On the Exponential Process associated with a CARMA-type Process

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Abstract

We study the correlation decay and the expected maximal increments of the exponential processes determined by continuous-time autoregressive moving average type processes of order (p, q). We consider two background driving processes, namely fractional Brownian motions and Lévy processes with exponential moments. The results presented in this paper are significant extensions of those very recent works on the Ornstein-Uhlenbeck-type case $(p = 1, q = 0)$, and we develop more refined techniques to meet the general (p, q). In a concluding section, we discuss the perspective role of exponential CARMA-type processes in stochastic modeling of the burst phenomena in telecommunications and the leverage effect in financial econometrics.

Keywords: Continuous-time ARMA process, Fractional Brownian motion, Lévy process, Exponential process, Correlation decay, Maximal increments.

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1 Introduction

Empirically observed data from financial markets usually are of high-frequency, and those from environmental sciences (hydrology and geology in particular) usually are irregularly spaced. Moreover, they exhibit characteristics of long-memory and/or non-Gaussianity. Therefore, the traditional ARMA or ARIMA time-series models must be treated in the continuous-time setting and to reflect the above characteristics, and various modelings have been proposed and studied in detail; see recent papers by [26, 27, 28, 29] for FBM driven models and [7, 8, 9] for Lévy driven models, and the references therein. In this paper, our model is based on Continuous-time Autoregressive-Moving-Average type (CARMA-type, for brevity) processes driven by fractional Brownian Motions and by Lévy processes, as those papers cited above. We shall study the exponential processes associated with these CARMA-type processes.

We mention that, in recent years, it has been of great interest to study the exponential functionals and the exponential processes determined by Brownian motion and by Lévy processes, see [10, 11, 17], with the view toward application in financial economics. There also appear papers by [2, 21] to study the exponential processes associated with Ornstein-Uhlenbeck-type (OU-type for short) processes driven by fractional Brownian Motions and by Lévy processes, which are motivated

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by the connection to Kahane-Mandelbrot theory of turbulence. Furthermore, these exponential processes play roles in the context of stochastic volatility modelling (see, for example, [22, Chapter 11]). Indeed, the exponential CARMA-type processes driven by Lévy processes have been used very recently to study the leverage effect between the stock return and the volatility of continuous-time financial data, [16].

We study two fundamentally important properties of such an exponential processes Z. We study the correlation decay

$$\mathrm{Cov}\left(Z(t), Z(t+s)\right) \text{ as } s \uparrow \infty,$$

and estimate the upper bound for the expected maximal increments

$$E\left[\max_{t\leq s\leq t+r} |Z(s) - Z(t)|\right] \text{ as } r \downarrow 0.$$

The first result is useful to understand the spectral structure of the process. The second result is of intrinsic importance to the path variation (hence toward various applications) of the process.

In Section 2, we present some preliminaries. In Sections 3 and 4, we state the main results, respectively, on the CARMA processes driven by a fractional Brownian motion and by a Lévy process. We present all proofs of our results in Section 5. In a concluding Section 6, we discuss the advantage of our exponential CARMA-type process on stochastic modeling of the burst phenomena in telecommunications and the leverage effect in financial econometrics.

Finally, we should mention that, though the CARMA-type process given in Section 2 below may lead one to feel that it could be a superposition of OU-type processes. This is *not* true, since the superpositions proposed in, say, [3, 5] mean to a sum of independent OU-type processes, while the expression in Section 2 below for a CARMA-type process is a sum of the same driving force and the parameters λ_i in the expression are complex-valued. Therefore, the existing techniques for superposed OU-type processes are not applicable to the CARMA-type processes discussed in this paper, and we develop more refined techniques to prove the main results of this paper.

2 Preliminaries

2.1 Fractional Brownian motions and Lévy processes

We begin with a review on the definitions and properties of two background driving processes, fractional Brownian motion (*FBM* for short) and Lévy process (*LP* for short).

Definition 2.1 *Let* $0 < H < 1$. *A fractional Brownian motion* $\{B^H(t)\}_{t\in\mathbb{R}}$ *is a real-valued centered Gaussian process, with* $B^H(0) = 0$ *and* $\mathrm{Cov}\left(B^H(s), B^H(t)\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),\ (t,s) \in \mathbb{R}^2$.

It is well known that *FBM* has stationary increments and self-similarity with index H. We may and will assume that the sample path of a *FBM* is everywhere continuous. For more on *FBM* we refer to e.g. [15].

Definition 2.2 *A real-valued process* $L := \{L(t)\}_{t\geq 0}$ *with* $L(0) = 0$ *is a Lévy process on* $\mathbb{R}$, *if*
(i) *For any choice of* $n \geq 1$ *and* $0 \leq t_0 < t_1 \cdots < t_n$, *the random variables* $L(t_0), L(t_1) - L(t_0), L(t_2) - L(t_1), \ldots, L(t_n) - L(t_{n-1})$ *are independent.*
(ii) $L(t) - L(s)$ *has the same distribution as* $L(t-s), 0 \leq s < t < \infty$, *and*
(iii) L *is continuous in probability.*

We refer to [25] for intensive study on Lévy processes. We may and will assume that the sample path of a LP is everywhere right-continuous ($L(t+) = L(t)$) with left-limits $L(t-)$, and we may also assume that it is quasi-left-continuous (see [25, p.279]).

It is well-known that a LP is characterized by its generating triplet (b, a^2, ν), where $b \in \mathbb{R}$ is the drift part, $a^2 \geq 0$ is the variance of the Gaussian part, and ν is the Lévy measure of the jump part, i.e. it is a σ-finite measure on $\mathbb{R} \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}\setminus\{0\}} (1 \wedge |x|^2)\nu(dx) < \infty.$$

The characteristic function of $L(1)$ is determined as

$$\xi(\theta) := \log E[e^{i\theta L(1)}] = ib\theta - \frac{a^2}{2}\theta^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{i\theta x} - 1 - i\theta x 1_{\{|x|\leq 1\}})\nu(dx), \quad \theta \in \mathbb{R}, \tag{2.1}$$

and that for $L(t)$ is given as $e^{t\xi(\theta)}$. Hence we know the distribution of $L(t)$ for any $t \geq 0$.

The following form of the Lévy-Itô decomposition is written as [1, p.108],

$$L(t) = bt + aB(t) + \int_{|x|<1} x\tilde{N}([0,t], dx) + \int_{|x|\geq 1} xN([0,t], dx),$$

where B is the standard Brownian motion on $\mathbb{R}$, and $N(ds, dx)$ (resp. $\tilde{N}(ds, dx)$) is the Poisson (resp. compensated Poisson) random measure on $\mathbb{R}_+ \bigotimes (\mathbb{R}\setminus\{0\})$ with intensity measure $ds \bigotimes \nu(dx)$.

In this paper, since the second moment is essential, we assume that $E[(L(1))^2] < \infty$ and is scaled so that $\mathrm{Var}(L(1)) = 1$.

In order to obtain causal stationarity CARMA processes we define a two-sided (in time) Lévy process

$$L(t) = \begin{cases} L^1(t) & \text{if } t \geq 0 \\ L^2((-t)-) & \text{if } t < 0, \end{cases} \tag{2.2}$$

where $\{L^1(t)\}_{t\geq 0}$ and $\{L^2(t)\}_{t\geq 0}$ are independent copies of $\{L(t)\}_{t\geq 0}$.

2.2 CARMA processes driven by *FBM* and by *LP*

Now we define a CARMA-type process $Y := Y(p, q)$ as follows. We use similar notations as those in [8] and in [26]. The following framework of CARMA processes are the same for both two cases. Let $0 \leq q < p$, the process $Y(p, q)$ is represented by observation and state equations;

$$Y(t) = \beta' X(t), \quad t \geq 0, \tag{2.3}$$

$$dX(t) = AX(t)dt + \delta_p dW(t), \tag{2.4}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_p \end{bmatrix}, \quad X(t) = \begin{bmatrix} X^{(0)}(t) \\ X^{(1)}(t) \\ \vdots \\ X^{(p-2)}(t) \\ X^{(p-1)}(t) \end{bmatrix}, \quad \delta_p = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 \\ \beta_1 \\ \vdots \\ \beta_{p-2} \\ \beta_{p-1} \end{bmatrix}$$

and the coefficients β_j satisfy $\beta_j = 0$ for $j > q$.

