Probability Theory I: Syllabus and Exercise

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§ This course is suitable for those who have taken Basic Probability; some knowledge of Real Analysis is recommended(will be reviewed in the course).

§ The content and exercise are adapted from R Durrett: Probability: Theory and Examples, Third Edition(2005, Duxbury) Chapters 1,2,4.

§ Grading: HmWk/OnBb: 20 percent; Exams(mid-term and final): 80 percent. Notice: failure rate may be up to 20 percent

§ Ex's of Ch 1: p8 1.6,1.9,1.11(BP),1.12(BP);p11 2.4(RA),2.5(RA),2.7(RA);p12 2.8,2.9;p14 3.6;p15 3.8(!);p21 3.11(BP),3.13(RA),3.14(RA),3.18(RA);p28 4.6(BP),4.9(BP);p34 4.18,4.19,4.20;p37 5.2(look);p38 5.3(look);p45 5.3,5.8;p48 6.3(RA),6.4(RA);p51 6.8;p54 6.13,6.16(RA),6.18;p60 7.4;p68 8.3,8.9;p69 8.10

§ Ex's of Ch 2: p79 1.2, 1.3,;p83 2.2;p88 2.6,2.8;p89 2.9,2.10,2.11,2.15;p95 3.3,3.4;p99 3.13;p101 3.16;p102 3.20,3.21;p104 3.22,exam3.10;p105 3.23,3.25;p109 3.28;p113 4.1(BP);p114 4.5,4.6,4.7;p119 4.9,4.10,4.11,4.13;p137 6.1;p154 7.2;p159 7.5,7.6,7.8;p163 9.1;p167 9.5,9.6;p170 9.7,9.8

§ Ex's of Ch 4: p220 1.2; p223 1.3,1.4;p225 1.6,1.7;p226 1.9,1.10(BP),1.11;p229 2.2;p235 2.5,2.6,2.8,2.12;p236 2.14;p241 3.7,3.8;p246 3.12;p251 4.8,4.9;p260 5.2,5.3;p261 5.6,5.7;p262 5.8;p263 6.1;p268 6.3;p271 7.1;p273 7.4,7.6,7.8

Introductory

Why we need Probability Theory? Why we need go beyond *probability calculations*?!

§ the general definition of random variables(rv's) and expectations.

§ the precise definition of "with probability one" (as that in SLLN)

§ the general proof of CLT and its extension.

§ the general definition of conditional expectations and "fair games" (martingales).

Chapter 1: Laws of Large Numbers

§ Kolmogorov's definition of probability space $(\Omega, \mathcal{F}, P)$ as a measure space of total measure 1, probability measure

§ basic properties of pm: monotonicity, subadditivity, monotone continuity

§ examples: discrete, unit interval, finite products, infinite products

§ state space S, a separable complete metric space with Borel $\sigma-$algebra $\mathcal{B}(S)$

§ Kolmogorov's definition of an S-valued random variable(rv) as an S-valued measurable mapping.

§ probability distribution of a rv, distribution function of a (real-valued) rv, random vector

§ notation of skipping ω

§ properties of a df

§ characterization of a df, K-existence theorem

§ types of rv's: in terms of df

§ types of rv's: in terms of pd

§ decomposition theorem

§ examples: BPR for discrete and absolutely continuous rv's, one example of singularly continuous rv

§ one important estimate: tail distribution of $N(0,1)$

§ a property $\mathcal{P}$ holds with probability one (w pr 1)=holds almost sure (a.s.)

§ equivalence of two rv's; in "pointwise" sense and in distributional sense

§ the $\sigma-$algebra generated by a rv, by a family of rv's(in particular a random sequence)

§ operations of rv's

§ Kolmogorov's definition of the expectation EX as the Lebesgue integral of X w.r.t. P over Ω

§ integrability= 1st moment exists

§ properties of E: linear, ordered, additive

§ inequalities from RA: Jensen, Hólder, Cauchy-Schwarz,

§ Chebyshev inequality

§ a one-sided Chebyshev inequality and its application(BPR)

§ limit theorems from RA: Fatou's Lemma, MCT, BCT, LDCT

§ "change of variables" formula

§ Lemma: If $Y \geq 0$ and $p > 0$, $E(Y^p) = \int 0^\infty p y^{p-1} P(Y > 0) dy$.

