l^m(\theta,\varphi) = c_{lm}\exp(im\varphi)P_l^m(\cos\theta) \ , \ -l \le m \le l \ , \ \ l = 0,1,..., \tag{2.2}$$

where

$$c_{lm} = (-1)^m\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} \ , \ \ -l \le m \le l \ ,$$

and $P_l^m(\cos\theta)$ denotes the associated Legendre polynomial of degree l, m, i.e.

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x) \ , \tag{2.3}$$

where

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

is the Legendre polynomial. The spherical harmonics have the following properties

$$\begin{aligned} \int_0^{\pi}\int_0^{2\pi}\overline{Y_l^m}(\theta,\varphi)Y_{l'}^{m'}(\theta,\varphi)\sin\theta d\theta d\varphi &= \delta_l^{l'}\delta_m^{m'} \ , \\ \overline{Y_l^m}(\theta,\varphi) &= (-1)^m Y_l^{-m}(\theta,\varphi) \ , \\ Y_l^m(\pi-\theta,\varphi+\pi) &= (-1)^l Y_l^m(\theta,\varphi) \ , \end{aligned} \tag{2.4}$$

where $\delta_l^{l'}$ represent the Kronecker delta. The random coefficients in the Laplace series (2.1) can be obtained through inversion arguments in the form of mean-square stochastic integrals

$$\begin{aligned} a_l^m(r) &= \int_0^{\pi}\int_0^{2\pi} T(r,\theta,\varphi)\overline{Y_l^m}(\theta,\varphi)r^2\sin\theta d\theta d\varphi \\ &= \int_{s(1)}\widetilde{T}(ru)\overline{Y_l^m}(u)\sigma_1(du) \ , \ u = \frac{x}{\|x\|} \in s(1) \ , \ r = \|x\| \ , \end{aligned} \tag{2.5}$$

see, for example, Leonenko (1999), for more details.

The field $T(r,\theta,\varphi) = \widetilde{T}(x), x \in \mathbb{R}^3$, is said to be isotropic in $\mathbb{R}^3$, if $\mathrm{E}\widetilde{T}(x)^2 < \infty$, and its first and second order moments are invariant with respect to the group of rotations on the sphere, i.e.

$$\mathrm{E}\widetilde{T}(x) = \mathrm{E}\widetilde{T}(gx), \ \mathrm{E}\widetilde{T}(x)\widetilde{T}(y) = \mathrm{E}\widetilde{T}(gx)\widetilde{T}(gy) \ ,$$

for every $g \in SO(3)$, the group of rotations in $\mathbb{R}^3$. The restriction of an isotropic random field $\widetilde{T}(x)$, $x \in \mathbb{R}^3$ on a sphere $s(r)$ is an isotropic random field on the sphere. This is equivalent to saying that the covariance function $\mathrm{E}T(r,\theta,\varphi)T(r,\theta',\varphi')$ depends only on the angular distance $\theta = \theta_{PQ}$ between the points $P = (r,\theta,\varphi)$ and $Q = (r,\theta',\varphi')$. The restriction of the isotropic field on the sphere is spherical field on $s(r)$ if and only if

$$\mathrm{E}a_l^m(r)\overline{a_{l'}^{m'}}(r) = \delta_l^{l'}\delta_m^{m'}C_l(r) \ , \ -l \le m \le l \ , \ -l' \le m' \le l' \ , \tag{2.6}$$

or

$$\mathrm{E}|a_l^m(r)|^2 = C_l(r) \ , \ m = 0, \pm 1, ..., \pm l \ . \tag{2.7}$$

The functional series $\{C_1(r), C_2(r), ..., C_l(r), ...\}$ is called the angular power spectrum of the isotropic random field $T(r,\theta,\varphi)$. From (2.1), (2.5) and (2.6) we deduce that

$$\text{Cov}(T(r,\theta,\varphi)T(r,\theta',\varphi')) = \frac{1}{4\pi}\sum_{l=1}^{\infty}(2l+1)C_l(r)P_l(\cos\theta) \; , \tag{2.8}$$

where

$$\sum_{l=1}^{\infty}(2l+1)C_l(r) < \infty \; ,$$

for every fixed $r > 0$. If $T(r,\theta,\varphi)$ is an isotropic Gaussian field on the sphere $s(r)$, then the coefficients $a_l^m(r)$ are complex-valued independent Gaussian random processes with

$$\mathrm{E}a_l^m(r) = 0 \; , \; \mathrm{E}a_l^m(r)E\overline{a_{l'}^{m'}}(r) = \delta_m^{m'}\delta_l^{l'}C_l(r).$$

A random field $\widetilde{T}(x), \quad x \in \mathbb{R}^3$, with $\mathrm{E}\widetilde{T}(x)^2 < \infty$, is called homogenous (in the wide sense) if its first two moments are invariant with respect to the Abelian group of shifts in $\mathbb{R}^3$. The restriction of the HIRF $\widetilde{T}(x)$, $x \in \mathbb{R}^3$on the sphere $s(r)$ is an spherical field on $s(r)$ if and only if

$$\mathrm{E}a_l^m(r)\overline{a_{l'}^{m'}}(r) = \delta_l^{l'}\delta_m^{m'}C_l(r,s) \tag{2.9}$$

with

$$C_l(r,s) = 2\pi^2\int_0^{\infty}\frac{J_{l+\frac{1}{2}}(\mu r)}{(\mu r)^{1/2}}\frac{J_{l+\frac{1}{2}}(\mu s)}{(\mu s)^{1/2}}G(d\mu) \; , \tag{2.10}$$

$l = 1, 2, ...$, where G is a finite measure on the Borel sets of $[0,\infty)$ such that

$$\sigma^2 = \text{Var}\left\{\widetilde{T}(0)\right\} = \int_0^{\infty} G(d\mu) < \infty \; ,$$

and $J_\nu(z)$ again is the Bessel function of the first kind of order ν.