In the above, $W := \{W(t)\}_{t\in\mathbb{R}}$ is the driving process, which is *BM* σB ($\sigma > 0$ and B is the standard *BM*) in the classical context. In this work, we will firstly consider W to be *FBM* and secondly consider W to be *LP*. The first one is Gaussian with long-range dependence, in case $1/2 < H$, and the second one is non-Gaussian Markovian.

When we take the driving process to be *FBM* σB^H ($\sigma > 0$ and B^H is the *FBM* defined above), we write $Y := Y(p, H, q)$ to be the associated CARMA process, which has appeared in [26], yet without loss of essential significance we skip the drift term appeared in [26, (3)]. The solution of (2.4) can be written as

$$X(t) = e^{At}X(0) + \sigma \int_0^t e^{A(t-u)}\delta_p dB^H(u),$$

where $e^{At} = I + \sum_{n=1}^{\infty}\left\{(At)^n(n!)^{-1}\right\}$ and I is the identity matrix. Here we confine to the causal stationary solution ([26, p.182]). If the eigenvalues of A all have negative real parts, a strictly stationary solution of (2.4) is given by the form;

$$X(t) = \sigma \int_{-\infty}^t e^{A(t-u)}\delta_p dB^H(u), \quad t \geq 0.$$

This solution can be extended over all real t in natural way. Thus the CARMA-type process $Y := Y(p, H, q)$ has the moving-average form:

$$Y(t) = \sigma \int_{-\infty}^t g(t-u)dB^H(u), \quad t \in \mathbb{R}, \tag{2.5}$$

where

$$g(t) = \begin{cases} \beta' e^{At}\delta_p & \text{if } t \geq 0 \\ 0 & otherwise. \end{cases} \tag{2.6}$$

The kernel g here is given in the closed form and for our purpose it is desirable to get a more tractable expression. In [8, p.483] it is shown that, when all eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)$ of A have negative real parts and are all distinct, g has the form

$$g(t) = \sum_{i=1}^p \frac{\beta(\lambda_i)}{\alpha^{(1)}(\lambda_i)} e^{\lambda_i t}, \tag{2.7}$$

where $\alpha(z) = z^p - \alpha_p z^{p-1} - \cdots - \alpha_1$, $\alpha^{(1)}(\cdot)$ denotes its first derivative, $\beta(z) = 1 + \beta_1 z + \beta_2 z^2 + \cdots + \beta_q z^q$. Consequently, $Y(t)$ can be written as

$$Y(t) = \sigma \sum_{i=1}^p \frac{\beta(\lambda_i)}{\alpha^{(1)}(\lambda_i)} \int_{-\infty}^t e^{\lambda_i(t-u)}dB^H(u). \tag{2.8}$$

In this form one can see the clear difference between CARMA-type processes and superposition of OU-type processes. A more general expression for (2.6) is obtained in [29] (see also [8]), and (2.7) is the most explicit case. In this paper we work in the form (2.8), namely we always assume the condition below.

Assumption 2.1 *All eigenvalues of $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)$ of A have negative real parts and are all distinct, and hence the kernel g in (2.6) has expression (2.7).*

The autocovariance function of Y, which characterizes the memory property of the process, is shown in [26, Theorem 2]. Under Assumption 2.1, for $h \geq 0$, the autocovariance function of Y is equal to

$$\begin{aligned} \gamma_Y(h) &:= \operatorname{Cov}(Y(t+h), Y(t)) \\ &= \frac{\sigma^2}{2}\Gamma(2H+1)\sum_{i=1}^{p}\frac{\beta(\lambda_i)\beta(-\lambda_i)}{\alpha^{(1)}(\lambda_i)\alpha(-\lambda_i)}u(H,\lambda_i,h), \end{aligned} \tag{2.9}$$

where

$$u(H,\lambda,h) = 2(-\lambda)^{1-2H}\cosh(\lambda h) + \lambda^{1-2H}e^{\lambda h}P(2H,\lambda h) - (-\lambda)^{1-2H}e^{-\lambda h}P(2H,-\lambda h)$$

and $P(a,z) = \int_0^z e^{-u}u^{a-1}du/\Gamma(a), z \in \mathbb{C}$. For $H \neq 1/2$ as $h \to \infty$, we have the asymptotic behavior

$$\gamma_Y(h) = \sigma^2 H(2H-1)\frac{\beta^2(0)}{\alpha^2(0)}h^{2H-2} + O(h^{2H-3}). \tag{2.10}$$

This shows that $Y(p,H,q)$ with $H \in (\frac{1}{2},1)$ exhibits long-range dependence. When $H \in (\frac{1}{2},1)$ the autocovariance function of Y has another expression

$$\gamma_Y(h) = \sigma^2 H(2H-1)\int_{-\infty}^{0}\int_{-\infty}^{h} g(h-u)g(-v)|u-v|^{2H-2}dudv, \tag{2.11}$$

in which we easily see the sign of $\gamma_Y(h)$. Finally, we note that the $Y(1,H,0)$ process is the fractional Ornstein-Uhlenbeck (*FOU* for short) process,

$$Y(t) = \sigma\int_{-\infty}^{t} e^{\alpha_1(t-u)}dB^H(u),$$

and its autocovariance function is previously known (e.g. [12]) to be

$$\gamma_Y(h) = \sigma^2 c_H\int_{-\infty}^{\infty} e^{ihx}\frac{|x|^{1-2H}}{\alpha_1^2+x^2}dx, \quad c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}.$$

Thus we see again that CARMA process $Y(p,H,q)$ is a natural extension of *FOU* process.

Next, we replace the driving *FBM* σB^H in the above construction by a *LP* L to obtain a corresponding CARMA-type process $Y := Y(p,L,q)$ driven by L. Recall we suppose the second moment for L and $\operatorname{Var}(L(1)) = 1$. We remark that a Lévy driven CARMA-type process has been studied in [7, 8].

We again suppose Assumption 2.1. Then, similar to the *FBM* case, the process $\{X(t)\}_{t\in\mathbb{R}}$ defined by $X(t) = \int_{-\infty}^{t} e^{A(t-u)}\delta_p dL(u)$, $t \in \mathbb{R}$, is the causal strictly stationary solution of (2.4) with W now taken to be L, and the corresponding CARMA process is

$$Y(t) = \int_{-\infty}^{t} g(t-u)dL(u), \tag{2.12}$$

where $g(t)$ is given by (2.6). The following expression which we use later is useful, since $X(0)$ is independent of $\{L(t)\}_{t\geq 0}$,

$$Y(t) = \beta' e^{At}X(0) + \int_0^t \beta' e^{A(t-u)}\delta_p dL(u). \tag{2.13}$$

Under the Assumption 2.1 we have the expression (2.7) for the kernel g, which yields a corresponding expression, as that in the FBM-driven case, we have

$$Y(t) = \sum_{i=1}^{p} \frac{\beta(\lambda_i)}{\alpha^{(1)}(\lambda_i)} \int_{-\infty}^{t} e^{\lambda_i(t-u)} dL(u).$$

It is immediate that $EY(t) = -\beta' A^{-1}\delta_p E[L(1)]$, and due to [8, Remark 5] the autocovariance function has a form,

$$\gamma_Y(h) = \sum_{i=1}^{p} \frac{\beta(\lambda_i)\beta(-\lambda_i)}{\alpha^{(1)}(\lambda_i)\alpha(-\lambda_i)} e^{\lambda_i |h|}. \tag{2.14}$$

We thus see that the Lévy driven CARMA process exhibits short memory. Note that, despite a Lévy driven CARMA process is different from the superposition of p independent Lévy driven OU processes, as seen in (2.14) the correlation structures nevertheless look similar.

Since Y is an integral with a Lévy process, the distribution of $Y(t)$ is given by the cumulant generating function;

$$\log E[e^{i\theta Y(t)}] = \int_0^\infty \xi(\theta g(u)) du,$$

where ξ is given in (2.1).

