One-day Workshop on Probability and Finance

July 10 (Thursday), 2008

Room 308, New Mathematics Building, National Taiwan University

About this Workshop

The meeting aims to address some researches on Probability Theory and Financial Mathematics. It consists of four one-hour lectures and a session of short communications. Informal walk-in talks are welcome for the SC session. No registration is required.

One-Hour Lectures, Morning Session	
10:00 - 11:00	**Horng-Tzer Yau** (Harvard University)
11:15 - 12:15	**Chuan-Hsiang Han** (National Tsing Hua University)
Lunch at the venue	
One-Hour Lectures, Afternoon Session	
13:30 - 14:30	**Ching-Tang Wu** (National Chiao Tung University)
14:45 - 15:45	**Guan-Yu Chen** (National Chiao Tung University)
Short Communications	
16:00 - 17:30	Each talk is less than 20 minutes. Now-scheduled: **N.-R. Shieh** (NTU)

Sponsors :

Taida Institute of Mathematical Science
(http://www.tims.ntu.edu.tw)
Mathematics Division, National Center for Theoretical Sciences (Taipei Office)
(http://math.cts.ntu.edu.tw/)
Department of Mathematics, National Taiwan University
(http://www.math.ntu.edu.tw/)

Local semicircle law and complete delocalization for Wigner random matrices

Horng-Tzer Yau

Joint work with L. Erdős, B. Schlein (Munich)

WIGNER ENSEMBLE

$H = (h_{jk})$ is a hermitian $N \times N$ matrix, $N \gg 1$.

$$h_{jk} = \frac{1}{\sqrt{N}}(x_{jk} + iy_{jk}), \quad (j < k), \quad h_{jj} = \sqrt{\frac{2}{N}}\, x_{jj}$$

where x_{jk}, y_{jk} $(j < k)$ and x_{jj} are independent with distributions

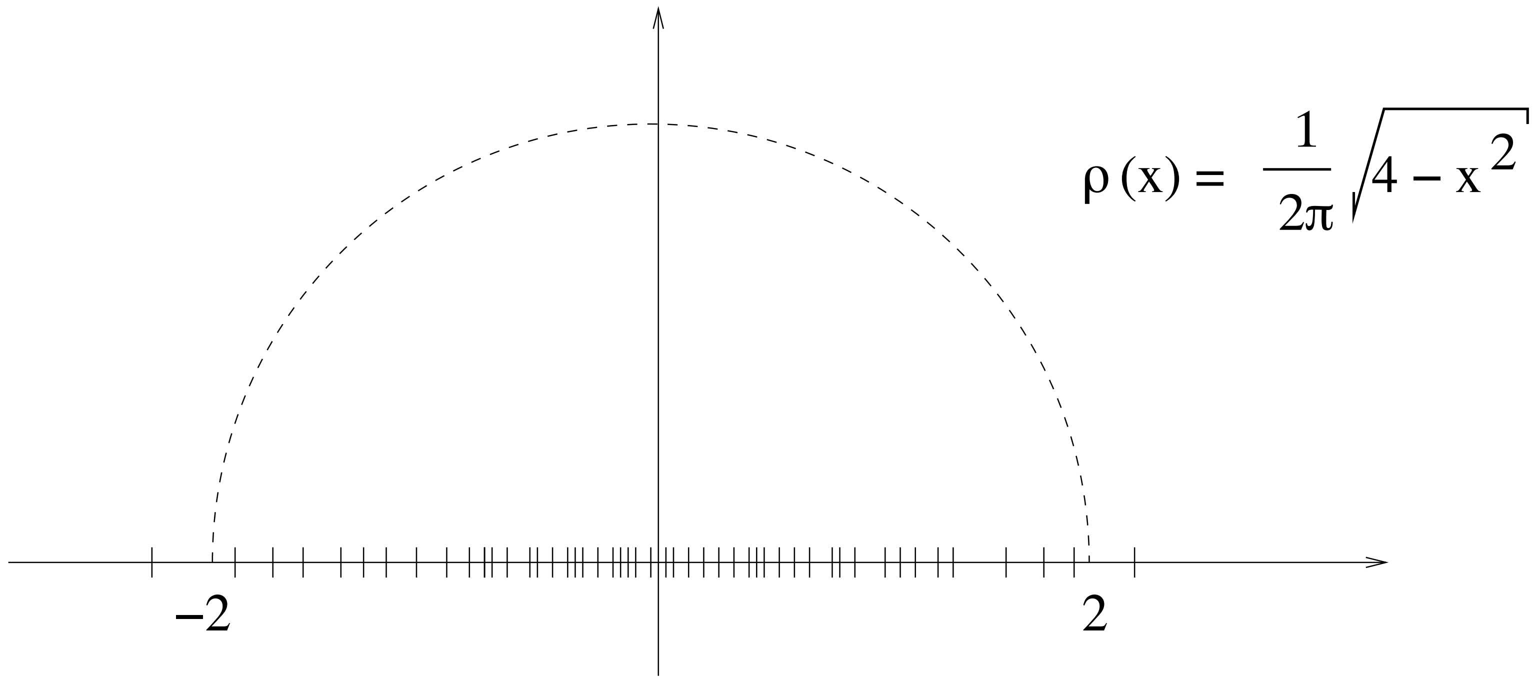
$$x_{jk},\ y_{jk} \sim \mathrm{d}\nu := e^{-g(x)}\mathrm{d}x,$$

Normalization: $\mathbf{E}\, x_{jk} = 0$, $\mathbf{E}\, x_{jk}^2 = \frac{1}{2}$.

Example: $g(x) = x^2$ is GUE.

Normalization ensures that $\mathrm{Spec}(H) = [-2, 2] + o(1)$

Results hold for real symmetric matrices as well, e.g. for GOE.



Eigenvalues: $E_1 \leq E_2 \leq \ldots\ldots E_N$

Typical eigenvalue spacing is $E_i - E_{i-1} \sim \frac{1}{N}$.

MAIN QUESTIONS

1) Density of states (DOS) — Wigner semicircle law.

2) Eigenvalue spacing distribution (Wigner-Dyson statistics and level repulsion);

3) (De)localization properties of eigenvectors.

RELATIONS:

• 2) is finer than 1) [bulk vs. individual ev.]

• Level repulsion $\Longleftrightarrow$ Delocalization ??? [Big open conjecture]

Motivation in background: Random Schrödinger operators in the extended states regime.

DENSITY OF STATES

$\mathcal{N}(I) := \#\{\mu_n \in I\}$ number of evalues μ_n of H in $I \subset \mathbb{R}$.

Smoothed density of states around E with window size η:

$$\varrho_\eta(E) = \frac{1}{N\pi} \operatorname{Im}\operatorname{Tr} \frac{1}{H - E - i\eta} = \frac{1}{N\pi} \sum_\alpha \frac{\eta}{(\mu_\alpha - E)^2 + \eta^2}$$

$\varrho_\eta(E)$ and $\mathcal{N}(I)$ with $I = [E - \frac{\eta}{2}, E + \frac{\eta}{2}]$ are closely related.