The covariance function $\text{Cov}\left\{\widetilde{T}(x), \widetilde{T}(y)\right\}$ of a mean-square continuous HIRF $\widetilde{T}(x), x \in \mathbb{R}^3$, depends only on the Euclidean distance

$$\begin{aligned} \rho &= |x-y| = \sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos\gamma} \; , \\ \cos\gamma &= \frac{<x,y>}{\|x\|\|y\|} \; , \; x = (\rho_1, u_1) \; , \; y = (\rho_2, u_2) \; , \end{aligned}$$

and by the addition theorem for Bessel functions can be represented as

$$\begin{aligned} B(\rho) &= \int_0^{\infty}\frac{\sin(\mu\rho)}{\mu\rho}G(d\mu) \\ &= 2\pi^2\sum_{l=1}^{\infty}\sum_{m=-l}^{l}Y_l^m(u_1)\overline{Y_l^m}(u_2)\int_0^{\infty}\frac{J_{l+\frac{1}{2}}(\mu\rho_1)}{(\mu\rho_1)^{1/2}}\frac{J_{l+\frac{1}{2}}(\mu\rho_2)}{(\mu\rho_2)^{1/2}}G(d\mu) \; . \end{aligned}$$

By Karhunen's Theorem, each mean-square continuous homogenous isotropic spherical random field with zero mean has a spectral representation

$$\widetilde{T}(x) = m + \sum_{l=1}^{\infty}\sum_{m=-l}^{l}Y_l^m(\theta,\varphi)a_l^m(r), \tag{2.11}$$

with

$$a_l^m(r) = \pi\sqrt{2}\int_0^{\infty}\frac{J_{l+\frac{1}{2}}(\mu r)}{\sqrt{(\mu r)}}Z_l^m(d\mu) \; , \tag{2.12}$$

where Z_l^m, $-l \leq m \leq l$, $l = 1, 2, ..$, are the family of complex-valued random measures on Borel sets of $[0, \infty)$ such that

$$\mathrm{E}Z_l^m(A) = 0 \ , \ \mathrm{E}Z_l^m(A)\overline{Z_{l'}^{m'}}(B) = \delta_l^{l'}\delta_m^{m'}G(A \cap B) \tag{2.13}$$

If there exists an isotropic spectral density $g(\mu) \geq 0$ such that

$$\frac{G(d\mu)}{d\mu} = \|s(1)\|\mu^2 g(\mu) \ , \ \mu^2 g(\mu) \in L_1([0, \infty)) \ , \tag{2.14}$$

where $\|s(1)\| = 4\pi$(the area of the unit sphere). Then (2.11) holds with

$$a_l^m(r) = (2\pi)^{3/2} \int_0^\infty \sqrt{\mu} J_{l+\frac{1}{2}}(\mu r)\sqrt{g(\mu)} W_l^m(d\mu) \ , \tag{2.15}$$

where

$$\mathrm{E}W_l^m(A)\overline{W_{l'}^{m'}}(B) = \delta_l^{l'}\delta_m^{m'}|A \cap B| \ ,$$

that is W_l^m, $-l \leq m \leq l$, $l = 1, 2, ...$ is a family of white noise random measures (Gaussian is field is Gaussian). The restriction of an HIRF $\widetilde{T}(x)$, $x \in \mathbb{R}^3$ on a sphere $s(r)$ is an isotropic random field on the sphere. In this particular case, the covariance function of this isotropic field T on $s(r)$ is representable in the form (2.8) with the angular power spectrum

$$C_l(r) = 2\pi^2 \int_0^\infty \frac{J^2_{l+\frac{1}{2}}(\mu r)}{\mu r} G(d\mu) \ , \ l = 1, 2, ... \tag{2.16}$$

or

$$C_l(r) = (2\pi)^3 \int_0^\infty \frac{J^2_{l+\frac{1}{2}}(\mu r)}{\mu r} \mu^2 g(\mu) d\mu, \tag{2.17}$$

whenever (2.14) is satisfied. For the correlation function of HIRF $\widetilde{T}(x)$, $x \in \mathbb{R}^3$of the form $\rho(r) = e^{-ar}, r \geq 0$, the isotropic spectral density $g(\mu) = a(\mu^2 + a^2)^{-\frac{3}{2}}/(2\pi^{\frac{3}{2}})$.

We introduce the following conditions for the spherical random fields on $\mathbb{R}^3$:
A″. Let the mother field $\Lambda = \left\{\tilde{\Lambda}(x), x \in s_2(1)\right\}$, be isotropic random field and the sphere $s_2(1) = \left\{x \in \mathbb{R}^3 : \|x\| = 1\right\} \subset \mathbb{R}^3$ such that

$$\mathrm{E}\tilde{\Lambda}(x) = 1, \mathrm{Var}\tilde{\Lambda}(x) = \sigma_\Lambda^2 < \infty, \Lambda(x) > 0, x \in s_2(1),$$

$$\mathrm{Cov}(\Lambda(\theta, \varphi), \Lambda(\theta', \varphi')) = \frac{1}{4\pi}\sum_{l=1}^\infty (2l+1) C_l P_l(\cos\theta) \ ,$$

$$\sum_{l=1}^\infty (2l+1) C_l < \infty \ ,$$

and $\tilde{\Lambda}^{(i)}(x), x \in s_2(1), i = 0, 1, 2, \ldots$ be a sequence of independent fields on $x = (1, \theta, \varphi) \in s_2(1)$ such that

$$\tilde{\Lambda^{(i)}}(x) \stackrel{d}{=} \tilde{\Lambda^{(i)}}(b^i x),$$

where $b > 1$ is a scaling factor, and we interpret $b^i x$ by $b^i x := (1, b^i \underset{\pi}{\times} \theta, b^i \underset{2\pi}{\times} \varphi) \in s_2(1)$, where the modulus algebra is used accordingly.