The purpose of this paper is to study the exponential process associated with Y,

$$Z := \{Z(t)\}_{t\in\mathbb{R}} := \{e^{Y(t)}\}_{t\in\mathbb{R}},$$

where Y is $Y(p, H, q)$ and $Y(p, L, q)$. For the OU-type case, i.e. $Y(1, H, 0)$ and $Y(1, L, 0)$, the associated process is investigated very recently in [21] and, respectively, in [2]. The present work is a certain continuation of these two papers, yet we need to develop more refined techniques to meet the general (p, q).

Finally, we mention that, upon various modelling needs, for example [7, 22, 28, 29], we are led to consider the situation: the kernel g is *non-negative*, and/or the correlations for the exponential process Z are *non-negative*. For instance, in stochastic volatility modelling, the volatility processes are known to be clustering, which suggests positivity of their autocorrelation functions. Thus we also impose

Assumption 2.2 *Assume that the kernel g defined in (2.6) is non-negative.*

A condition for g to be non-negative is given in [28, 29]. In our case, Assumption 2.2 implies non-negativity for the correlations of our exponential processes, as we show in the following sections.

3 The main result I: the FBM-driven case

Recall that we assume Assumptions 2.1 throughout. Firstly we study the correlation decay of the exponential processes.

Lemma 3.1 *Let $Z(t) := e^{Y(t)}$ be the exponential process determined by a stationary Gaussian process $Y(t)$. Then*

$$\mathrm{Cov}(Z(0), Z(s)) \gtreqless 0 \quad \textit{if and only if} \quad \mathrm{Cov}(Y(0), Y(s)) \gtreqless 0.$$

Moreover, assume that as $s \to \infty$, $\mathrm{Cov}(Y(0), Y(s)) \to 0$. Then

$$\mathrm{Cov}(Z(0), Z(s)) = \exp\left[\mathrm{Var}(Y(0))\right] \left\{\mathrm{Cov}(Y(0), Y(s)) + o(\mathrm{Cov}(Y(0), Y(s)))\right\}. \tag{3.1}$$

Due to the stationarity the same result holds for $\mathrm{Cov}(Z(t), Z(t+s))$, $t \in \mathbb{R}$ in Lemma 3.1. Now one can see that Assumption 2.2 implies non-negative correlation of the process Z since by (2.11) the correlation of Y is non-negative under this assumption. The following corollary is immediate from Lemma 3.1 with the covariance decay (2.10) for $Y(p, H, q)$.

Corollary 3.1 *Let $Y := Y(p, H, q)$ process with $H \in (\frac{1}{2}, 1)$. Assume 2.1 and 2.2, then the stationary process $Z(t) := e^{Y(t)}$ has the following covariance structure as $s \to \infty$;*

$$\begin{aligned} 0 \le \mathrm{Cov}(Z(t), Z(t+s)) &= \exp(\gamma_Y(0)) \left\{ \gamma_Y(s) + O((\gamma_Y(s))^2) \right\} \\ &= \exp(\gamma_Y(0)) \left\{ \sigma^2 H(2H-1) \frac{\beta^2(0)}{\alpha^2(0)} s^{2H-2} + O(s^{2H-3}) \right\}, \end{aligned}$$

where $\gamma_Y(\cdot)$ is the autocovariance function of Y.

Remark 3.1 Assumption 2.2 is only used to assure the non-negativity for the autocorrelation function of Z; indeed, if we only concern the correlation decay in Lemma 3.1 and Corollary 3.1, then this assumption can be dropped.

Next, we estimate the expected maximum increments of the process Z associated with $Y(p, H, q)$. We need two lemmas. The first one is Lemma 2.3 of [21], which is based on Statement 4.8 of [23]. The second one requires new proof.

Lemma 3.2 *Let $H \in [\frac{1}{2}, 1)$. Then for any $r \ge 0$ and $t \in \mathbb{R}$ we have*

$$E\left[\left(\max_{t \le s \le t+r} |B^H(s)|\right)^m\right] \le \begin{cases} r^{Hm} \frac{2\sqrt{2}}{\sqrt{\pi}} (m-1)!! & \text{if } m \text{ is odd} \\ r^{Hm} 2(m-1)!! & \text{if } m \text{ is even.} \end{cases} \tag{3.2}$$

Lemma 3.3 *Let $H \in [\frac{1}{2}, 1)$, $m = 1, 2, \ldots$ and $Y := Y(p, H, q)$, and assume 2.1. Then the stationary process Y has the following bound for m-th moment of maximal increments for all $r \ge 0$*

$$\frac{E[\max_{0 \le s \le r} |Y(s) - Y(0)|^m]}{m!} \le C(\boldsymbol{\lambda}) r^{Hm}, \tag{3.3}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ are eigenvalues of A and $C(\boldsymbol{\lambda})$ is a constant, which depends on $\boldsymbol{\lambda}$ and does not depend on any m or H.

Now, we state the harder result in this section as follows.

Theorem 3.1 *Let $H \in [\frac{1}{2}, 1)$ and $Y := Y(p, H, q)$. Define the stationary process $Z(t) := e^{Y(t)}$. Then there exists a constant $\overline{C}(H, \boldsymbol{\lambda})$ such that for all r with $r^H < 1/2$ and all $t \ge 0$,*

$$E\left[\max_{t \le s \le t+r} |Z(s) - Z(t)|\right] \le \overline{C}(H, \boldsymbol{\lambda}) r^H. \tag{3.4}$$

Here $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ are eigenvalues of A and $\overline{C}(H, \boldsymbol{\lambda})$ depends on H and $\boldsymbol{\lambda}$.

Remark 3.2 To our knowledge, the maximal inequality in the above is new even when the driving process is Brownian motion($H = 1/2$).

4 The main result II: the Lévy-driven case

We begin with the correlation decay. Under Assumption 2.1, the CARMA process $Y := Y(p, L, q)$ is strictly stationary and has causal stationary representations (2.12), as that mentioned in Section 2.

We assume that the Lévy process L given in (2.1) *has no Gaussian part*, i.e. $a^2 = 0$, and that we denote the drift part by bt and the Poisson random measure of the jump part by $N(ds, dx)$ for which the intensity measure is $ds \bigotimes \nu(dx)$. From now on, we always use γ to denote the following *finite and positive* quantity,

$$\gamma := p \sup_{i \in \{1,\dots,p\}} \left| \frac{\beta(\lambda_i)}{\alpha^{(1)}(\lambda_i)} \right|,$$

by which we bound g as $0 \le g(u) \le \gamma$, $\forall u \ge 0$.

We start with two lemmas; the second one is the moment generating function of Y, namely the moment of $Z = e^Y$.

Lemma 4.1 *Let g the kernel given in* (2.7)*. For every $t > 0$ it follows that*

$$\int_{-\infty}^{0} g(t-u)du \le M_1 e^{-M_2 t}, \qquad \int_{-\infty}^{0} g^2(t-u)du \le M_3 e^{-M_4 t},$$
$$\int_{0}^{t} g(t-u)du \le M_5, \qquad \int_{0}^{t} g^2(t-u)du \le M_6,$$

where $M_i,\ i = 1, \dots, 6$ are some positive constants.

Lemma 4.2 *Let $Y := Y(p, L, q)$ and the kernel g as in* (2.7)*. Assume 2.1 and 2.2, and suppose, for a given $\theta > 0$, that*

$$\int_{x>1} e^{\theta\gamma x} x \nu(dx) < \infty \quad \textit{and} \quad \int_{x<-1} |x| \nu(dx) < \infty. \tag{4.1}$$

Then, for each $t > 0$ we have

$$\begin{aligned} &E[e^{\theta Y(t)}] = E[(Z(t))^\theta] \\ &= \exp\left\{ \theta b \int_{-\infty}^{t} g(t-s)ds \right\} \\ &\times \exp\left\{ \int_{-\infty}^{t} \int_{\mathbb{R}} \left[\exp(\theta g(t-s)x) - 1 - \theta g(t-s)x 1_{\{|x| \le 1\}} \right] \nu(dx)ds \right\}. \end{aligned} \tag{4.2}$$

We state the covariance decay of the exponential process Z as follows.