WIGNER SEMICIRCLE LAW

For any fixed $I \subset \mathbb{R}$,

$$\lim_{N\to\infty} \frac{\mathbf{E}\,\mathcal{N}(I)}{N} = \int_I \varrho_{sc}(x)\mathrm{d}x, \qquad \varrho_{sc}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\, \mathbf{1}(|x| \le 2)$$

Similar statement for $\varrho_\eta(E)$, window size $\eta = O(1)$ fixed.

Fluctations and almost sure convergence are also known.

The Wigner-Dyson statistics (universal distribution of eigenvalue spacing) requires info on individual evalues on a scale $\eta \sim 1/N$.

It is believed to hold for general Wigner matrices, but proven only for Gaussian and related models and the proofs use explicit formulas for the joint ev. distribution. [Dyson, Deift, Johansson]

GOALS:

(i) Prove Semicircle Law for any scales $\eta \gg N^{-1}$.

(ii) Prove that eigenvectors are delocalized.

Theorem 1: Fix $\kappa, \varepsilon > 0$. Let $\eta \gg \frac{(\log N)^8}{N}$, then, as $N \to \infty$,

$$\mathbf{P}\left\{ \sup_{|E| \le 2-\kappa} \left| \frac{\mathcal{N}[E - \frac{\eta}{2}, E + \frac{\eta}{2}]}{N\eta} - \varrho_{sc}(E) \right| \ge \varepsilon \right\} \le e^{-c(\log N)^2}$$

i.e. Semicircle Law holds for energy windows $\sim 1/N$ (mod logs)

Theorem 2: Fix $\kappa > 0$, then

$$\mathbf{P}\left\{ \exists \mathbf{v}, \|\mathbf{v}\|_2 = 1,\ H\mathbf{v} = \mu\mathbf{v},\ |\mu| \le 2-\kappa,\ \|\mathbf{v}\|_\infty \ge \frac{(\log N)^5}{N^{1/2}} \right\} \le e^{-c(\log N)^2}$$

i.e. almost all eigenfunctions are fully delocalized.

ASSUMPTIONS on the single site distribution $d\nu = e^{-g(x)}dx$

(i) $\sup g'' < \infty$

(ii) There exists $\delta > 0$ such that $\int e^{\delta x^2} \mathrm{d}\nu(x) < \infty$

(iii) $\mathrm{d}\nu$ satisfies the logarithmic Sobolev inequality

$$\int u \log u \, \mathrm{d}\nu \leq C \int |\nabla \sqrt{u}|^2 \, \mathrm{d}\nu$$

Item (i) was needed for a concentration Lemma. J. Bourgain has informed us that this lemma also holds if (i) is replaced by a decay stronger than Gaussian (e.g. bounded r.v.).

Lemma: [Upper bound]. Assume $g'' < \infty$. Let $|I| \geq \frac{\log N}{N}$, then

$$\mathbf{P}\{\mathcal{N}(I) \geq KN|I|\} \leq e^{-cKN|I|}$$

for large K. Similar result holds for $\mathbf{P}\{\varrho_\eta(E) \geq K\}$.

Proof: Decompose

$$H = \begin{pmatrix} h & \mathbf{a}^* \\ \mathbf{a} & B \end{pmatrix}, \qquad h \in \mathbb{C}, \;\; \mathbf{a} \in \mathbb{C}^{N-1}, \;\; B \in \mathbb{C}^{(N-1)\times(N-1)}$$

Let $\lambda_\alpha, \mathbf{u}_\alpha$ be the ev's of B and define

$$\xi_\alpha := N|\mathbf{a} \cdot \mathbf{u}_\alpha|^2, \qquad\qquad \mathbf{E}\xi_\alpha = 1$$

For the (1,1) matrix element of $G_z = (H - z)^{-1}$, $z = E + i\eta$:

$$G_z(1,1) = \frac{1}{h - z - \mathbf{a} \cdot (B - z)^{-1}\mathbf{a}} = \left[h - z - \frac{1}{N} \sum_{\alpha=1}^{N-1} \frac{\xi_\alpha}{\lambda_\alpha - z} \right]^{-1}$$

$$|G_z(1,1)| \le \frac{1}{\left|\mathrm{Im}\left[h - z - \frac{1}{N}\sum_\alpha \frac{\xi_\alpha}{\lambda_\alpha - z}\right]\right|} \le \frac{\eta^{-1}}{1 + \frac{1}{N}\sum_\alpha \frac{\xi_\alpha}{(\lambda_\alpha - E)^2 + \eta^2}} \le \frac{N\eta}{\sum\limits_{\alpha\,:\,\lambda_\alpha \in I} \xi_\alpha}$$

for any interval $I = [E - \eta, E + \eta]$.

$$\mathcal{N}_I \le C\eta\,\mathrm{Im}\,\mathrm{Tr} G_z \le C\eta \sum_{k=1}^{N} |G_z(k,k)|$$

Repeating the above construction for each k,

$$\mathcal{N}_I \le CN\eta^2 \sum_{k=1}^{N} \Big| \sum_{\alpha\,:\,\lambda_\alpha^{(k)} \in I} \xi_\alpha^{(k)} \Big|^{-1}$$

so to get an upper bound on $\mathcal{N}_I$, we need a lower bound on $\sum \xi_\alpha$.

Good news: For the decomposition

$$H = \begin{pmatrix} h & \mathbf{a}^* \\ \mathbf{a} & B \end{pmatrix},$$

the eigenvalues μ_α of H and λ_α of B are *interlaced*:

$$\mu_1 \le \lambda_1 \le \mu_2 \le \lambda_2 \le \dots$$

so the number of $\lambda_\alpha \in I$ is $\mathcal{N}(I) \pm 1$.

$$\mathcal{N}_I \le CN\eta^2 \sum_{k=1}^{N} \Big| \sum_{\alpha\,:\,\lambda_\alpha^{(k)} \in I} \xi_\alpha^{(k)} \Big|^{-1}$$

Suppose

$$\sum_{\alpha\,:\,\lambda_\alpha^{(k)} \in I} \xi_\alpha^{(k)} \ge c\,\#\{\lambda_\alpha^{(k)} \in I\} \ge c\mathcal{N}(I)$$

(recalling $\mathbf{E}\,\xi = 1$ and hoping for weak correlation) then we had

$$\mathcal{N}(I) \lesssim \frac{N^2\eta^2}{\mathcal{N}(I)} \quad\Longrightarrow\quad \mathcal{N}(I) \lesssim N\eta$$

Lower bound on $\sum_\alpha \xi_\alpha$:

Recall $\xi_\alpha = N|\mathbf{a}\cdot\mathbf{u}_\alpha|^2$. Note that $\mathbf{a}$ is indep of $\lambda_\alpha, \mathbf{u}_\alpha$. The ξ_α's are not independent, but almost, so their sum has a strong concentration property.