Define the finite product fields on $s_2(1)$

$$\tilde{\Lambda}_k(x) = \prod_{i=0}^k \tilde{\Lambda}^{(i)}(b^i x),$$

and the random measure on Borel σ-algebra $\mathcal{B}$ of $s_2(1)$

$$\mu_k(B) = \int_B \Lambda_k(y)dy, \quad k = 0, 1, 2, \ldots, \quad B \in \mathcal{B}, \tag{2.18}$$

in the 3-dimensional spherical coordinate, the above is then in the form

$$\mu_k(B) = \int\int_{x=(1,\theta,\varphi)} \Lambda_k(1, \theta, \varphi) \sin\theta d\theta d\varphi, \quad k = 1, 2, \ldots.$$

We denote

$$\mu_k \to^D \mu, k \to \infty,$$

the weak convergence of the measures μ_k to some non-degenerate measure μ, that is

$$\int_{s_2(1)} f(y)\mu_k(dy) \to \int_{s_2(1)} f(y)\mu(dy), k \to \infty,$$

for all continuous functions $f(y), y \in s_2(1)$.

The Rényi function of μ on $s_2(1)$ is now defined as

$$T(q) = \liminf_{m\to\infty} \frac{\log_2 \mathrm{E} \sum_l \mu\left(S_l^{(m)}\right)^q}{\log_2 \left|S_l^{(m)}\right|}$$

where $\{S_l^{(m)}, l, m\}, l = 0, 1, \ldots, 2^m - 1$, is the mesh formed by the m-th level dyadic decomposition of the spherical surface $s_2(1)$.

We then reformulate the Rényi function $T(q)$ in our basic scheme Theorem 2.1 in the following spherical form. It is a direct consequence of Theorem 2.1 (the $s_2(1)$ is identified as a 2-dimensional surface) and thus the proof is omitted; we refer it as a theorem since its formulation is of individual interest.

Theorem 2.2. *Suppose that the condition $\boldsymbol{A}''$ holds and the isotropic random field is the restriction of the HIRF $\Lambda(x), x \in \mathbb{R}^3$ with correlation function $\rho_\Lambda(\|x - y\|) = \rho(r)$ on the sphere $s_2(1)$. We assume the similar assumptions as those in Theorem 2.1.*

The Rényi function $T(q)$ of the limit measure μ on $s_2(1)$ is given by

$$T(q) = q - 1 - \frac{1}{2}\log_b \mathrm{E}\Lambda^q(t), q \in Q.$$

3. Some important scenarios

In this section we consider several scenarios for multifractal random fields on the n-dimensional Euclidean space $\mathbb{R}^n, n \geq 2$. When n is reduced to be 1, they are consistent with the previous known results of the exponential OU-type stationary processes studied in a series of papers by Anh, Leonenko, and Shieh (2008a,b, 2009a,b, 2010).

Firstly, we provide the the lognormal scenario for multifractal products of HIRF's, which follow the prominent lognormal hypothesis of Kolmogorov (1962) in turbulent cascades. In fact, this lognormal scenario has its origin in Kahane (1985,1987). We reformulate Theorem 2.1 for this special case in order to have a precise scaling law for the moments.

$\mathbf{B}'$. Consider a mother field of the form

$$\Lambda(x) = \exp\left\{Y(x) - \frac{1}{2}\sigma_Y^2\right\},$$

where $Y(x)$, $x \in \mathbb{R}^n$ is a zero-mean Gaussian, measurable, separable random field with covariance function

$$R_Y(r) = \sigma_Y^2 \rho_Y(r), \quad \rho_Y(0) = 1.$$

Let us consider the following more specific case. The Gaussian solution of the stochastic Laplace or stochastic Helmholtz equation studied in Kelbert, Leonenko and Ruiz-Medina (2005); the equation is in the from:

$$\left(\alpha^2 I-\Delta\right)^{\nu/2} Y(x)=\varepsilon(x),\ x\in\mathbb{R}^n,\ n\geq 2,\ \nu>0,$$

where $\varepsilon(x),\quad x\in\mathbb{R}^n$, is white noise with variance σ^2, Δ is the Laplacian and I is identity operator. The authors show that the homogeneous isotropic solution to this equation has the covariance function $R_Y(r)$, which belongs to Matérn class, as one can see Stein (2000, pp. 49-51), and takes the following form:

$$R_Y(r)=\frac{\pi^{n/2}}{\alpha^{2\nu-n}2^{\nu-n/2-1}\Gamma(\nu)}\frac{\sigma^2}{(2\pi)^n}K_{\nu-n/2}(r\alpha)(r\alpha)^{\nu-n/2},\ r>0,\ \alpha>0,\ \nu>n/2,$$

where

$$K_\lambda(x)=\frac{1}{2}\int_0^\infty u^{\lambda-1}e^{-\frac{1}{2}x(u+\frac{1}{u})}du,\ x>0,$$

is the modified Bessel function of the third kind or McDonald function with index λ.For $\lambda=r+1/2$, where r is an nonnegative integer, the Bessel function $K_\lambda(x)$ has the closed form

$$K_{r+1/2}(x)=\sqrt{\frac{\pi}{2x}}e^{-x}\sum_{k=0}^{r}\frac{(r+k)!}{(r-k)!k!}(2x)^{-k}.$$

Thus, for $\nu-\frac{n}{2}=\frac{1}{2}$, (up to constant)

$$R_Y(r)=const\frac{1}{\alpha}e^{-\alpha r},r\geq 0,$$

and condition (2.3) holds; for instance, $\nu=\frac{3}{2}$ for $n=2$, and $\nu=2$ for $n=3$.