Proposition 4.1 *Let $Z := \{Z(t)\}_{t\ge0} := \{e^{Y(t)}\}_{t\ge0}$ be the exponential process associated with the CARMA process $Y(p, L, q)$. Assume 2.1 and 2.2, and suppose that the Lévy measure ν of L satisfies*

$$\int_{x>1} e^{2\gamma x} x \nu(dx) < \infty \quad \textit{and} \quad \int_{x<-1} x^2 \nu(dx) < \infty. \tag{4.3}$$

Then, there exist positive constants c_0 and c_1 such that

$$0 \le \mathrm{Cov}(Z(0), Z(s)) \le c_0 e^{-c_1 s} \tag{4.4}$$

for all $s > 0$.

Due to the stationarity the same result holds for $\mathrm{Cov}(Z(t), Z(t+s)), t \in \mathbb{R}$. At here the assumption on the Lévy measure is stronger than that in Lemma 4.2; yet, this is needed for the existence of $E[Z(0)Z(s)]$.

Remark 4.1 (1). Proposition 4.1 extends the previous OU-type case ($p = 1, q = 0$) [2, Proposition 1] to the general CARMA-type, and its proof in Section 5 also corrects the previous flaw. Moreover, it now only requires the exponential decay of the Lévy measure $\nu(dx)$ on $x > 1$ and the second moment on $x < -1$.
(2). In [16, Remark 5.1], the authors also obtained the exponential decay of the covariance function. However, their condition on the Lévy measure (in their Propositions 4.3 and 4.4) is stronger than ours, and we are able to show the *positivity* of the covariance function which is not achieved by their method. The positivity of the covariance function is important in both the applications to the telecommunications (see Section 6 below) and to the volatility clustering (see, for example, [22]).
(3). If we replace (4.1) and (4.3) with a stronger condition, namely suitable exponential decay of the Lévy measure $\nu(dx)$ as x tends to both $\pm\infty$, then Proposition 4.1 can be valid without Assumption 2.2.
(4). The covariance decay of $Z = e^Y$ is closely related with quantity

$$U(\theta_1, \theta_2; t) = E[e^{i(\theta_1 Y(t)+\theta_2 Y(0))}] - E[e^{i\theta_1 Y(t)}]E[e^{i\theta_1 Y(0)}], \quad -\infty < \theta_1, \theta_2 < \infty,$$

in [24, p.212 and pp.580-581], which characterizes the mixing property of any stationary infinitely divisible process. Thus, we can apply and assert that $Y(p, L, q)$ is mixing, suppose that the exponential integrability imposed on the Lévy measure $\nu(dx)$ is valid for whole range θ. In this aspect, we mention that: in [16, Theorem 3.1(i)] the authors prove, by a suitable application of [20, Theorem 4.3], that both Y and Z can be strongly mixing, under somewhat different conditions.

Now we present the harder estimate, the expected maximum increments of the exponential process $Z := e^Y$.

Theorem 4.1 *Let $Y := Y(p, L, q)$ and the kernel g be as in (2.6). Assume for a given $\theta > 0$, that*

$$\int_{x>1} e^{2\theta\gamma x} x\nu(dx) < \infty \quad \text{and} \quad \int_{x<-1} x^2 \nu(dx) < \infty. \tag{4.5}$$

Then, for $t_0 > 0$ small enough, there exists a positive constant $\overline{C}$ such that

$$E\left[\max_{0\le t\le t_0} |(Z(t))^\theta - (Z(0))^\theta|\right] \le \overline{C}\sqrt{t_0}.$$

Here $\overline{C}$ depends on the kernel g and Lévy measure $\nu(dx)$ and drift parameter b.

Remark 4.2 1. As one may see from the proof given in the next Section 5, what we require on the kernel g for Theorem 4.1 is that $|g(u)| \le \gamma, \forall u \ge 0$ (may also allow g to be $\pm$). This can be achieved, for example, when $Re(\lambda_j) < 0$ (yet not necessarily all λ_j are distinct), as one may see in a general expression for g given in [29]).
2. Theorem 4.1 is still valid when the Lévy triplet has Gaussian part; we simply combine Theorems 3.1 (with $H = 1/2$) and 4.1 and use the Lévy-Itô decomposition theorem. We leave it to readers for the detail.

5 Proofs

Proof of Lemma 3.1.
Since the distribution of $(Y(0), Y(s))$ is a bivariate Gaussian, its moment generating function taken at 1 yields

$$\begin{aligned} E[Z(0)Z(s)] &= E[e^{Y(0)+Y(s)}] \\ &= \exp\left\{\frac{1}{2}(1,1)\Sigma(1,1)'\right\}, \end{aligned}$$

where

$$\Sigma = \begin{pmatrix} \mathrm{Var}\,(Y(0)) & \mathrm{Cov}\,(Y(0), Y(s)) \\ \mathrm{Cov}\,(Y(0), Y(s)) & \mathrm{Var}\,(Y(s)) \end{pmatrix}.$$

Then by using $\mathrm{Cov}\,(Y(0), Y(s)) \to 0$ as $s \to \infty$ and $e^x = \sum_{n=0}^{\infty} x^n/n!$ for $|x| < \infty$, we have

$$\begin{aligned} &\mathrm{Cov}\,(Z(0), Z(s)) \\ &= \exp\{\mathrm{Var}\,(Y(0))\}\,[\exp\{\mathrm{Cov}(Y(0), Y(s))\} - 1] \qquad (5.1) \\ &= \exp\{\mathrm{Var}\,(Y(0))\}\left[\mathrm{Cov}(Y(0), Y(s)) + \frac{(\mathrm{Cov}\,(Y(0), Y(s)))^2}{2!} + \frac{(\mathrm{Cov}\,(Y(0), Y(s)))^3}{3!} + \cdots\right] \\ &= \exp\{\mathrm{Var}\,(Y(0))\}\left[\mathrm{Cov}(Y(0), Y(s)) + o\left(\{\mathrm{Cov}(Y(0), Y(s))\}^2\right)\right]. \end{aligned}$$

□

Proof of Lemma 3.3.
Without loss of generality set $\sigma = 1$. By Assumption 2.1 we use the fact $\mathrm{Re}(\lambda_i) < 0$ for all $i = 1, 2, \ldots, p$ without mention. The increment has the following form via the integral by parts formula (see Theorem 2.21 of [30]).

$$\begin{aligned} Y(s) - Y(0) &= \sum_{i=1}^{p} \frac{\beta(\lambda_i)}{\alpha^{(1)}(\lambda_i)} \left(\int_{-\infty}^{s} e^{\lambda_i(s-u)} dB^H(u) - \int_{-\infty}^{0} e^{-\lambda_i u} dB^H(u)\right) \\ &= \sum_{i=1}^{p} \frac{\beta(\lambda_i)}{\alpha^{(1)}(\lambda_i)} \left(B^H(s) + \lambda_i \int_{0}^{s} e^{\lambda_i(s-u)} B^H(u) du + (e^{\lambda_i s} - 1) Y_i(0)\right), \end{aligned}$$

where for simplicity we denote $\int_{-\infty}^{0} e^{-\lambda_i u} dB^H(u)$ by $Y_i(0)$. Take absolute value of this to obtain

$$\begin{aligned} |Y(s) - Y(0)| &\le \sum_{i=1}^{p} \left|\frac{\beta(\lambda_i)}{\alpha^{(1)}(\lambda_i)}\right| \Bigg(|B^H(s)| + \max_{0\le u\le s} |B^H(u)| \frac{|\lambda_i|}{\mathrm{Re}(\lambda_i)} (e^{\mathrm{Re}(\lambda_i)s} - 1) \\ &\qquad + (1 - e^{\mathrm{Re}(\lambda_i)s}) |Y_i(0)|\Bigg) \\ &\le C\left(|B^H(s)| + \max_{0\le u\le s} |B^H(u)|(1 - e^{\mu s}) + (1 - e^{\mu s}) \sum_{i=1}^{p} |Y_i(0)|\right), \end{aligned}$$

where $C = p \sup_i \left|\frac{\beta(\lambda_i)}{\alpha^{(1)}(\lambda_i)} \frac{\lambda_i}{\mathrm{Re}(\lambda_i)}\right|$ and $\mu = \inf_i \mathrm{Re}(\lambda_i) < 0$. Take m-th product of this to get

$$|Y(s) - Y(0)|^m$$

$$\leq C^m \sum_{j+k+\ell=m; j,k,\ell\geq 0} \frac{m!}{j!\,k!\,\ell!} |B^H(s)|^j \left\{ \max_{0\leq u\leq s} |B^H(u)|(1-e^{\mu s}) \right\}^k \left\{ (1-e^{\mu s}) \sum_{i=1}^p |Y_i(0)| \right\}^\ell$$

$$\leq C^m \sum_{j+k+\ell=m; j,k,\ell\geq 0} \frac{m!}{j!\,k!\,\ell!} \left(\max_{0\leq u\leq s} |B^H(u)| \right)^{j+k} (1-e^{\mu s})^{k+\ell} \left(\sum_{i=1}^p |Y_i(0)| \right)^\ell .$$