Lemma: Let $g'' < \infty$ or $\operatorname{supp}\nu$ compact, then

$$\mathbf{P}\left(\sum_{\alpha\in A} \xi_\alpha \le \delta|A|\right) \le e^{-c|A|}$$

Note $$\sum_{\alpha\in A}\xi_\alpha = N\sum_{\alpha\in A}|\mathbf{a}\cdot\mathbf{u}_\alpha|^2 = N|P_A\mathbf{a}|^2, \qquad P_A = \text{proj}$$

Lemma: Let $\mathbf{z} = (z_1, \ldots z_N)$, $z_j = x_j + iy_j$, $x_j, y_j \sim \mathrm{d}\nu(x)$. Let P be a projection of rank m in $\mathbb{C}^N$. Then

$$\mathbf{E}\, e^{-c(P\mathbf{z},P\mathbf{z})} \le e^{-c'\mathbf{E}(P\mathbf{z},P\mathbf{z})} = e^{-c'm}$$

Proof: Brascamp-Lieb or Bourgain's decoupling method.

Proof of the local semicircle law: Consider the Stieltjes transform

$$m(z) = \int \frac{\varrho(x)\mathrm{d}x}{x - z}$$

The Stieltjes tr. of the semicircle law satisfies

$$m_{sc}(z) + \frac{1}{m_{sc}(z) + z} = 0$$

This fixed point equation is stable away from the spectral edge.

Let $m(z)$ be the Stieltjes tr. of the empirical density of H, and $m^{(k)}(z)$ that of the minor $B^{(k)}$:

$$m(z) = \frac{1}{N}\mathrm{Tr}\frac{1}{H - z}, \qquad m^{(k)}(z) = \frac{1}{N-1}\mathrm{Tr}\frac{1}{B^{(k)} - z}$$

Then from the expansion

$$m(z) = \frac{1}{N}\sum_{k=1}^{N} G_z(k,k) = \frac{1}{N}\sum_{k=1}^{N} \frac{1}{h_{kk} - z - \mathbf{a}^{(k)} \cdot \frac{1}{B^{(k)}-z}\mathbf{a}^{(k)}}$$

obtain

$$m = \frac{1}{N}\sum_{k=1}^{N} \frac{1}{h_{kk} - z - \left(1 - \frac{1}{N}\right)m^{(k)} - X_k}$$

with

$$X_k = \mathbf{a}^{(k)} \cdot \frac{1}{B^{(k)} - z}\mathbf{a}^{(k)} - \underbrace{\mathbf{E}_k\Big[\mathbf{a}^{(k)} \cdot \frac{1}{B^{(k)} - z}\mathbf{a}^{(k)}\Big]}_{=(1-\frac{1}{N})m^{(k)}} = \frac{1}{N}\sum_{\alpha=1}^{N-1} \frac{\xi_\alpha^{(k)} - 1}{\lambda_\alpha^{(k)} - z}$$

(recall $\xi_\alpha^{(k)} = N|\mathbf{a}^{(k)} \cdot \mathbf{u}_\alpha^{(k)}|^2$, $\mathbf{E}_k\xi_\alpha^{(k)} = 1$)

$$m = \frac{1}{N}\sum_{k=1}^{N} \frac{1}{h_{kk} - z - \left(1 - \frac{1}{N}\right)m^{(k)} - X_k}$$

(i) $\mathbf{P}\{h_{kk} \geq \varepsilon\} \leq e^{-\delta\varepsilon^2 N}$

(ii) By interlacing property $\left| m - \left(1 - \frac{1}{N}\right)m^{(k)} \right| = o(1)$

(iii) **Lemma:** $\mathbf{P}\{|X_k| \geq \varepsilon\} \leq e^{-c\varepsilon(\log N)^2}$

Then, away from an event of tiny prob, we have

$$m = -\frac{1}{N}\sum_{k=1}^{N} \frac{1}{m + z + \delta_k}$$

where the random variables δ_k satisfy $|\delta_k| \leq \varepsilon$. From stability of the equation $m_{sc} = -\frac{1}{m_{sc}+z}$, we get $|m - m_{sc}| \leq C\varepsilon$.

Proof of Lemma: Forget k.

$$X = \frac{1}{N}\sum_{\alpha=1}^{N-1}\frac{\xi_\alpha - 1}{\lambda_\alpha - z}, \qquad \xi = |\mathbf{b}\cdot\mathbf{v}_\alpha|^2,$$

With a high prob in the prob. space of the minor, we have $\#\{\lambda_\alpha \in I\} \le N\eta(\log N)^2$. Fix such an event and play with $\mathbf{a}$.

Compute

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\Big[e^{-\beta}\log\mathbf{E}\, e^{e^\beta X}\Big] = e^{-\beta}\mathbf{E}\, u\log u \le Ce^{-\beta}\mathbf{E}\,|\nabla\sqrt{u}|^2, \qquad u := \frac{e^{e^\beta X}}{\mathbf{E}\, e^{e^\beta X}}$$

and

$$e^{-\beta}\mathbf{E}\,|\nabla\sqrt{u}|^2 \le e^\beta\mathbf{E}\left[u\sum_k\left|\frac{\partial X}{\partial b_k}\right|^2\right] = \frac{e^\beta}{N^2}\mathbf{E}\Big[u\sum_\alpha\frac{\xi_\alpha}{|\lambda_\alpha - z|^2}\Big] \le \frac{e^\beta}{N\eta}\mathbf{E}\,[uY]$$

with $Y = \frac{1}{N}\sum_\alpha\frac{\xi_\alpha}{|\lambda_\alpha - z|}$.

$$e^{-\beta}\mathbf{E}\,|\nabla\sqrt{u}|^2 \le \frac{e^{\beta}}{N\eta}\mathbf{E}\,[uY], \qquad Y = \frac{1}{N}\sum_\alpha \frac{\xi_\alpha}{|\lambda_\alpha - z|}$$

Use entropy inequality

$$\mathbf{E}\,[uY] \le \gamma^{-1}\mathbf{E}\,u\log u + \gamma^{-1}\mathbf{E}\,e^{\gamma Y}$$

(with optimal $\gamma \sim e^{\beta}/N\eta$) and log-Sobolev once more to get

$$\mathbf{E}\,|\nabla\sqrt{u}|^2 \le \mathbf{E}\,e^{\gamma Y}$$

Integrate the inequality

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\Big[e^{-\beta}\log\mathbf{E}\,e^{e^{\beta}X}\Big] \le e^{-\beta}\mathbf{E}\,e^{\gamma Y}$$

from $-\infty$ to $\beta_0 \sim \frac{1}{2}\log(N\eta) - 2\log\log N$.

The boudary term at $\beta = -\infty$ vanishes since $\mathbf{E}\,X = 0$, thus

$$\log \mathbf{E}\, e^{e^{\beta_0 X}} \leq \mathbf{E}\, e^{\delta Y}, \qquad Y = \frac{1}{N} \sum_\alpha \frac{\xi_\alpha}{|\lambda_\alpha - z|}$$

with $\delta \sim 1/(\log N)^4 \ll 1$.

Since $\xi_\alpha = |\mathbf{b} \cdot \mathbf{v}_\alpha|^2$ has finite exponential moment, if there are not too many λ_α near E, then Y has finite exponential moment for a small δ.

This controls the exponential moment of X.

EXTENDED STATES: EIGENVECTOR DELOCALIZATION

No concept of absolutely continuous spectrum.

$\mathbf{v} \in \mathbb{C}^N$, $\|\mathbf{v}\|_2 = 1$ is **extended** if $\|\mathbf{v}\|_p \sim N^{\frac{1}{p}-\frac{1}{2}}$, $p \neq 2$.