Under condition $\mathbf{B}'$, we have the following moment generating functions:

$$\begin{aligned}
M(\zeta) &= \mathrm{E}\exp\left\{\zeta\left(Y(x)-\frac{1}{2}\sigma_Y^2\right)\right\}=e^{\frac{1}{2}\sigma_Y^2(\zeta^2-\zeta)},\quad \zeta\in\mathbb{R}^1,\\
M(\zeta_1,\zeta_2;\|x_1-x_2\|) &= \mathrm{E}\exp\left\{\zeta_1\left(Y(x_1)-\frac{1}{2}\sigma_X^2\right)+\zeta_2\left(Y(x_2)-\frac{1}{2}\sigma_Y^2\right)\right\}\\
&= \exp\left\{\frac{1}{2}\sigma_Y^2\left[\zeta_1^2-\zeta_1+\zeta_2^2-\zeta_2\right]+\zeta_1\zeta_2R_Y(\|x_1-x_2\|)\right\},\quad \zeta_1,\zeta_2\in\mathbb{R}^1,
\end{aligned}$$

It turns out that, in this case,

$$\begin{aligned}
M(1) &= 1;\quad M_\theta(2)=e^{\sigma_Y^2};\quad \sigma_\Lambda^2=e^{\sigma_Y^2}-1;\\
\mathrm{Cov}(\Lambda(x_1),\Lambda(x_2)) &= M(1,1;\|x_1-x_2\|)-1=e^{R_Y(\|x_1-x_2\|)}-1
\end{aligned}$$

and

$$\log_b \mathrm{E}\Lambda(x)^q=\frac{(q^2-q)\sigma_Y^2}{2\log b},\quad q>0.$$

Using Theorem 2.1, we obtain

Theorem 3.1. *Suppose that condition* $\boldsymbol{B}'$ *holds with the correlation function*

$$0<|\rho_Y(r)|\leq Ce^{-\gamma r},r>0,$$

for some positive C *and* γ.

Then, for any $b>\exp\left\{\frac{\sigma_Y^2}{n}\right\}$, *the measures*

$$\mu_k\to^D\mu,k\to\infty,$$

and the Rényi function of measure μ is given by

$$T(q) = q\left(1+\frac{\sigma_Y^2}{2n\log b}\right) - q^2\left(\frac{\sigma_Y^2}{2n\log b}\right) - 1, q \in [1,2].$$

Moreover, for $n = 3$, if $Y(x)$ in the condition $\boldsymbol{B}'$, is a spherical isotropic random field obtained as restriction of the HIRF $Y(x), x \in \mathbb{R}^3$ with correlation function $\rho(\|x-y\|) = \rho_Y(r)$ on the sphere $s_2(1)$, then the random measures (2.18) generated by the spherical fields $\Lambda(x) = \exp\left\{Y(x) - \frac{1}{2}\sigma_Y^2\right\}$, $x \in s_2(1)$, converge weakly to the random measure μ, with the Rényi function

$$T(q) = q\left(1+\frac{\sigma_Y^2}{4\log b}\right) - q^2\left(\frac{\sigma_Y^2}{4\log b}\right) - 1, q \in [1,2].$$

In the theory of turbulence cascades, the log-gamma scenario is known as an alternative to the lognormal scenario; see Saito (1992) for details and discussions. Now, we propose a homogeneous isotropic random field version of the log-gamma scenario. We will use a specific construction of the gamma-correlated random field $Z(x)$, $x \in \mathbb{R}^n$, as that in Leonenko (1999) and the references therein). However, for our present purpose, we will extend the construction in there to allow two parameters $\beta > 0$ and $\lambda > 0$ (in the usual literatures the parameter $\lambda = 1$). Accordingly, there exists a HIRF $Z(x)$, $x \in \mathbb{R}^n$, with given marginal density

$$p(u) = \frac{\lambda^\beta}{\Gamma(\beta)} u^{\beta-1} e^{-\lambda u}, \quad u > 0, \lambda > 0, \beta > 0 \tag{3.1}$$

and fixed correlation function $\gamma = \rho_Z(\tau)$.

The bivariate density function of $Z(x)$ can be constructed via the bilinear expansion

$$p(u,w;\gamma) = p(u)p(w)\left[1 + \sum_{k=1}^{\infty} \gamma^k e_k(u) e_k(w)\right] =$$

$$= \lambda^2 p_0(u\lambda, w\lambda; \gamma), \quad ,0 < \gamma < 1, \tag{3.2}$$

where

$$e_k(u) = e_k^{(\beta)}(u) = L_k^{(\beta-1)}(u) \left\{\frac{k!\Gamma(\beta)}{\Gamma(\beta+k)}\right\}^{1/2}, \quad k = 0,1,2,\ldots,$$

where $L_k^{(\beta)}(u)$ are the generalized Laguerre polynomials of index β for $k \geq 0$, defined as

$$L_k^{(\beta)}(u) = (k!)^{-1} u^{-\beta} e^u \frac{d^k}{du^k}\{e^{-u}u^{\beta+k}\}.$$