Moreover, taking expectation of maxima of this, we have

$$\begin{aligned} & E\left[\max_{0\leq s\leq r} |Y(s)-Y(0)|^m \right] \\ & \leq C^m \sum_{j+k+\ell=m; j,k,\ell\geq 0} \frac{m!}{j!\,k!\,\ell!} E\left[\max_{0\leq u\leq s} |B^H(u)|^{j+k} \left(\sum_{i=1}^p |Y_i(0)| \right)^\ell \right] (1-e^{\mu s})^{k+\ell}. \end{aligned} \tag{5.2}$$

Let $\overline{Y}(0) := \int_{-\infty}^0 e^{-\eta s} dB^H(s)$ with $\eta = \sup_i \mathrm{Re}(\lambda_i) < 0$. To analyze expectation is (5.2) we start to bound $E[|Y_n(0)|^{2\ell}]$, $n=1,\dots,p$ by $E[|\overline{Y}(0)|^{2\ell}]$. Define the real and the imaginary part of $Y_n(0)$ as

$$\begin{aligned} Y_n(0) &= \int_{-\infty}^0 e^{-(i\mathrm{Im}(\lambda_n)+\mathrm{Re}(\lambda_n))u} dB^H(u) \\ &= \int_{-\infty}^0 e^{-\mathrm{Re}(\lambda_n)u} \cos(\mathrm{Im}(\lambda_n)u) dB^H(u) - i \int_{-\infty}^0 e^{-\mathrm{Re}(\lambda_n)u} \sin(\mathrm{Im}(\lambda_n)u) dB^H(u) \\ &=: RY_n(0) - iIY_n(0), \quad n=1,\dots,p. \end{aligned}$$

Noticing $|Y_n(0)|^2 = (RY_n(0))^2 + (IY_n(0))^2$ we have

$$E[|Y_n(0)|^{2\ell}] \leq 2^{2\ell-1} \left(E[(RY_n(0))^{2\ell}] + E[(IR_n(0))^{2\ell}] \right).$$

Since $RY_n(0)$ and $IY_n(0)$ are both Gaussian and their variances are smaller than $E[|\overline{Y}(0)|^2]$, we further bound $E[|Y_n(0)|^{2\ell}]$ as

$$E[|Y_n(0)|^{2\ell}] \leq 2^{2\ell} E[|\overline{Y}(0)|^{2\ell}] \leq 2^{2\ell}(2\ell-1)!! M_1^{2\ell}, \tag{5.3}$$

where $M_1 = \sqrt{\mathrm{Var}(\overline{Y}(0))}$. Due to this together with Lemma 3.2, the expectation in each term of the sum is bounded as

$$\begin{aligned} & E\left[\max_{0\leq u\leq s} |B^H(u)|^{j+k} \left(\sum_{i=1}^p |Y_i(0)| \right)^\ell \right] \\ & \leq \sqrt{E\left[\max_{0\leq u\leq s} |B^H(u)|^{2(j+k)} \right]} \sqrt{E\left[p^{2\ell-1} \sum_{i=1}^p |Y_i(0)|^{2\ell} \right]} \\ & \leq \sqrt{E\left[\max_{0\leq u\leq s} |B^H(u)|^{2(j+k)} \right]} \sqrt{E\left[(2p)^{2\ell} |\overline{Y}(0)|^{2\ell} \right]} \\ & \leq \sqrt{2(2(j+k)-1)!! r^{2H(j+k)}} \sqrt{(2\ell-1)!!(2pM_1)^{2\ell}} \\ & \leq \sqrt{2^{j+k}(j+k)! r^{2H(j+k)}} \sqrt{2^\ell \ell! (2pM_1)^{2\ell}} \\ & = \sqrt{(j+k)!\ell!} (\sqrt{2} r^H)^{j+k} (2\sqrt{2}pM_1)^\ell. \end{aligned} \tag{5.4}$$

Here M_1 depends on the parameter H and η by the definition. However since $H \in [\frac{1}{2}, 1)$, we can make M_1 to attain a certain bound regardless of H. In fact

$$\mathrm{Var}(\overline{Y}(0)) = 2\sigma^2 c_H \int_{-\infty}^{\infty} \frac{|x|^{1-2H}}{\eta^2 + x^2} dx, \quad c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}.$$

Note that as $H \uparrow 1$ the integral diverges, but simultaneously $c_H \downarrow 0$ in the same rate. Therefore, we conclude that $\mathrm{Var}(\overline{Y}(0))$ is bounded regardless of $H \in [\frac{1}{2}, 1)$.

Now substituting (5.4) into (5.2) and dividing by $m!$, we have

$$\begin{aligned}
&\frac{E\left[\max_{0\le s\le r} |Y(s) - Y(0)|^m\right]}{m!} \\
&\le \frac{C^m}{\sqrt{m!}} \sum_{j+k+\ell=m; j,k,\ell\ge 0} \frac{m!}{j!\,k!\,\ell!} \sqrt{\frac{(j+k)!\,\ell!}{m!}} (\sqrt{2} r^H)^j (\sqrt{2} r^H (1 - e^{\mu s}))^k (2\sqrt{2} p M_1 (1 - e^{\mu s}))^\ell \\
&\le \frac{1}{\sqrt{m!}} C^m \left\{ \sqrt{2} r^H + \sqrt{2} r^H (1 - e^{\mu s}) + 2\sqrt{2} p M_1 (1 - e^{\mu s}) \right\}^m \\
&\le \frac{r^{Hm}}{\sqrt{m!}} C^m \left\{ \sqrt{2} + \sqrt{2}(1 - e^{\mu s}) + 2\sqrt{2} p M_1 \frac{(1 - e^{\mu s})}{r^H} \right\}^m .
\end{aligned}$$

Here we observe that $(1 - e^{\mu s})/r^H \le |\mu|$ for any $0 \le s \le r$ and any $H \in [\frac{1}{2}, 1)$ and that $c^m/m! \le e^c$ for any $c > 0$ and any $m = 1, 2, \ldots$. Hence we can take a universal constant $C(\boldsymbol{\lambda}) > 0$ and obtain

$$\frac{E\left[\max_{0\le s\le r} |Y(s) - Y(0)|^m\right]}{m!} \le C(\boldsymbol{\lambda}) r^{Hm}.$$

□

Proof of Theorem 3.1.
Once we have Lemma 3.3, the proof of the theorem can be proceeded exactly as that of of Theorem 2.1 in [21]; thus we omit it. □

Proof of Lemma 4.1.
The expression (2.7) of g yields

$$|g(t-u)| \;=\; \left| \sum_{i=1}^{p} \frac{\beta(\lambda_i)}{\alpha^{(1)}(\lambda_i)} e^{\lambda_i (t-u)} \right| \le \frac{\gamma}{p} \sum_{i=1}^{p} e^{\mathrm{Re}(\lambda_i)(t-u)}.$$

Then, since we assume that $\mathrm{Re}(\lambda_i) < 0, i = 1, \ldots, p$, the results are obtained by basic calculations. □

