Stichastic Calculus: Syllabus and Exercise

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§ This course is suitable for those who have taken Probability Theory (I); some knowledge of Real Analysis is needed.

§ The content and exercise are adapted from

1. R. Durrett: Probability: Theory and Examples, Third Edition(2005, Duxbury) Chapters 4, 7.

2. B. Okensdal: Stochastic Differential Equations, Sixth Edition(2003, Springer) Chapters 3,4,5,7,8.

§ Grading: Exams(mid-term and final): 80 percent, Homework/Report: 20 percent; subject to change. Notice: classroom participation is noted.

§ Ex's of I[1, Ch 4]: p220 1.2; p223 1.3,1.4;p225 1.6,1.7;p226 1.9,1.10(BP),1.11;p229 2.2;p235 2.5,2.6,2.8,2.12;p236 2.14;p241 3.7,3.8;p246 3.12;p251 4.8,4.9;p260 5.2,5.3;p261 5.6,5.7;p262 5.8;p263 6.1;p268 6.3;p271 7.1;p273 7.4,7.6,7.8

§ Ex's of II[1, Ch 4]: p377 1.2(path non-differentiability), 1.3(quadratic variation);p382 2.3;p385 3.2; p386 3.5;p393 read:example 4.4;p397 read:(5.8);p398 read:(5.9), 5.3;p399 5.7

I. Martingales, adapted from [1, Ch. 4]

§ Notice: In [1, ch. 4] the time index is discrete, while in this course we need to *the continuous-time.*

§ Review from PT I: conditional expectation, filtration, martingale (fair game process), submartingale, supermartingale

§ a Hilbert space viewpoint of conditional expectation

§ stopping (optional) times

§ predictable sequence

§ discrete-time " stochastic integral" process $(H \cdot X)_n$

§ stopped rv, stopped process; the stopped process is still a martingale

§ upcrossings of a sequence over an interval $[a, b]$

§ a key property: the upper bound for EU_n, U_n: the upcrossings up to time n

§ martingale convergence theorem

§ Doob-Meyer decomposition theorem, the discrete-time case

§ an example from SRW

§ no "bounded infinite oscillations"

§ a conditional form of BC Lemma

§ martingale property of a Galton-Watson branching process

§ a theorem on EX_N, N bounded by k

§ Doob's inequality (1)

§ Doob's inequality (2)

§ martingale $L^p, 1 < p < \infty$, convergence theorem

§ uniform integrability and L^1 convergence: from RA

§ martingale L^1 convergence theorem

§ Lévy's convergence theorem and 0–1 law

§ the stopped process is still u.i.

§ a theorem on EX_N, N any stopping time

§ stopped σ-algebra $\mathcal{F}_N$; a monotone lemma

§ optional stopping theorem

§ backwards (reversed) martingale and its convergence theorem

§ continuous-time martingale, the "cadlag" requirement

§ stopping times for continuous-time case

§ the continuous-time Doob-Meyer decomposition theorem, ?!

§ the space of square integrable martingales

II. Brownian Motions, adapted from [1, Ch. 7]

§ math def of BM: Wiener process

§ remarks on the sample paths

§ finite dim'l distributions of BM

§ translation and scaling invariance of BM

§ BM as a Gaussian process, inversion invariance

§ construction of BM on "canonical space"

§ Kolmogorov's path-continuity criterion

§ Markov property

§ Blumenthal's 0–1 law, two applications

§ hitting time of BM

§ strong Markov property

§ zero set and level sets

§ first passage time

§ reflection principle

§ non-differentiability of Brownian paths

§ quadratic variation of Brownian paths

§ $U(t) = B^2(t) - t$ is a martingale

§ $V(t) = \exp(\lambda B(t) - (1/2)\lambda^2 t)$ is a martingale

§ from RW to BM (Donsker's invariance principle)

§ martingale CLT

III, Basic Itô Calculus, ,adapted from [2, Ch. 3 and Ch. 4]

§ Ito integral: first step

§ Ito isometry

§ Ito integral: next step

§ two basic examples

§ basic properties

§ Ito integral process: path continuity

§ Ito integral process: martingale property

§ Ito integral: final step

§ multi-dim'l Ito integral

§ Stratonovich integral

§ Ito process

§ Ito formula, how to operate $(dX)^2$

§ two basic examples

§ integration by parts formula

§ multi-dim'l case

§ $f(B_t)$

§ Bessel process

§ Ito representation theorem

§ martingale representation theorem

§ a dense lemma for $L^(dP)$

§ exponential martingale

§ quadratic variation of Ito process

§ BM local time and Tanaka formula

IV. Stochastic Diffusion Equations (SDE) , adapted from [2, Ch. 5]

§ 1-dim'l SDE and geometric Brownian Motion

§ existence and uniqueness theorem for SDE solution, remind of ODE

§ proof of the uniqueness, based on and Gronwall inequality

§ proof of the existence, based on Picard's approximation

§ strong solution and weak solution, why we need weak solution

§ Tanaka equation

§ geometric Brownian Motion

§ Ornstein-Uhlenbeck equation (Langevin equation)

§ Brownian bridge

§ BM on the ellipse

V. Itô Diffusions, adapted from [2, Ch. 7 and Ch. 8]

§ (time-homogeneous) Ito diffusion

§ Markov property

§ strong Markov property

§ generator of Ito diffusion

§ basic example: generator for BM and space-time BM

§ Dynkin formula

§ Kolmogorov equation

§ semigroup of operators

§ resolvent operator

§ Feynman-Kac formula

§ martingale property of Ito diffusion

§ martingale problem

§ Ito process versus Ito diffusion

§ (random) time-change and the equivalence theorem

§ time-change of BM

§ Lévy characterization of BM

§ quadratic variation for multi-dim'l Ito process

§ absolute continuity and Random-Nykodym derivative, revisited

§ Girsanov Theorem (various version)

§ toward a beginning: Stochastic Finance (Itô Legacy)