E.g. For GUE, all eigenvectors have $\|\mathbf{v}\|_4 \sim N^{-1/4}$ (symmetry)
Question: in general for Wigner? [T. Spencer]

Our Theorem 2 answers to this in the strongest possible norm, with log corrections, for all eigenvectors (away from the edge)

Theorem 2: Fix $\kappa > 0$, then

$$\mathbf{P}\left\{\exists \mathbf{v}, \|\mathbf{v}\|_2 = 1,\ H\mathbf{v} = \mu\mathbf{v},\ |\mu| \leq 2-\kappa,\ \|\mathbf{v}\|_\infty \geq \frac{(\log N)^5}{N^{1/2}}\right\} \leq e^{-c(\log N)^2}$$

Proof. Decompose as before $$H = \begin{pmatrix} h & \mathbf{a}^* \\ \mathbf{a} & B \end{pmatrix},$$

Let $H\mathbf{v} = \mu\mathbf{v}$ and $\mathbf{v} = (v_1, \mathbf{w})$, $\mathbf{w} \in \mathbb{C}^{N-1}$. Then

$$hv_1 + \mathbf{a}\cdot\mathbf{w} = \mu v_1, \qquad \mathbf{a}v_1 + B\mathbf{w} = \mu\mathbf{w} \quad \Longrightarrow \quad \mathbf{w} = (\mu - B)^{-1}\mathbf{a}v_1$$

From the normalization, $1 = \|\mathbf{w}\|^2 + |v_1|^2$, we have

$$|v_1|^2 = \frac{1}{1 + \frac{1}{N}\sum_\alpha \frac{\xi_\alpha}{(\mu-\lambda_\alpha)^2}} \le \frac{1}{\frac{1}{N}\frac{1}{(q/N)^2}\sum_{\alpha\in A}\xi_\alpha}, \qquad (\xi_\alpha := N|\mathbf{a}\cdot\mathbf{u}_\alpha|^2)$$

where recall $\lambda_\alpha, \mathbf{u}_\alpha$ are the ev's of B and let

$$A = \Big\{\alpha \;:\; |\lambda_\alpha - \mu| = \frac{q}{N}\Big\} \qquad q \sim (\log N)^8$$

Concentration ineq. and lower bound on the local DOS imply

$$\sum_{\alpha\in A}\xi_\alpha \ge c|A| \ge cq$$

with very high probability, thus

$$|v_1|^2 \le \frac{q}{N} \quad \Longrightarrow \quad \|\mathbf{v}\|_\infty \le N^{-1/2} \quad \text{modulo logs}$$

SUMMARY

- All results for general Wigner matrices, no Gaussian formulas

- We established the Semicircle Law for the DOS on scale $\frac{(\log N)^8}{N}$ (optimal modulo logs)

- All eigenvectors are fully delocalized away from the spectral edges. Optimal estimate on the sup norm (modulo logs)

OPEN QUESTIONS:

- Are all conditions necessary (strong decay plus log-Sobolev)?

- Wigner-Dyson distribution of level spacing [DREAM...]

Large Deviations, Small Default Probabilities and Importance Sampling

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TIMS

July 10, 2008

Outline

- Credit Derivatives: market data and issues
- Approach I - reduced form: copula method
- Approach II - structural form: first passage time problem
- Modification: stochastic correlation
- Conclusions and Future works

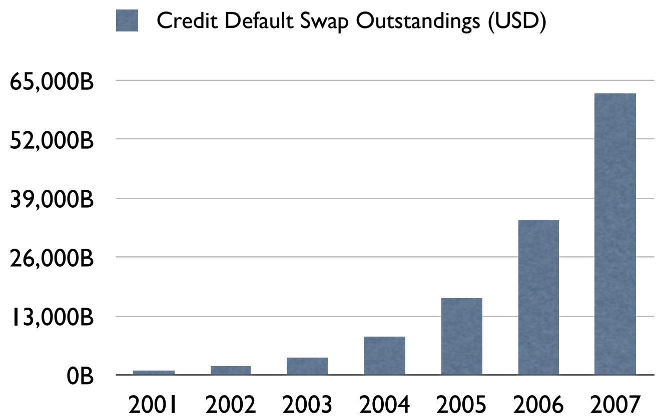
Introduction of Credit Derivatives

- A contract between two parties whose values are contingent on the creditworthiness of underlying asset (s).

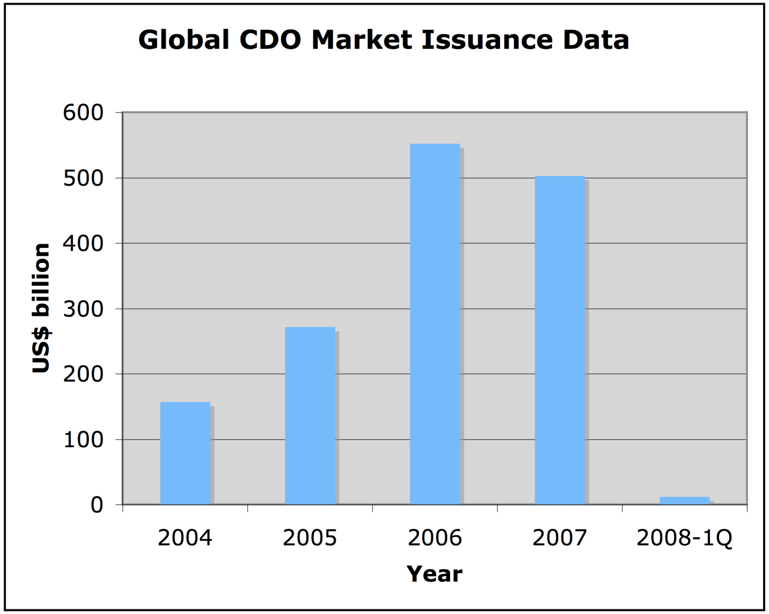
- Single-name: only one reference asset, like CDS (Credit Default Swaps).

- Multi-name: several assets in one basket, like CDO (Collateralized Debt Obligations) or BDS (Basket Default Swaps).

Credit Default Swap



ISDA Market Survey



Source: Securities Industry and Financial Markets Association.

A Example: Credit Swap Evaluation

$$premium =$$
$$\mathbb{E}\left\{(1-R)\times B(0,\tau)\times \mathbf{I}(\tau<T)\right\} /$$
$$\mathbb{E}\left\{\sum_{j=1}^{N} \triangle_{j-1,\,j}\times B(0,t_j)\times \mathbf{I}(\tau>t_j)\right\}$$

Notations: τ: default time, R: recovery rate, $B(0,t)$: discount factor, $\triangle_{j-1,\,j}$: time increment.

Some Mathematical Issues

- Modeling default time
- Modeling correlations between default times
- Estimating joint default probability: rare event in high dimension

Approaches to Modeling Default Times

- Intensity-Based (Reduced Form)

View firm's default as extraneous, modeling the hazard rate of the firm.

$$\mathbb{P}(\tau \le t) = F(t) = 1 - \exp\left\{-\int_0^t h(s)ds\right\}.$$

- Asset Value-Based (Structural Form)

First passage time problem: in 2-d

$$\begin{cases} dS_{1t} = \mu_1 S_{1t}\, dt + \sigma_1 S_{1t}\, dW_{1t} \\ dS_{2t} = \mu_2 S_{2t}\, dt + \sigma_2 S_{2t}\, d(\rho W_{1t} + \sqrt{1-\rho^2}\, W_{2t}) \end{cases}$$

Joint default occurs if $S_{1t} < B_1$ and $S_{2t} < B_2$ for some $t \le T$.