Proof of Lemma 4.2.
Since g is a non-negative bounded continuous function with $g \in L^1 \cap L^2$, due to approximation Lemma in [13, p.91], we may take a step function $g_n(x) = \sum_{i=1}^{n-1} a_i 1_{(s_i, s_{i+1}]}(x)$, where $a_i \ge 0,\ i = 1, \ldots, n$ and $-\infty < s_1 < s_2 < \cdots < s_n < \infty$ such that

$$g_n(x) \le g_{n+1}(x) \le g(x),$$

$$\lim_{n\to\infty} g_n(x) = g(x) \quad a.e. \quad \text{and} \quad g_n \to g \ \in L^1 \cap L^2.$$

Let $Y_n(t) = \int_{-\infty}^{t} g_n(t-u) dL(u)$, then by [25, p.165], with the drift b and without the Gaussian part at there, we see that

$$E\left[e^{\theta \int_{-\infty}^{t} g_n(t-u) dL(u)} \right]$$

$$= \exp\left\{\theta b\int_{-\infty}^{t} g_n(t-u)du + \int_{-\infty}^{t}\int_{\mathbb{R}}[e^{\theta g_n(t-u)x} - 1 - \theta g_n(t-u)x1_{\{|x|\le 1\}}]\nu(dx)du\right\}. \tag{5.5}$$

The integrand of the last integral is bounded by

$$\begin{aligned}
&\theta^2 e^{\theta\gamma|x|}|g_n(t-u)|^2|x|^2 1_{\{|x|\le 1\}} + \theta|g_n(t-u)||x|\left(e^{\theta\gamma x}1_{\{x>1\}} + 1_{\{x<-1\}}\right) \\
&\le \theta^2 e^{\theta\gamma|x|}|g(t-u)|^2|x|^2 1_{\{|x|\le 1\}} + \theta g(t-u)|x|\left(e^{\theta\gamma x}1_{\{x>1\}} + 1_{\{x<-1\}}\right),
\end{aligned}$$

which is integrable. Hence by the dominated convergence theorem the right hand side of $E[e^{\theta\int_{-\infty}^{t} g_n(t-u)dL(u)}]$ converges to (4.2) and (4.2) is continuous at $\theta = 0$. Then due to definition of stochastic integral, (4.2) is turned out to be the moment generating function of Y. □

Proof of Proposition 4.1.
Due to the expression (2.13), where $X(0)$ and $\{L(t)\}_{t\ge 0}$ are independent, it follows that

$$\begin{aligned}
&\mathrm{Cov}(Z(0), Z(t)) \\
&= E[e^{Y(0)+Y(t)}] - E[e^{Y(0)}]E[e^{Y(t)}] \\
&= E[e^{\beta' X(0)+\beta' e^{At}X(0)}]E\left[\exp\left\{\int_0^t g(t-u)dL(u)\right\}\right] \\
&\quad -E\left[e^{\beta' X(0)}\right]E\left[e^{\beta' e^{At}X(0)}\right]E\left[\exp\left\{\int_0^t g(t-u)dL(u)\right\}\right] \\
&= \underbrace{E\left[\exp\left\{\int_0^t g(t-u)dL(u)\right\}\right]}_{\mathrm{I}}\underbrace{\left(E\left[e^{\beta' X(0)+\beta' e^{At}X(0)}\right] - E\left[e^{\beta' X(0)}\right]E\left[e^{\beta' e^{At}X(0)}\right]\right)}_{\mathrm{II}}.
\end{aligned}$$

We begin to see the calculation of the quantity I. By Lemma 4.2 and Taylor expansion it follows that

$$\begin{aligned}
\mathrm{I} &\le \exp\left\{b\int_0^t g(t-u)du + \int_0^t\int_{\mathbb{R}}\left|e^{g(t-u)x} - 1 - g(t-u)x1_{\{|x|\le 1\}}\right|\nu(dx)du\right\} \\
&\le \exp\left\{b\int_0^t g(t-u)du + \int_0^t\int_{|x|\le 1} e^{g(t-u)|x|}g^2(t-u)|x|^2\nu(dx)du\right. \\
&\qquad \left. + \int_0^t\int_{x>1} e^{g(t-u)x}g(t-u)x\nu(dx)du + \int_0^t\int_{x<-1} g(t-u)|x|\nu(dx)du\right\} \\
&\le \exp\left\{b\int_0^t g(t-u)du + \int_0^t g^2(t-u)du\int_{|x|\le 1} e^{\gamma}|x|^2\nu(dx)\right. \\
&\qquad \left. + \int_0^t g(t-u)du\left(\int_{x>1} e^{\gamma x}x\nu(dx) + \int_{x<-1}|x|\nu(dx)\right)\right\}.
\end{aligned}$$

Here in the second step we use the inequalities;

$$e^x - 1 \le e^x x \quad \text{and} \quad 1 - e^{-x} \le x, \quad x \ge 0. \tag{5.6}$$

In view of Lemmas 4.1 and 4.2, the quantity I is proven to be bounded uniformly in t. Again by Lemma 4.2 we have

$$E[e^{\beta'(I+e^{At})X(0)}]$$

$$= \exp\left\{ b\int_{-\infty}^{0} (g(t-u)+g(-u))du \right\}$$
$$\times \exp\left\{ \int_{-\infty}^{0}\int_{\mathbb{R}} [e^{(g(t-u)+g(-u))x} - 1 - (g(t-u)+g(-u))x1_{\{|x|\le 1\}}]\nu(dx)du \right\}$$

and

$$\begin{aligned}
&E[e^{\beta' X(0)}]E[e^{\beta' e^{At}X(0)}] \\
&= \exp\left\{ b\int_{-\infty}^{0} (g(t-u)+g(-u))du \right\} \\
&\quad \times \exp\left\{ \int_{-\infty}^{0}\int_{\mathbb{R}} [e^{g(-u)x} - 1 - g(-u)x1_{\{|x|\le 1\}}]\nu(dx)du \right\} \\
&\quad \times \exp\left\{ \int_{-\infty}^{0}\int_{\mathbb{R}} [e^{g(t-u)x} - 1 - g(t-u)x1_{\{|x|\le 1\}}]\nu(dx)du \right\}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\mathrm{II} \;=\;& \underbrace{\exp\left\{ b\int_{-\infty}^{0} (g(t-u)+g(-u))du \right\}}_{\mathrm{II}a} \\
&\times \underbrace{\left[\exp\left\{ \int_{-\infty}^{0}\int_{\mathbb{R}} [e^{(g(t-u)+g(-u))x} - e^{g(-u)x} - e^{g(t-u)x} + 1]\nu(dx)du \right\} - 1\right]}_{\mathrm{II}b} \\
&\times \underbrace{\exp\left\{ \int_{-\infty}^{0}\int_{\mathbb{R}} [e^{g(-u)x} - 1 - g(-u)x1_{\{|x|\le 1\}}]\nu(dx)du \right\}}_{\mathrm{II}c} \\
&\times \underbrace{\exp\left\{ \int_{-\infty}^{0}\int_{\mathbb{R}} [e^{g(t-u)x} - 1 - g(t-u)x1_{\{|x|\le 1\}}]\nu(dx)du \right\}}_{\mathrm{II}d}.
\end{aligned}$$

We study $\mathrm{II}_a, \mathrm{II}_b$, and II_c and II_d in order. Due to Lemma 4.1, the quantity II_a is bounded uniformly in t. Again by (5.6) it is easy that

$$\begin{aligned}
\mathrm{II}_b \;:=\;& \exp\left\{ \int_{-\infty}^{0}\int_{\mathbb{R}} (e^{g(t-u)x} - 1)(e^{g(-u)x} - 1)\nu(dx)du \right\} - 1 \\
\le\;& \exp\left\{ \int_{-\infty}^{0}\int_{|x|\le 1} e^{g(t-u)x} g(t-u)\cdot e^{g(-u)x} g(-u)|x|^2\nu(dx)du \right. \\
&\left. + \int_{-\infty}^{0}\int_{x>1} e^{g(t-u)x+g(-u)x} g(t-u)x\nu(dx) + \int_{-\infty}^{0}\int_{x<-1} g(t-u)|x|\nu(dx)du \right\} - 1 \\
\le\;& \exp\left\{ \int_{-\infty}^{0} g(t-u)du \int_{\mathbb{R}} \left(\gamma e^{2\gamma|x|}|x|^2 1_{\{|x|\le 1\}} + e^{2\gamma|x|}x1_{\{x>1\}}\right)\nu(dx) \right. \\
&\left. + \int_{-\infty}^{0} g(t-u)du \int_{x<-1} |x|\nu(dx) \right\} - 1.
\end{aligned}$$