One can show that

$$e_0^{(\beta)}(u) \equiv 1, \quad e_1^{(\beta)}(u) = (\beta - u)\,\beta^{-1/2}, \ldots$$

It is known that

$$p_0(u,w;\gamma) = \left(\frac{uw}{\gamma}\right)^{\frac{\beta-1}{2}} \exp\left\{-\frac{u+w}{1-\gamma}\right\} I_{\beta-1}\left(2\frac{\sqrt{u\cdot w\cdot\gamma}}{1-\gamma}\right)\frac{1}{\Gamma(\beta)(1-\gamma)}, \tag{3.3}$$

where

$$I_\nu(z) = \sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^{2k+\nu}\frac{1}{k!\,\Gamma(k+\nu+1)}.$$

being the modified Bessel function of the first kind of order ν. In our case $\gamma = \rho_Z(\tau)$. The existence of gamma-correlated HIRF with marginal density (3.1), bivariate density (3.3) and correlation function $\gamma = \rho_Z(\tau)$, in which $\rho_Z(\tau)$ is a continuous positive-definite function with $\rho_Z(0) = 1$ can be proven by using the maximum entropy principle, see Joe (1997) or Dozzi and Leonenko (2011).

Let $Z(x)$, $x \in \mathbb{R}^n$, be homogeneous isotropic random field with one dimensional densities (3.1) and two-dimensional densities of the form (3.3), and let $\gamma = \gamma(\|x-y\|)$ be a continuous non-negative definite kernel on $\mathbb{R}^n \times \mathbb{R}^n$. Then becomes

$$\mathrm{E}e_k^{(\beta)}(Z(x)) = 0, \quad \mathrm{E}\ e_m^{(\beta)}(Z(x))\ e_k^{(\beta)}(Z(y)) = \delta_m^k\ \gamma^m(\|x-y\|). \tag{3.4}$$

As we shall see below this field can be given constructive for every $\alpha = p/2$, where $p \geq 1$ is integer. So the class of Gamma correlated random fields is not empty. To see this we consider the so-called χ -squared random field:

$$Z_p(x) = \frac{1}{2}(\eta_1^2(x) + \cdots + \eta_p^2(x)), x \in \mathbb{R}^n, \quad p \geq 1, \tag{3.5}$$

where $Y_1(x), \ldots, Y_p(x)$ are independent copies of homogeneous Gaussian field $Y(x)$, with

$$\mathrm{E}Y(x) = 0, \quad \mathrm{E}Y^2(x) = 1, \quad \mathrm{Cov}(Y(0),\ Y(x)) = R(\|x\|), x \in \mathbb{R}^n$$

Let the random variables $(X_1, Y_1), \ldots, (X_p, Y_p)$ be i.i.d. with common standard normal bivariate distribution with correlation coefficient ρ. Then it can be shown that the r.v. $\frac{1}{2}(X_1^2, Y_1^2)$ has the characteristic function

$$\varphi_\beta(t, s, \gamma) = [1 - it - is - t\ s\ (1-\gamma)]^{-\beta}. \tag{3.6}$$

with $\gamma = \rho^2$ and $\beta = \frac{1}{2}$. Consequently, the function (3.6) is the characteristic function of $\frac{1}{2}(X_1^2 + \cdots + X_p^2,\ Y_1^2 + \cdots + Y_p^2)$ with $\beta = p/2$. In what follows, the Gamma correlated random field $Z(x)$, $x \in \mathbb{R}^n$ may be realized for $\beta = p/2$, $\gamma(\|x\|) = R^2(\|x\|)$, as χ-squared random field (3.5). Note that

$$\mathrm{E}Z_p(x) = \frac{p}{2}, \quad \mathrm{Var}Z_p(x) = \frac{p}{2}, \quad \mathrm{Cov}(Z_p(0),\ Z_p(x)) = \frac{p}{2}R^2(\|x\|). \tag{3.7}$$

The one-dimensional and two-dimensional densities of χ-squared random field $\xi_p(x)$ are given by (3.1) and (3.3) respectively with $\beta = p/2$. We obtain

$$\mathrm{E}e_k^{(p/2)}(Z_p(x)))\ e_m^{(p/2)}(Z_p(x)) = \delta_m^k\ R^{2m}(\|x-y\|).$$

Note that the positive definiteness is generally not sufficient for a given function to be the covariance function of a χ-squared random field (3.5), since the covariance function of the χ-squared random field (3.5) must be also nonnegative as it follows from (3.7).
$\mathbf{B}''$. Consider a mother field of the form

$$\Lambda(x) = \exp\left\{Z(x) - c_Z\right\}, \quad c_Z = \log\frac{1}{\left(1-\frac{1}{\lambda}\right)^\beta}, \tag{3.8}$$

where $Z(x), x \in \mathbb{R}^n$ is a gamma-correlated HIRF with marginal density (3.1), bivariate density (3.3) and correlation function $\rho_Z(r)$.