Reduced Form Approach: Copula Method*

Default Times Modeling: $\left\{\tau_i = F_i^{-1}(U_i)\right\}_{i=1}^{n}$,
U's are (standard) uniform random variables.

A n-dimensional Copula is a distribution function on $[0,1]^n$ with uniform marginal distributions.

Through a **copula function**, one can build up correlations between default times.

*Cherubini, Luciano, Vecchiato (2004), Nelson(2006).

Gaussian Copula

- Li (2000) introduced Gaussian copula

  $$C(u_1, u_2, \cdots, u_n; \Sigma) = \Phi_\Sigma(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \cdots, \Phi^{-1}(u_n)),$$

  where Σ denotes the variance-covariance matrix.

- Laurent and Gregory (2003) introduced Gaussian Factor Copula so that parameters numbers are reduced from $O(n^2)$ to $O(n)$.

- Easy to compute but lack of economic sense.

Structural Form Approach: Review

- Merton (1974) applied Black-Scholes Option Theory (1973). Default time only happens at maturity.

- Black and Cox (1976) proposed the first passage time problem (1-dim) to model default event.

- Zhou (2001) extended to 2-dim case.

Credit Risk Modeling: Structural Form Approach

Multi-Names Dynamics: for $1 \le i \le n$

$$
\begin{aligned}
dS_{it} &= \mu_i S_{it} dt + \sigma_i \, S_{it} dW_{it}, \\
d \left\langle W_{it}, W_{jt} \right\rangle &= \rho_{ij} dt.
\end{aligned}
$$

Each default time τ_i for the i^{th} name is defined as $\tau_i = \inf\{t \ge 0 : S_{it} \le B_i\}$, where B_i denotes the i^{th} debt level.
The i^{th} default event is defined as $\{\tau_i \le T\}$.

Joint Default Probability: First Passage Time Problem

Q: How to compute, for any finite n names,

$$DP = \mathbb{E}\left\{\Pi_{i=1}^{n}\mathbf{I}_{(\tau_i \leq T)} \mid \mathcal{F}_t\right\}?$$

Explicit Formulas exist for 1 and 2 names cases so far...(no mention for stochastic correlation/volaility...)

Multi-Dimensional Girsanov Theorem

Given a Radon-Nikodym derivative

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = Q_T^h = e^{\left(\int_0^T h(s,S_s)\cdot d\tilde{W}_s - \frac{1}{2}\int_0^T ||h(s,S_s)||^2 ds\right)},$$

$\tilde{W}_t = W_t + \int_0^t h(s, S_s)ds$ is a vector of Brownian motions under $\tilde{\mathbb{P}}$. Thus

$$DP = \tilde{\mathbb{E}}\left\{\prod_{i=1}^n \mathbf{I}_{(\tau_i \le T)} Q_T^h\right\}.$$

If $h = -\frac{1}{DP}\sigma^T \nabla DP$, zero variance for the new estimator.

Monte Carlo Simulations: Importance Sampling

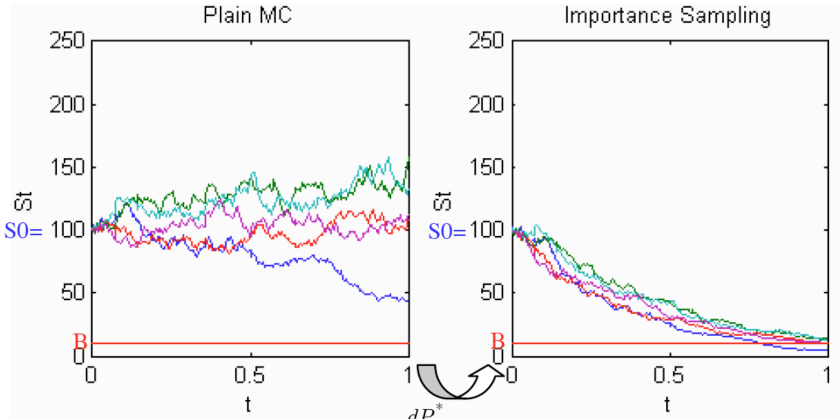
An importance sampling method is to select a constant vector $h = (h_1, \cdots, h_n)$ to satisfy the following n conditions

$$\tilde{E}\{S_{iT}|\mathcal{F}_0\} = B_i, i = 1, \cdots, n.$$

Each h_i can be uniquely determined by **the linear system**
$\Sigma_{j=1}^{i}\rho_{ij}h_j = \frac{\mu_i}{\sigma_i} - \frac{\ln B_i/S_{i0}}{\sigma_i T}$, for $i = 1, \cdots, n$.

Trajectories under different measures
Single Name Case

Simulation of the stock price :



$$\frac{dP^*}{d\widetilde{P}} = Q_T$$

Single Name Default Probability

B	BMC	Exact Sol	Importance Sampling
50	0.0886 (0.0028)	0.0945	0.0890 (0.0016)
20	0 (0)	$7.7 * 10^{-5}$	$7.2 * 10^{-5}(2.3 * 10^{-6})$
1	0 (0)	$1.3 * 10^{-30}$	$1.8 * 10^{-30}(3.4 * 10^{-31})$

Number of simulations are 10^4 and the Euler discretization takes time step size $T/400$, where T is one year. Other parameters are $S_0 = 100, \mu = 0.05$ and $\sigma = 0.4$.

Three-Names Joint Default Probability

ρ	BMC	Importance Sampling
0.3	$0.0049(6.98 * 10^{-4})$	$0.0057(1.95 * 10^{-4})$
0	$3.00 * 10^{-4}(1.73 * 10^{-4})$	$6.40 * 10^{-4}(6.99 * 10^{-5})$
-0.3	$0(0)$	$2.25 * 10^{-5}(1.13 * 10^{-5})$

Parameters are $S_{10} = S_{20} = S_{30} = 100, \mu_1 = \mu_2 = \mu_3 = 0.05$, $\sigma_1 = \sigma_2 = 0.4, \sigma_3 = 0.3$ and $B_1 = B_2 = 50, B_3 = 60$. Standard errors are shown in parenthesis.

Effect of Correlation! Debt to Asset Ratios (B_i/S_{i0}) are not small.

We propose an algorithm to compute the joint default prob.

In fact, the choice of our new measure is optimal in Large Deviations Theory.

Large Deviations Theory:
Cramer's Theorem

Let $\{X_i\}$ be real-valued IID r.v.'s under $\mathbb{P}$ and $\mathbb{E}X_1 < \infty$. For any $x \geq \mathbb{E}X_1$, we have

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\left[\frac{S_n}{n} \geq x\right] = -\Gamma^*(x) = -\inf_{y\geq x} \Gamma^*(y).$$

1. $S_n = \sum_{i=1}^{n} X_i$: sample sum
2. $\Gamma(\theta) = \ln \mathbb{E}\left[e^{\theta X_1}\right]$: the cumulant function
3. $\Gamma^*(x) = \sup_{\theta\in\Re} [\theta\, x - \Gamma(\theta)]$: Legendre transform of Γ (also called rate function).