Here in the second inequality we use (5.6). Then due to Lemma 4.1 and Taylor expansion we have

$$\mathrm{II}_b \le \exp\left\{ c\int_{-\infty}^{0} g(t-u)du \right\} - 1 \le c' e^{-c'' t}, \tag{5.7}$$

where c, c' and c'' are some positive constants. Next we see the boundedness of II_c and II_d. Since proofs are similar we only show the boundedness of II_c. Again by Lemmas 4.1 and 4.2 with (5.6) we have

$$\begin{aligned}
\log \mathrm{II}_c \le & \int_{-\infty}^0 \int_{|x|\le 1} e^{g(-u)|x|} (g(-u)x)^2 \nu(dx)du \\
& + \int_{-\infty}^0 \int_{x>1} e^{g(-u)x} g(-u)x\nu(dx)du \\
& + \int_{-\infty}^0 \int_{x<-1} \left(1 - e^{g(-u)x}\right) \nu(dx)du \\
\le & \int_{-\infty}^0 g^2(-u)du \int_{|x|\le 1} e^{\gamma|x|}|x|^2\nu(dx) \\
& + \int_{-\infty}^0 g(-u)du \left(\int_{x>1} e^{\gamma x} x\nu(dx) + \int_{x<-1} |x|\nu(dx)\right) < \infty.
\end{aligned}$$

The same calculation holds for II_d, and II_c and II_d are turned out to be bounded uniformly in t. Hence with boundedness of I and II_a, the equation (5.7) yields the results. □

Proof of Theorem 4.1.
In the following proof, the c, c', c'', C, C', C'', *etc*, will denote positive constants for which the exact values are irrelevant and may vary from line to line. By the Lévy-type stochastic integral of Y in Section 2 (recall that we have assumed there is no Gaussian part), for $t > 0$,

$$Y(t) \quad = \quad \beta' e^{At} X(0) + \int_0^t g(t-u)dL(u)$$

where g is the CARMA kernel defined in Section 2. The Lévy-Itô decomposition for L mentioned in Section 2, written in the differential form yields,

$$dL(s) = bds + \int_{|x|<1} x\tilde{N}(ds, dx) + \int_{|x|\ge 1} xN(ds, dx).$$

This together with the state equation (2.4) gives

$$\begin{aligned}
dY(t) &= \left(\beta' A e^{At} X(0) + \int_0^t g'(t-s)dL(s)\right)dt + g(0)dL(t) \\
&= \beta' AX(t)dt + g(0)dL(t) \\
&= (\beta' AX(t) + g(0)b)dt + g(0)\int_{|x|<1} x\tilde{N}(dt, dx) + g(0)\int_{|x|\ge 1} xN(dt, dx),
\end{aligned}$$

where $g(0) = \beta'\delta_p$. We apply Itô's formula for Lévy-type stochastic integrals, as that in [1, p.226, Theorem 4.4.7] with $f(x) = e^{\theta x}$ to obtain

$$\begin{aligned}
e^{\theta Y(t)} - e^{\theta Y(0)} \quad = & \quad \int_0^t \theta e^{\theta Y(s-)}(\beta' AX(s-) + g(0)b)ds \\
& + \int_0^t \int_{|x|\ge 1} [e^{\theta(Y(s-)+g(0)x)} - e^{\theta Y(s-)}]N(ds, dx) \\
& + \int_0^t \int_{|x|<1} [e^{\theta(Y(s-)+g(0)x)} - e^{\theta Y(s-)}]\tilde{N}(ds, dx)
\end{aligned}$$

$$+ \quad \int_0^t \int_{|x|<1} [e^{\theta(Y(s-)+g(0)x)} - e^{\theta Y(s-)} - \theta g(0)xe^{\theta Y(s-)}]\nu(dx)ds.$$

We rearrange the above expression to get the following semimartingale decomposition, *confer* [1, p.252],

$$e^{\theta Y(t)} - e^{\theta Y(0)} = M(t) + V(t),$$

where the martingale part $M(t)$ and the finite variation part $V(t)$ are, respectively, expressed by

$$M(t) = \int_0^t \int_{x\in\mathbb{R}} e^{\theta Y(s-)}\big(e^{\theta g(0)x} - 1\big)\tilde{N}(ds, dx)$$

and

$$V(t) = \int_0^t \Big\{e^{\theta Y(s-)}\Big(\theta\beta' AX(s-) + \theta g(0)b + \int_{|x|\geq 1}(e^{\theta g(0)x} - 1)\nu(dx) \\ + \int_{|x|<1}(e^{\theta g(0)x} - 1 - \theta g(0)x)\nu(dx)\Big)\Big\}ds.$$

We estimate the expected maximums of M and V separately.
(i) Estimate for $E[\max_{0\leq t\leq t_0}|V(t)|]$.
Due to a basic inequality $|e^y - 1 - y| \leq c(y^2/2)$ and inequalities (5.6) with Fubini's theorem,

$$E\left[\max_{0\leq t\leq t_0}|V(t)|\right] \\ \leq \int_0^{t_0}\Big\{\theta E[e^{\theta Y(s-)}|\beta' AX(s-)|] + E[e^{\theta Y(s-)}]\Big(\theta g(0)|b| + \frac{c}{2}\theta^2 g^2(0)\int_{|x|<1}x^2\nu(dx) \\ +2\int_{x>1}e^{\theta\gamma x}\nu(dx) + \theta g(0)\int_{x<-1}|x|\nu(dx)\Big)\Big\}ds.$$

We observe that $s \to Y(s-)$ is also strictly stationary, $E[f(Y(s-))] = E[f(Y(s))] = E[f(Y(0))]$ for each nonnegative f and each s by the left-quasi-continuity of the Lévy process L. Therefore,

$$E\left[\max_{0\leq t\leq t_0}|V(t)|\right] \leq \int_0^{t_0}\Big(\theta\sqrt{E[e^{2\theta Y(0)}]E[(\beta' AX(0))^2]} + E[e^{\theta Y(0)}]c'\Big)ds \leq c'' t_0.$$

(ii) Estimate for $E\left[\max_{0\leq t\leq t_0}|M(t)|\right]$.
By the Burkholder's inequality for martingale with jumps as that given in [6, p.213] with $p = 1$ there, we have, for each $t_0 > 0$,

$$E\left[\max_{0\leq t\leq t_0}|M(t)|\right] \leq CE\left[[M]^{1/2}(t_0)\right] \leq C'\sqrt{E\left[[M](t_0)\right]},$$

where $[M]$ denotes the quadratic variation process of the martingale process M; we have also used a basic inequality that $E\sqrt{V} \leq \sqrt{EV}$ for every nonnegative random variable V. By the quadratic variation of stochastic integrals, see [1, p.230] without the Gaussian part there, and our expression of $M(t)$ given above, we have

$$[M](t_0) = \int_0^{t_0}\int_{x\in\mathbb{R}} e^{2\theta Y(s-)}\big(e^{\theta g(0)x} - 1\big)^2 N(ds, dx).$$

Since the process $s \to Y(s-)$ is a left-continuous process, by the compensation formula as that given in [18, Theorem 4.4], we have

$$E\left[[M](t_0)\right] = \int_0^{t_0} \int_{x\in\mathbb{R}} E\big[e^{2\theta Y(s-)}\big]\big(e^{\theta g(0)x} - 1\big)^2 \nu(dx)ds.$$

As we have argued in the proof of the drift part (i), for each s, $E\big[e^{2\theta Y(s-)}\big] = E\big[e^{2\theta Y(0)}\big]$. Under our integrability assumptions on $\nu(dx)$, similar calculations as before show that

$$\int_{x\in\mathbb{R}} \big(e^{\theta g(0)x} - 1\big)^2 \nu(dx) < \infty.$$

Thus,

$$E\left[\max_{0\le t\le t_0} |M(t)|\right] \le C'\sqrt{E\left[[M](t_0)\right]} \le C''\sqrt{t_0}.$$

As a consequence,

$$E\left[\max_{0\le t\le t_0} |e^{\theta Y(t)} - e^{\theta Y(0)}|\right] \le E\left[\max_{0\le t\le t_0} |M(t)|\right] + E\left[\max_{0\le t\le t_0} |V(t)|\right] \le C''\sqrt{t_0} + c'' t_0,$$

which is of $O(\sqrt{t_0})$ as t_0 is small. □

6 Concluding remarks

We have presented the study on the exponential process

$$Z := \{Z(t)\}_{t\in\mathbb{R}} := \{e^{Y(t)}\}_{t\in\mathbb{R}},$$

where Y is $Y(p, H, q)$ and $Y(p, L, q)$, the CARMA (p, q) process driven by FBM and by LP with exponential moments, under Assumptions 2.1 and 2.2. In this concluding section, we discuss its advantageous role in stochastic modelling of finance and telecommunication.