The covariance function of the general gamma-correlated random field then takes the form

$$R_Z\left(r\right) = \frac{\beta}{\lambda^2}\rho_Z\left(r\right).$$

Under condition $\mathbf{B}''$, we obtain the following moment generating function

$$M\left(\zeta\right) = \mathrm{E}\exp\left\{\zeta\left(Z\left(x\right) - c_Z\right)\right\} = \frac{e^{-c_Z\zeta}}{\left(1-\frac{\zeta}{\lambda}\right)^\beta}, \quad \zeta < \lambda, \lambda > 1, \beta \geq 1. \tag{3.9}$$

and the bivariate moment generating function is

$$M\left(\zeta_1, \zeta_2; \|x_1 - x_2\|\right) = \mathrm{E}\exp\left\{\zeta_1\left(Z(x_1) - c_Z\right) + \zeta_2\left(Z(x_2) - c_Z\right)\right\}$$
$$= e^{-c_Z(\zeta_1+\zeta_2)} / \left[1 - \frac{\zeta_1}{\lambda} - \frac{\zeta_2}{\lambda} + \frac{\zeta_1\zeta_2}{\lambda^2}\left(1-\rho_Z\left(\tau\right)\right)\right]^\beta, \quad \zeta_1 < \lambda, \zeta_2 < \lambda, \lambda > 1,$$

Thus, the correlation function of the mother process (3.8) takes the form

$$\rho_{\Lambda}(r) = \left[\frac{e^{-2c_Z}}{\left[1-\frac{2}{\lambda}+\frac{2}{\lambda^2}\left(1-\rho_Z(\tau)\right)\right]^{\beta}} - 1 \right] \Big/ \left[\frac{e^{-2c_Z}}{\left[1-\frac{1}{\lambda}\right]^{\beta}} - 1 \right], \tag{3.10}$$

where c_Z is defined in (3.8). It turns out that, in this case,

$$\log_b \mathrm{E}\Lambda(x)^q = \frac{1}{\log b}\left[-q \log \frac{1}{\left(1-\frac{1}{\lambda}\right)^{\beta}} - \beta \log\left(1-\frac{q}{\lambda}\right) \right],$$

and we can formulate the following

Theorem 3.2. *Suppose that condition* $\boldsymbol{B}''$ *holds and* $\lambda > 2$*, and the correlation function*

$$0 < \rho_Z(r) \le Ce^{-\gamma r}, r > 0,$$

for some positive C *and* γ*.*

Then, for any

$$(\beta, \lambda) \in L_{\beta,\lambda} : b > \left\{ 1 + \frac{\frac{1}{\lambda^2}}{1-\frac{2}{\lambda}} \right\}^{\frac{\beta}{n}} \cap \{\lambda > 2\},$$

the measures

$$\mu_k \to^{D} \mu, k \to \infty,$$

and the Rényi function of μ *is given by*

$$T(q) = q\left(1 + \frac{1}{n\log b}\log\frac{1}{\left(1-\frac{1}{\lambda}\right)^{\beta}}\right) + \left(\frac{\beta}{n\log b}\right)\log\left(1-\frac{q}{\lambda}\right) - 1,$$

where

$$q \in Q = \{0 < q < \lambda, \lambda > 2\} \cap [1, 2] \cap L_{\beta,\lambda}.$$

B‴. Consider a mother field of the form

$$\Lambda(x) = \exp\left\{U(x) - c_U\right\}, x \in \mathbb{R}^n, \tag{3.11}$$

where

$$U(x) = -\frac{1}{Z(x)}, \ x \in \mathbb{R}^n,$$

and $Z(x),\ x \in \mathbb{R}^n$ is a gamma-correlated HIRF with marginal density (3.1), bivariate density (3.3) and correlation function $\rho_Z(r)$.

Note that the field $U(x) = -\frac{1}{Z(x)},\ x \in \mathbb{R}^n$ has the negative inverted gamma marginal density

$$p(u) = \frac{\lambda^{\beta}}{\Gamma(\beta)}(-u)^{-\beta-1}e^{\lambda/u}, u < 0,$$

whose moment generating function is

$$M(\zeta) = \mathrm{E}e^{\zeta U(x)} = \frac{2\lambda^{\beta}}{\Gamma(\beta)}\left(\frac{\zeta}{\lambda}\right)^{\beta/2} K_{\beta}\left(2\sqrt{\lambda}\sqrt{\zeta}\right), \ \zeta > 0, \tag{3.12}$$

where $K_{\lambda}(x)$ is modified Bessel function of the third kind or McDonald function We have then

$$c_U = -\log\frac{\Gamma(\beta)}{2\lambda^{\beta/2}K_{\beta}(2\sqrt{\lambda})}. \tag{3.13}$$

and

$$\log_b \mathrm{E}\Lambda(x)^q = \frac{1}{\log b}\left[-qc_U + \log\frac{2\lambda^{\beta/2}}{\Gamma(\beta)} + \log\{q^{\frac{\beta}{2}}K_{\beta}\left(2\sqrt{q\lambda}\right)\} \right], q > 0. \tag{3.14}$$

Theorem 3.3. *Suppose that condition B''' holds with the correlation function*

$$0 < \rho_Z(r) \leq Ce^{-\gamma r}, r > 0,$$

for some positive C and γ. Then, for any

$$(\beta, \lambda) \in L_{\beta,\lambda} : b > \left\{ \frac{\Gamma(\beta) 2^{\frac{\beta}{2}-1} K_\beta \left(2\sqrt{2\lambda}\right)}{\lambda^{\beta/2} \left[K_\beta \left(2\sqrt{\lambda}\right)\right]^2} \right\}^{\frac{1}{n}},$$

the measures

$$\mu_k \to^D \mu, k \to \infty,$$

and the Rényi function of measure μ is given by

$$T(q) = q(1 + \frac{c_U \log \frac{2\lambda^\beta}{\Gamma(\beta)}}{n \log b}) - \frac{1}{n \log b} \log \left\{ q^{\frac{\beta}{2}} K_\beta \left(2\sqrt{q\lambda}\right) \right\} - (1 + \frac{\log \frac{2\lambda^{\beta/2}}{\Gamma(\beta)}}{n \log b}), q \in Q = [1, 2] \cap L_{\beta,\lambda}.$$

4. Proofs

Proof of Theorem 2.1.