Tie to Importance Sampling

Define an expo. change of measure $\mathbb{P}_\theta$ by

$$\begin{aligned} \frac{d\mathbb{P}_\theta}{d\mathbb{P}} &= exp\left(\theta\, S_n - n\, \Gamma(\theta)\right), \\ p_n := \mathbb{P}\left[\frac{S_n}{n} \geq x\right] &= \mathbb{E}_\theta\left[\mathbf{I}_{\frac{S_n}{n}\geq x} exp\left(-\theta\, S_n + n\, \Gamma(\theta)\right)\right]. \end{aligned}$$

The optimal 2^{nd} moment ($M_n^2(\theta, x)$) of the new estimator can be shown as

$M_n^2(\theta_x, x) \approx p_n^2$, where $\Gamma^*(x) = \theta_x\, x - \Gamma(\theta_x)$.

Under the optimal measure, the event is not rare any more! (Note: $\mathbb{E}_{\theta_x}[S_n/n] = x$.)

Large Deviation Principle (LDP)

A $\mathcal{X}$-valued seq. $\{Z^\varepsilon\}_\varepsilon$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfies a LDP with the rate function I if
(1) Upper Bound: for any closed subset F of $\mathcal{X}$, $\limsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}[Z^\varepsilon \in F] \le -\inf_{x\in F} I(x)$
(2) Lower Bound: for any open subset G of $\mathcal{X}$, $\limsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}[Z^\varepsilon \in G] \ge -\inf_{x\in G} I(x)$

If $F \subseteq \mathcal{X}$ s.t. $\inf_{x\in F^0} I(x) = \inf_{x\in \bar{F}} := I_F$, then

$$\limsup_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}[Z^\varepsilon \in F] = -I_F.$$

Freidlin-Wentzell Theorem

The solution of

$$
\begin{aligned}
dX_t^\varepsilon &= b(X_t^\varepsilon)dt + \sqrt{\varepsilon}\sigma(X_t^\varepsilon)dW_t, \\
X_0^\varepsilon &= x,
\end{aligned}
$$

satisfies a LDP with the rate function

$I(f) =$

$$\frac{1}{2}\int_0^T < \dot{f}(t) - b(f(t)), a^{-1}(f(t))(\dot{f}(t) - b(f(t))) > dt$$

for some nice function f, or $I(f) = \infty$ otherwise. $a(x) = \sigma(x)\,\sigma'(x)$.

Single-Name Default Prob. Approximation

$$
\begin{aligned}
& \mathbb{P}\left[\inf_{0\le t\le T} S_t = S_0\, e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t} \le B\right] \\
= \;& \mathbb{E}\left[\mathbf{I}\left(\inf_{0\le t\le T} \varepsilon\left(\mu-\frac{\sigma^2}{2}\right)t + \varepsilon\sigma W_t \le -1\right)\right] \\
:= \;& P_\varepsilon \qquad (\textbf{scaling by } \ln(B/S_0) = \frac{-1}{\varepsilon}) \\
\approx \;& \exp\left(\frac{-1}{\varepsilon^2\, 2\, \sigma^2\, T}\right). \qquad (\text{ by } \textbf{F-W Thm} \text{ })
\end{aligned}
$$

Importance Sampling: 2^{nd} Moment Approximation

$$\tilde{\mathbb{E}}\left[\mathbf{I}\left(\inf_{0\le t\le T} S_t \le B\right) e^{2\,h\,\tilde{W}_T - h^2\,T}\right]$$

$$S_t = S_0\, e^{\left(\mu - \frac{\sigma^2}{2} - \sigma h\right)t + \sigma \tilde{W}_t},\ h = \frac{\mu}{\sigma} - \frac{\ln B/S_0}{\sigma\,T}$$

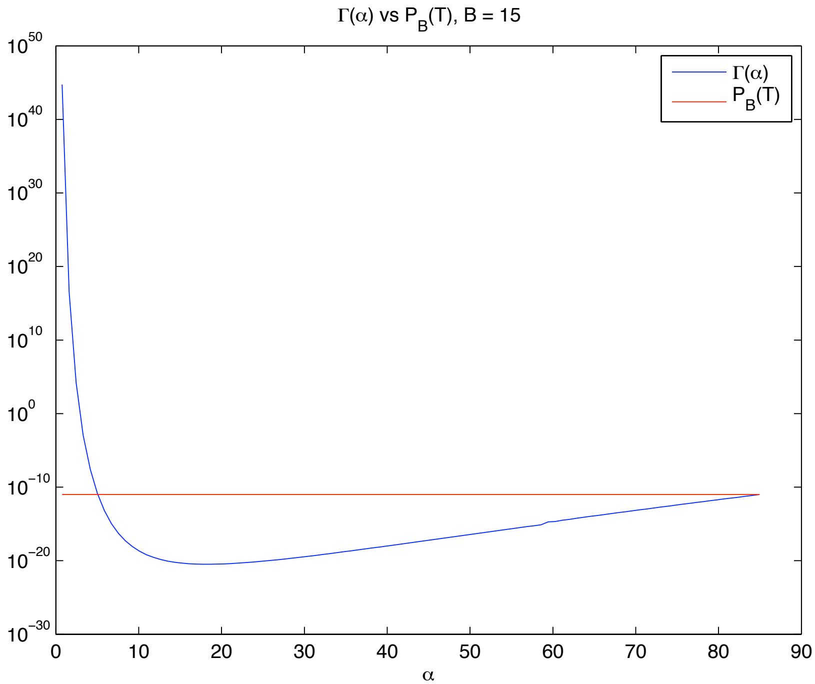
$$= \hat{\mathbb{E}}\left[\mathbf{I}\left(\inf_{0\le t\le T} S_0\, e^{\left(\mu - \frac{\sigma^2}{2} + \sigma h\right)t + \sigma \hat{W}_t} \le B\right)\right] e^{h^2\,T}$$

2^{nd} Moment Approximation (Cont.)

$$
\begin{aligned}
= \quad & \hat{E}\left[\mathbf{I}\left(\inf_{0\le t\le T}\left(\varepsilon\left(2\mu-\frac{\sigma^2}{2}\right)+\frac{1}{T}\right)t+\varepsilon\sigma\hat{W}_t\le -1\right)\right] \\
& \times e^{\left(\frac{r}{\sigma}+\frac{1}{\varepsilon\sigma T}\right)^2 T} \qquad (\textbf{scaling} \text{ by } \ln(B/S_0)=\frac{-1}{\varepsilon}) \\
:= \quad & M_\varepsilon^2 \\
\approx \quad & \exp\left(\frac{-1}{\varepsilon^2\sigma^2 T}\right). \qquad (\text{ by } \textbf{F-W Thm} \text{ })
\end{aligned}
$$

Theorem: By $M_\varepsilon^2 \approx (P_\varepsilon)^2$ we observe the optimality of chosen measure.