§ BP Review: calculations of mean(expectation), moments(at least 2nd moment) and variance of some rv's

§ independence: two events, two rv's, tow sub- σ-algebras

§ the case of more than two: pairwise and totally

§ criterions for independence

§ (joint) pd of n independent rv's

§ $Eh(X, Y)$, for two independent X, Y

§ pd of $X + Y$, for two independent X, Y; examples

§ Kolmogorov existence(extension, construction) theorem for independent rv's, finitely many and infinitely many

§ convergence modes of random sequence: a.s. convergence, convergence in probability, convergence in the mean(in mean square, in $L^p(\Omega, P), 1 \geq p < \infty$), convergence in distributional sense

§ Cauchy criterion for convergence

§ additive property of variance for uncorrelated rv's

§ WLLN for L^2 uncorrelated random sequence, the iid case

§ Bernstein's approximation theorem

§ the triangular array and its WLLN

§ examples to be read

§ truncation

§ WLLN for the truncated array

§ WLLN for iid L^1 random sequence

§ St Petersburg "paradox"

§ Monte Carlo

§ $\{A_n \;\; i.o.\}$: the events happen infinitely often; the events happen all but finite many times

§ Lemma: $X_n \to 0 \;\; a.s.$ iff $\forall \epsilon > 0, \;\; P\{|X_n| > \epsilon \;\; i.o.\} = 0.$

§ Borel–Cantelli Lemma

§ proof of a RA theorem by BC Lemma

§ SLLN for iid rv's with 4th moment by BC Lemma

§ BC Lemma for independent events; a zero–one law

§ LLN does not hold when iid rv's with infinite mean

§ SLLN for pairwise independent rv's with same distribution and with 1st moment: an 1981 proof

§ a version of SLLN for iid rv's with infinite mean

§ SLLN for renewal processes

§ Glivenko–Cantelli theorem for empirical distributions

§ Kolmogorov 0–1 law

§ maximal inequality

§ "one-series" theorem

§ three-series theorem

§ LIL

§ random walks

Chapter 2: Central Limit Theorems

§ BPR: DeMoivre-Laplace CLT by hand proof

§ weak convergence of pm's

§ real line case: convergence in distributional sense

§ examples: DeM-L, G-C, $X_n = X + 1/n$, $pX_p, p \downarrow 0, X_p := G(p)$

§ theorem: weak convergence vs a.s. convergence

§ continuous mapping theorem

§ TFAE theorem for weak convergence

§ Helly's selection theorem (from RA)

§ tightness and its role in Helly's theorem

§ a criterion for tightness

§ characteristic function (chf) and its basic properties; why chf is better than mgf (you learn this BP)?

§ some calculations on chf's

§ fourier-Lévy inversion formula

§ absolutely continuous case in FL formula (fourier inverse transform in RA)

§ Lévy continuity theorem

§ chf for rv with n-th moment

§ chf for rv with 2nd moment

§ Polya's criterion of chf's; $\alpha-$stable distribution

§ moment problem

§ CLT: iid case

§ CLT for triangular arrays: Lindeberg-Feller theorem

§ Poisson convergence theorem

§ BSP Review: Poisson processes and compound Poisson processes

§ beyond CLT: stable laws(distributions), infinitely divisible laws

§ CLT for multidimensional state space $R^d, d > 1$

Remark: Chapter 3 on RWs is left to your own study; however the notions of stopping times in this chapter will be given in Chapter 4.

Chapter 4: Martingales

§ BP Review: a "fair game" view of SRW on the line

§ conditional probability and conditional expectation: a RA definition

§ lemma: existence and uniqueness in the above definition

§ conditional expectation w.r.t. a decomposition of Ω

§ conditional expectation of X w.r.t. Y; comparison of that in BP

§ basic properties of conditional expectations

§ conditional expectation of L^2 rv's: a Hilbert space point-of-view

§ conditional variance

§ filtration: an increasing sequence of sub-σ-algebras of Ω; natural filtration of a random sequence

§ definition of a martingale (fair game process), a submartingale, a supermartingale

§ remarks: continuous-time case; multiparameter-time case

§ stopping (optional) times

§ predictable sequence; the " stochastic integral" process $(H \cdot X)_n$

§ stopped rv, stopped process; the stopped process is still a martingale

§ upcrossings of a sequence over an interval $[a, b]$

§ a key property: the upper bound for EU_n, U_n: the upcrossings up to time n

§ martingale convergence theorem

§ Doob-Meyer decomposition theorem

§ an example from SRW

§ no “bounded infinite oscillations”

§ a conditional form of BC Lemma

§ martingale property of a Galton-Watson branching process

§ a theorem on EX_N, N bounded by k

§ Doob’s inequality (1)

§ Doob’s inequality (2)

§ martingale $L^p, 1 < p < \infty$, convergence theorem

§ uniform integrability and L^1 convergence: from RA

§ martingale L^1 convergence theorem

§ Lévy’s convergence theorem and 0–1 law

§ the stopped process is still u.i.

§ a theorem on EX_N, N any stopping time

§ stopped σ-algebra $\mathcal{F}_N$; a monotone lemma

§ optional stopping theorem

§ backwards (reversed) martingale and its convergence theorem

§ Hewitt-Savage 0–1 law

§ SLLN as a consequence of backwards martingale convergence theorem

§ square integrable martingales

§ martingale CLT

§ remarks: the continuous-time case, and the multiparameter-time case