For each Borel $B \subset B^n$, define $Y_k := \int_B \Lambda_k(y) dy$. Then it is seen that $Y_k,\ k = 0, 1, 2, \ldots,$ is a martingale. The assumption (i) asserts it converges both in the mean square and almost surely. Indeed, we have

$$E(Y_k - Y_{k-1})^2 = \iint_{(y,y') \in B \times B} E(\Lambda_k(y)\Lambda_k(y'))E([1 - \Lambda_k(y)][1 - \Lambda_k(y')])dydy',$$

which is equal to

$$\sum_k E(Y_k - Y_{k-1})^2 = \sum_k \iint_{(y,y')} \left[\prod_{i=0}^{k-1} \left(1 + \sigma^2 \rho \left(b^i \|y - y'\|\right)\right) \left(\sigma^2 \rho \left(b^k \|y - y'\|\right)\right) \right] dy\, dy'.$$

We claim that, under the exponential delay assumption on ρ, for $b > \sqrt[n]{1 + \sigma^2}$ the above sum is finite.

Write $\|y - y'\| = r$, and assume the condition (i) that $|\rho(r)| \leq Ce^{-\gamma r}$. For each $k \geq 1$, we have the following estimates:

$$\begin{aligned} &\Big(\prod_{i=0}^{k-1} |1 + \sigma^2 \rho \left(b^i \|y - y'\|\right)| \Big) |\sigma^2 \rho \left(b^k \|y - y'\|\right)| \\ \leq\ & \left(1 + \sigma^2\right)^k \times \left(\sigma^2 \rho \left(b^k \|y - y'\|\right)\right) \\ \leq\ & \left(1 + \sigma^2\right)^k \times \left(\sigma^2 C e^{-\gamma b^k r}\right). \end{aligned}$$

Using the spherical coordinates to integrate over y, y' together with a change of variable $\gamma b^k r = s$, we have

$$\sum_k E(Y_k - Y_{k-1})^2 \leq const \times \sum_k \left(1 + \sigma^2\right)^k \times \Big(\frac{\int_{s=0}^\infty e^{-s} s^{n-1} ds}{(\gamma b^k)^n}\Big),$$

the above sum is finite when $b^n > 1 + \sigma^2$. This asserts the claim.

By martingale L^2 convergence, we see that Y_k converges in mean square to a limit.

Allowing B to vary over the meshes of all dyadic decompositions of B^n in the spherical coordinate, we obtain the limiting measure μ. The measure μ is non-degenerate and $E\mu^2(B^n) < \infty$.

The scaling-invariant property of μ is a consequence of the construction of the μ_k, indeed

$$\mu_k(dy) = \Lambda_k(y)dy \overset{d}{=}$$
$$\Lambda(y)\Lambda(by)\cdots\Lambda(b^k y)dy \overset{d}{=}$$
$$\Lambda(y)\Lambda(y')\Lambda(by')\cdots\Lambda(b^{k-1}y')b^{-n}dy',\ by = y',$$

where $\overset{d}{=}$ means the equality in the distribution. Thus the scaling invariance of μ is a consequence of letting $k \to \infty$; whenever it is permitted to take the limit, as we have shown above.

By the assumption (ii), both $\mathrm{E}\Lambda^q(0) < \infty$ and $\mathrm{E}\mu^q(B^n) < \infty$. Assume for simplicity that $b = 2$ in the following. The scaling-invariant property of μ asserts that

$$\mathrm{E}\mu^q(dy) = 2^{-nq}\mathrm{E}\Lambda^q(0)\mathrm{E}\mu^q(2^n dy).$$

Summing Δ over the class $\mathcal{D}_m$ of all members in the m-th level dyadic decomposition, and making backward recursion, we have

$$\sum_{\Delta\in\mathcal{D}_m} \mathrm{E}\mu^q(\Delta) = \mathrm{E}\mu^q(B^n)2^{nm}2^{-nmq}\left(\mathrm{E}\Lambda^q(0)\right)^m,$$

then, Rényi function has the claimed expression with $b = 2$; we can use base b for any $b > 1$, by using decompositions based on b^{-1}-adic base. $\square$

Remark 4.1. *The above proof on the martingale L^2 convergence, suppose n being reduced to be 1, is complementary to that in Mannersalo et al (2002, pp. 892-3); in which the authors allow the weaker correlation decay (power decay), yet the positivity of the correlation is essential there, and moreover the arguments there are restricted to the 1-parameter (the process) case.*

Proof of Theorem 3.1

We consider the random field

$$\eta_q(x) = \Lambda^q(x) = G_q\left(\tilde{Y}(x)\right),$$

where

$$G_q(u) = \exp\left\{qu\sigma_X - \frac{q}{2}\sigma_X^2\right\},$$

and the Gaussian field $\tilde{Y}(x) = Y(x)/\sigma_X$.has the covariance function $\rho_Y(\|x\|)$.