The Optimal Variance Reduction: A Numerical Evidence



A Modification: Stochastic Correlation

$$\begin{cases} dS_t^1 = rS_t^1 dt + \sigma_1 S_t^1 dW_t^1 \\ dS_t^2 = rS_t^2 dt + \sigma_2 S_t^2 d(\rho(Y_t) dW_t^1 + \sqrt{1-\rho^2(Y_t)} dW_t^2) \\ dY_t = \frac{1}{\delta}(m - Y_t)dt + \frac{\sqrt{2}\beta}{\sqrt{\delta}} dZ_t \end{cases}$$

Joint default probability

$$P^\delta(t, x_1, x_2, y) := \mathbb{E}_{x_1,x_2,y}\left\{ \prod \mathbf{I}_{\{\min_{t \le u \le T} S_t^i \le B_i\}} \right\}$$

In this case, the construction of our IS method **fails!**

Full Expansion of P^δ

Theorem

$$P^\delta(t, x_1, x_2, y) = \sum_{i=0}^{\infty} \delta^i P_i(t, x_1, x_2, y),$$

where P's can be obtained recursively and the y variable can be factored out (separate).

Proof: by means of Singular Perturbation Techniques.
Accuracy results are ensured given smoothness of terminal condition.

Leading Order Term

$P_0(t, x_1, x_2)$ solves the **homogenized** PDE (y-independent).

$$\left(\mathcal{L}_{1,0} + \bar{\rho}\,\mathcal{L}_{1,1}\right) P_0(t, x_1, x_2) = 0$$

$\bar{\rho} =< \rho(y) >$, average taken wrt the **invartiant measure** of Y.

Differential operators are

$$\begin{aligned}
\mathcal{L}_{1,0} &= \frac{\partial}{\partial t} + \sum_{i=1}^{2} \frac{\sigma_i^2 x_i^2}{2} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{2} \mu_i x_i \frac{\partial}{\partial x_i} \\
\mathcal{L}_{1,1} &= \sigma_1 \sigma_2 x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2}.
\end{aligned}$$

Other Terms

$$P_{n+1}(t, x_1, x_2, y) = \sum_{i\geq 0, j\geq 1}^{i+j=n+1} \varphi_{i,j}^{(n+1)}(y)\, \mathcal{L}_{1,0}^{i}\, \mathcal{L}_{1,1}^{j}\, P_n$$

where a seq. of Poisson eqns to be solved:

$$\mathcal{L}_0\, \varphi_{i+1,j}^{(n+1)}(y) = \left(\varphi_{i,j}^{(n)}(y) - < \varphi_{i,j}^{(n)}(y) >\right)$$

$$\mathcal{L}_0\, \varphi_{i,j+1}^{(n+1)}(y) = \left(\rho(y)\, \varphi_{i,j}^{(n)}(y) - < \rho\, \varphi_{i,j}^{(n)} >\right),$$

where $\mathcal{L}_0 = \beta^2 \frac{\partial^2}{\partial y^2} + (m - y)\frac{\partial}{\partial y}$.

Numerical Result I: Stochastic Correlation

$\alpha = \frac{1}{\delta}$	BMC	Importance Sampling
0.1	$0.0037(6 * 10^{-4})$	$0.0032(1 * 10^{-4})$
1	$0.0074(9 * 10^{-4})$	$0.0065(2 * 10^{-4})$
10	$0.0112(1 * 10^{-3})$	$0.0116(4 * 10^{-4})$
50	$0.0163(1 * 10^{-3})$	$0.0137(5 * 10^{-4})$
100	$0.016(1 * 10^{-3})$	$0.0132(4 * 10^{-4})$

Parameters are $S_{10} = S_{20} = 100, B_1 = 50, B_2 = 40, m = \pi/4, \nu = 0.5, \rho(y) = |sin(y)|$.

Using homogenization in IS, note the effect of correlation.

Numerical Result II: Stochastic Correlation

$\alpha = \frac{1}{\delta}$	BMC	Importance Sampling
0.1	0(0)	$9.1 * 10^{-7}(7 * 10^{-8})$
1	0(0)	$7.5 * 10^{-6}(6 * 10^{-7})$
10	0(0)	$2.4 * 10^{-5}(2 * 10^{-6})$
50	$1 * 10^{-4}(1 * 10^{-4})$	$2.9 * 10^{-5}(3 * 10^{-6})$
100	$1 * 10^{-4}(1 * 10^{-4})$	$2.7 * 10^{-5}(2 * 10^{-6})$

Parameters are $S_{10} = S_{20} = 100, B_1 = 30, B_2 = 20, m = \pi/4, \nu = 0.5$.

Note the effect of correlation.

Conclusion

- Credit risk models are introduced.
- A simple yet efficient importance sampling method is proposed, justified by large deviations theory.
- Full expansion of joint default probability under stochastic correlation and its application to importance sampling.

Future Works

- Generalized to stochastic volatility models.

- Risk management of credit portfolios.

- Similar variance analysis for Gaussian copula models.

- Homogenization in Large Deviations.

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- TIMS, NTU.

- NSC.

Thank You!

Weak Brownian Motion and its Applications

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National Chiao Tung University

July 10, 2008
National Taiwan University

Outline

1. Motivation
2. Weak Brownian Motions
3. Martingale Marginal Property
4. Wiener Chaos
5. Future Works
6. References

Pricing Formula

In a financial model with interest rate 0, stock price process (S_t) and risk neutral probability measure $\mathbb{P}^*$, the price of European call option at time 0 is given by

$$\pi(K, T) = E^* \left[(S_T - K)^+ \right], \tag{1}$$

where T is the maturity and K is the strike price.

Two methods to discuss it:

- Stochastic analysis
- Dynamic analysis

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Relation between π and S_t

Method 1: Stochastic Analysis
u_t: marginal utility of S_t under $\mathbb{P}^*$, Then π is determined by the distribution function and the partial moment and

$$\pi_{xx}(\cdot,t) = u_t \text{ with density function } p(\cdot,t).$$

Method 2: Dynamic Analysis
Suppose S_t satisfies

$$dS_t = S_t(b_t\,dt + \sigma(S_t,t)\,dW_t),$$

then we have Dupire equation

$$\pi_t = \frac{1}{2}x^2\sigma^2(x,t)\pi_{xx} - x\sigma_t\pi_x. \tag{2}$$

Solve it!

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Solve it!

Another Point of View

Consider

$$\pi(K,T) = E^* \left[(S_T - K)^+ \right],$$

where (S_t) is a martingale with respect to $\mathbb{P}^*$.

Breeden and Litzenberger (1978) and Dupire (1997) show that

$$\mathbb{P}^*(S_T > K) = -\frac{\partial}{\partial K+}\pi(K,T),$$

where $\dfrac{\partial}{\partial K+}\pi(K,t)$ means the right-derivative of π with respect to K.

Question

Does there exist a stochastic process whose marginal (or k-marginal) is identical to the marginal of (S_t)?

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Stoyanov's Conjecture

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There exists a stochastic process X with $X_0 = 0$ satisfying

1. $X_t - X_s \sim \mathcal{N}(0, t-s)$ for all $s < t$.
2. $X_{t_2} - X_{t_1}$ and $X_{t_4} - X_{t_3}$ are independent for $0 \leq t_1 < t_2 \leq t_3 < t_4$.