(1) Telecommunication: In this context, we work on the normalized exponential CARMA-type process

$$\tilde{Z}(t) := e^{Y(t)-c},$$

where the constant $c = Ee^{Y(0)}$ so that the resulting $\tilde{Z}(t)$ is a positive-valued stationary process with mean 1 (we may choose c so that the mean of $\tilde{Z}(t)$ is any prescribed positive quantity). In view of the burst phenomenon of internet traffic, it is studied in [19] the infinite product of a re-scaled "mother process" Λ. The Λ is *any* mean 1, positive-valued, positive-correlated stationary process, for which the correlation decay (as time-lag tends to ∞) and the expected maximal increments (as time-lag tends to 0) are both of a certain power-decay. The exponential FBM driven OU-type process and the exponential LP driven OU-type process, respectively in [2] and [21], are used to model such a mother process. We would propose to use the $\tilde{Z}(t)$ as a more general mother process to create the infinite product; one advantage of the CARMA-type process over the OU-type process in this situation should be that we can have more adjusting parameters p, q to fit the *multifractality* of the various burst measures.
(2) Finance: In a very recent work [16], the authors use the exponential CARMA-type process driven by a Lévy process to study the leverage effect between the stock return and the volatility for continuous-time financial markets with jumps. Their §3 has features on: (i) the construction of

a proper return process via the underlying Lévy process L and the volatility exponential CARMA process driven by the derived Lévy process M, and (ii) the leverage of the return and the volatility conditioning on the jumps of L. As the long memory property of financial markets is well recognized (see, for example, [14]), it seems worthwhile to study [16] with fractional Brownian motion as the underlying process. However, both above features and related analysis seem not readily achieved; the issue will be addressed elsewhere.

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References

[1] Applebaum, D. (2004), *Lévy Processes and Stochastic Calculus* (Camb. Studies Adv. Math. **93**) (Cambridge University Press, Cambridge)

[2] Anh, V.V., Leonenko, N.N. and Shieh, N.-R. (2008), Multifractality of products of geometric Ornstein-Uhlenbeck-type processes. *Adv. in Appl. Probab.* **40**, 1129–1156.

[3] Barndorff-Nielsen, O.E. (2001), Superposition of Ornstein-Uhlenbeck type processes, *Theory Probab. Appl.* **45**, 175–194.

[4] Barndorff-Nielsen, O.E. and Shephard, N. (2001), Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **63**, 167–241.

[5] Bibby, M., Skovgaard, I. and Sørensen, M. (2005) Diffusion-type models with given marginal distribution and autocorrelation function, *Bernoulli* **11**, 191–220.

[6] Bichteler, K. (2002), *Stochastic Integration with Jumps*, Encyclopedia in Mathematics and its Applications **89** (Cambridge University Press, Cambridge)

[7] Brockwell, P.J. (2004), Representations of continuous-time ARMA processes. *J. Appl. Probab.* **41A**, 375–382.

[8] Brockwell, P.J. and Marquardt, T. (2005), Lévy-driven and fractionally integrated ARMA processes with continuous time parameter. *Statist. Sinica* **15**, 477–494.

[9] Brockwell, P.J. and Lindner, A. (2009), Existence and uniqueness of stationary Lévy-driven CARMA processes, *Stochastic Process. Appl.* **119**, 2660–2681.

[10] Carmona, Ph., Petit, F. and Yor, M. (1997), On the distribution and asymptotic results for exponential functionals of Lévy processes, in: Yor, M. ed. *Exponential Functionals and Principal Values Related to Brownian Motion* (Biblioteca de la Revista Matematica Ibero-Americana) 73–126

[11] Carmona, Ph., Petit, F. and Yor, M. (2001), Exponential functionals of Lévy processes, in: Barndorff-Nielsen, O.E., Mikosch, T. and Resnick, S.I. eds. *Lévy Processes: Theory and Applications* (Birkhäuser, Boston) 41–55.

[12] Cheridito, P., Kawaguchi, H. and Maejima, M. (2003), Fractional Ornstein-Uhlenbeck processes, *Electron. J. Probab.* **8**, 1–14.

[13] Chung, K.L. (2001), *A Course in Probability Theory* (Academic press, San Diego)

[14] Comte, F. and Renault, E. (1998), Long memory in continuous-time stochastic volatility models, *Mathematical Finance* **8**, 291–323.

[15] Embrechts, P. and Maejima, M. (2002), *Selfsimilar Processes* (Princeton Univ. Press, Princeton)

[16] Haug, S. and Czado, C. (2007), An exponential continuous-time GARCH process, *J. Applied Probab.* **44**, 960-976.

[17] Jeanblanc, M., Klöppel, S. and Miyahara, Y. (2007), Minmal f^q-martingale measures for expoenetial Lévy processes, *Ann. Appl. Probab.* **17**, 1615–1638.

[18] Kyprianou, A.E. (2006) *Introductory Lectures on Fluctuations of Lévy Processes with Applications* (Springer, Berlin)

[19] Mannersalo, P., Norros. I. and Riedi, R. (2002), Multifractal products of stochastic processes: construction and some basic properties, *Adv. in Appl. Probab.* **34**, 888–903.

[20] Masuda, H. (2004), On multidimensional Ornstein-Uhlenbeck processes driven by a general Lévy process, *Bernoulli* **10**, 97–120.

[21] Matsui, M. and Shieh, N.-R. (2009), On the Exponentials of Fractional Ornstein-Uhlenbeck Processes, *Electron. J. Probab.* **14**, 594–611.

[22] Shephard, N. (2005), *Stochastic Volatility: Selected Readings (Advanced Texts in Econometrics)* (Oxford Univ. Press, New York)

[23] Ruzmaikina, A.A. (2000), Stieltjes integrals of Hölder continuous functions with applications to fractional Brownian motion, *J. Statist. Phys.* **100**, 1049–1069.

[24] Samorodnitsky, G. and Taqqu, M.S. (1994), *Stable Non-Gaussion Random Processes: Stochastic Models with Infinite Variance: Stochastic modeling* (Chapman and Hall, New York)

[25] Sato, K. (1999), *Lévy Processes and Infinitely Divisible Distributions* (Cambridge University Press, Cambridge)

[26] Tsai, H. (2009), On continuous-time autoregressive fractionally integrated moving average processes. *Bernoulli* **15**, 178–194.

[27] Tsai, H. and Chan, K.S. (2005a), Maximum likelihood estimation of linear continuous time long memory processes with discrete time data, *J. R. Stat. Soc. Ser. B Stat. Methodol.* **67**, 703–716.

[28] Tsai, H. and Chan, K.S. (2005b), A note on non-negative continuous-time processes, *J. R. Stat. Soc. Ser. B Stat. Methodol.* **67**, 589–597.

[29] Tsai, H. and Chan, K.S. (2009), A note on the non-negativity of continuous-time ARMA and GARCH processes, *Statist. Comput.* **19**, 149-153.

[30] Wheeden, R.L. and Zygmund, A. (1977), *Measure and Integral* (Marcel Dekker, New York-Basel)