Let $H_k(u)$, $k = 0, 1, 2, ...,$ be Hermite polynomials with the leading coefficient equal to one, which form a complete system of orthogonal polynomials in the Hilbert space $L_2(\mathbb{R}, \varphi(u)\mathrm{d}u)$, where

$$\varphi(u) = \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}, \quad u \in \mathbb{R}.$$

Then $G_q(u)$ has the expansion

$$\begin{aligned} G_q(u) &= \sum_{k=0}^{\infty}\left(C_k(q)/k!\right)H_k(u), \\ C_k(q) &= \int_{\mathbb{R}} \frac{H_k(u)}{\sqrt{2\pi}} e^{qu\sigma_X - \frac{q}{2}\sigma_y^2 - \frac{u^2}{2}}du, \\ \sum_{t=0}^{\infty} C_k^2(q)/\,k! &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{2\left(q\sigma_X - \frac{q}{2}\sigma_y^2\right) - \frac{u^2}{2}}du < \infty. \end{aligned}$$

Taking into account the well-known formula

$$\mathrm{E}\left(H_k\left(\tilde{Y}(x)\right)H_j\left(\tilde{Y}(y)\right)\right) = k!\delta_k^j R^k(\|x-y\|), \quad k, j \geqslant 1,$$

where δ_k^j is the Kronecker symbol, we obtain

$$R_\Lambda(t,s) = \mathrm{Cov}\left(\Lambda^q(x), \Lambda^q(y)\right) = \sum_{k=1}^{\infty} \left(C_k^2(q)/\, k! \right) \rho_Y^k(\|x-y\|),$$

and

$$Ce^{-\gamma r} C_1^2(q) \le R_\Lambda(r) \leqslant Ce^{-\gamma r} \sum_{k\geqslant 1}^{\infty} C_k^2(q)/\, k!.$$

This completes the proof. □

Proof of Theorem 3.2

Similar to the proof of Theorem 3.1, we can show that the process $Z(x)$ has continuous sample paths. We now consider the process $\Lambda(x)$ as a non-linear transformation of a gamma-correlated process $Z(x)$, that is,

$$\Lambda(t) = G_q\left(Z(x)\right), \quad G_q(u) = \exp\left\{qu - qc_Z\right\}.$$

Let $\{e_k(u)\}_{k=0}^{\infty}$ be generalized Laguerre polynomials; then, for $\lambda > 2$,

$$G_q(u) \in L_2\left((0,\infty), p(u)du\right), q \in Q = Q = \{0 < q < \lambda, \lambda > 2\},$$

and we have the expansion

$$G_q(u) = \sum_{k=0}^{\infty} C_k(q) e_k(u), \quad C_k(q) = \int_0^{\infty} e_k(u) e^{-qc_X} e^{-qu(\lambda-2)} u^{\beta-1} du,$$

where

$$C_k(q) = \frac{e^{-qc_X}\lambda^\beta}{\Gamma(\beta)} \int_0^{\infty} e^{-qu(\lambda-1)} u^{\beta-1} e_k(u) du, \quad k = 0, 1, \ldots$$

and

$$Ce^{-\gamma r} C_1^2(q) \le \sigma_\Lambda^2 \rho_\Lambda(\tau) \le Ce^{-\gamma r} \sum_{k=1}^{\infty} C_k^2(q).$$

This completes the proof. □

Proof of Theorem 3.3

The proof again follows the main steps of that of Theorem 3.1, with necessary modifications. We only need to note that, in this case,

$$\Lambda(x) = G\left(Z\left(x\right)\right),\ G(u) = e^{-\frac{1}{u} - c_U},$$

where $Z(x)$ is a gamma-correlated process defined in A″. Then

$$\begin{aligned}
\mathrm{E}G^2\left(Z(x)\right) &= \frac{\lambda^\beta}{\Gamma(\beta)} \int_0^{\infty} e^{-\frac{2}{x} - 2c_X} x^{\beta-1} e^{-\lambda x} dx \\
&= \frac{\lambda^\beta}{\Gamma(\beta)} e^{-2c_X} \int_0^{\infty} e^{-\sqrt{2\lambda}\left(\frac{x}{\sqrt{2/\lambda}} + \frac{\sqrt{2/\lambda}}{x}\right)} x^{\beta-1} dx \\
&= \frac{e^{-2c_X}\alpha^\beta 2^{\beta/2}}{\Gamma(\beta)} K_\beta\left(\sqrt{2\lambda}\right) < \infty
\end{aligned}$$

for all $\beta > 0,\ \lambda > 0$.

We have again

$$Ce^{-\gamma r} C_1^2(q) \le \sigma_\Lambda^2 \rho_\Lambda(\tau) \le Ce^{-\gamma r} \sum_{k=1}^{\infty} C_k^2(q),$$

where now

$$C_k(q) = \frac{e^{-qc_X}\lambda^\beta}{\Gamma(\beta)} \int_0^\infty e^{-q(\frac{1}{u}-c_U)} u^{\beta-1} e_k(u) du.$$

This completes the proof. □

5. Conclusion and Perspective

In this paper, we address the Rényi function, which is a central core of Multifractal Analysis, for the infinite-products generated by several random fields on the multi-dimensional, 3-dimensional in particular, sphere. The scenarios for the random fields presented in this paper are log-normal, log-gamma, and log-negative-inverted-gamma.

The main motivation of this paper is to provide scenarios which may test the possible *non-Gaussianity* for the statistical distribution of Cosmic Microwave Background Radiation(CMBR); a challenging issue of cosmology presented for example in the review paper by Marinucci (2004) and the monograph by Marinucci and Peccati (2011, Chapter 1, §1.2).

There are statistical calibrations which are not able to present in this paper. The Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) data released at December 20, 2012 (arXiv:1212.5226) could be an extremely valuable database for future statistical testing of the models.

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School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF24 4AG, UK. E-mail: LeonenkoN@Cardiff.ac.uk

Department of Mathematics, Chinese University of Hong Kong, Shatin N.T. China. On leave from National Taiwan University. E-mail: shiehnr@ntu.edu.tw URL: http://www.math.ntu.edu.tw/~shiehnr/