But X is not a Brownian motion.

Thus,

We aim to see if there exists a stochastic process whose marginal (or k-marginal) is identical to the marginal of a Brownian motion, but is not a Brownian motion.

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A stochastic process X is called a weak Brownian motion of order k, if for all $(t_1, t_2, ..., t_k)$

$$(X_{t_1}, X_{t_2}, ..., X_{t_k}) \overset{\text{(law)}}{=} (B_{t_1}, B_{t_2}, ..., B_{t_k}),$$

where B is a Brownian motion.

Another formulation

$$E[f_1(X_{t_1}) \cdots f_k(X_{t_k})] = E[f_1(B_{t_1}) \cdots f_k(B_{t_k})]$$

for $f_1, ..., f_k \in C_0^1(\mathbb{R})$.

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Main Results

Theorem (Föllmer-W.-Yor (2000))

Let $k \in \mathbb{N}$. There exists a process $(X_t)_{0 \leq t \leq 1}$ which is not Brownian motion such that the k-dimensional marginals of X are identical to those of Brownian motion.

Theorem

For every $\varepsilon > 0$, there exists a probability measure $\mathbb{Q} \neq \mathbb{P}$ on $C([0,1])$ such that

1. $\mathbb{Q} \approx \mathbb{P}$
2. $\mathbb{Q} \perp \mathbb{P}$

and satisfies

$$\mathbb{Q} = \mathbb{P} \qquad \text{on } \mathcal{F}_J = \sigma(X_t : t \in J)$$

for any $J \subseteq [0,1]$ such that J^c contains some interval of length ε.

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Properties

Proposition

Let X be a weak Brownian motion of order k.

1. If $k \geq 2$, then X has a continuous version. Moreover, if X is a Gaussian process, then X is a Brownian motion
2. If $k \geq 4$, then $\langle X \rangle_t = t$. Moreover, if X is a martingale, then X is a Brownian motion.

Remark

A weak Brownian motion may not be a martingale, e.g.,

$$X_t = \begin{cases} W_t, & t \leq 1/2, \\ W_{\frac{1}{2}} + \left(\sqrt{2} - 1\right) W_{t-\frac{1}{2}}, & t > 1/2. \end{cases}$$

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Itô Integral

If X is a continuous weak Brownian motion of order $k \geq 1$ whose paths have quadratic variation

$$\langle X \rangle_t = t,$$

then the Itô integral

$$\int_0^t f(X_t)\, dX_t$$

exists as a pathwise limit of non-nticipting Riemann sums along dyadic partitions for any bounded $f \in C^1$ and satisfies the Itô's formula even though X may not be a semimartingale, see Föllmer (1981). Moreover,

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Characterization

1. W is a Brownian motion if and only if there exist an orthonormal basis (φ_n) on $L^2([-0,1])$ and a sequence of i.i.d. $\mathcal{N}(0,1)$-distributed random variables (ξ_n) such that

$$W_t = \sum_{k=1}^{\infty} \left(\int_0^t \varphi_n(u)\, du \right) \xi_n.$$

2. X is a weak Brownian motion of order k if and only if there exist an orthonormal basis (φ_n) on $L^2([-0,1])$ and a sequence of uncorrelated $\mathcal{N}(0,1)$-distributed random variables (η_n) such that

$$\sum_{k=1}^{\infty} \left(\lambda_1 \int_0^{t_1} \varphi_n(u)\, du + \cdots + \lambda_k \int_0^{t_k} \varphi_n(u)\, du \right) \eta_n$$

is Gaussian for all $\lambda_1, ..., \lambda_k \in \mathbb{R}$, $t_1 \le t_2 \le \cdots \le t_k$ and

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Martingale Marginal

Definition

The family of densities $\mathcal{Q} = \{q(x,t) : t > 0\}$ has martingale marginal property if there exists a probability space on which one may define a martingale (M_t) such that for every t, the law of M is given by the density $q(M,t)$.

Theorem (Strassen (1965))

A family of probability measures $(\mu_n)_{n\geq 0}$ has martingale marginal property if and only if for all $n \geq 0$, $\displaystyle\int |x| \, \mu_n(dx) < \infty$, and for any concave μ_n-integrable function ψ, the sequence $\left(\displaystyle\int \psi(x)\, \mu_n(dx)\right)$ is non-increasing (the values of the integrals may be $-\infty$).

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Generalizations

Remark

Doob (1968) proved the continuous version of above result.

Theorem (Rothschild and Stiglitz (1970, 1971))

$\{q(x,t) : t > 0\}$ *has martingale marginal property if and only if for all* K *and for all* $T_1 \leq T_2$

$$\int_0^\infty S q(S, T_2)\, dS \leq \int_0^\infty S q(S, T_1)\, dS$$

$$\int_0^\infty (S-K)^+ q(S, T_2)\, dS \geq \int_0^\infty (S-K)^+ q(S, T_1)\, dS.$$

Remark

This concept is relative to stochastic orders, see Föllmer and Schied (2004).

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Markov Martingales

Question

Given a family of densities $\{q(x,t) : t > 0\}$, does there exist a probability space on which one can define a Markov martingale (M_t) such that for every t, the law of M is given by the density $q(M,t)$?

Theorem (Kellerer (1972))

1. *Let $\{q(x,t) : t > 0\}$ be a family of marginal densities, with finite first moment, such that for $s < t$*

$$\int f(x)q(x,t)\,dx \geq \int f(x)q(x,s)\,dx$$

for all convex non-decreasing functions f, then there exists a Markov submartingale (M_t) with marginal densities $\{q(M_t,t) : t > 0\}$.

2. *Furthermore, if the means are independent of t, then (M_t) is a Markov martingale.*

Markov Martingales

Question

Given a family of densities $\{q(x,t) : t > 0\}$, does there exist a probability space on which one can define a Markov martingale (M_t) such that for every t, the law of M is given by the density $q(M,t)$?

Theorem (Kellerer (1972))

1. *Let $\{q(x,t) : t > 0\}$ be a family of marginal densities, with finite first moment, such that for $s < t$*

$$\int f(x)q(x,t)\,dx \geq \int f(x)q(x,s)\,dx$$

   for all convex non-decreasing functions f, then there exists a Markov submartingale (M_t) with marginal densities $\{q(M_t,t) : t > 0\}$.

2. *Furthermore, if the means are independent of t, then (M_t) is a Markov martingale.*

Constructions

Define the family of barycetre functions

$$\psi(x,t) = \frac{\int_x^\infty yq(y,t)\,dy}{\int_x^\infty q(y,t)\,dy}.$$

Suppose $\psi(x,t)$ is increasing in t and $q(x,t)$ is a family of zero mean densities.

Theorem (Madan and Yor (2002))

Let (B_t) be a standard Brownian motion. Define a stopping time

$$\tau_t = \inf\left\{s : \sup_{0\le u\le s} B_u \ge \psi(B_s,t)\right\}.$$

Then $M_t := B_{\tau_t}$ is an inhomogeneous Markov martingale with density q.

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Consequence of the Main Results

Let X represent the coordinate process and $L^2(\mathbb{P}) = L^2(C([0,1]), \mathbb{P})$.

Notation

For every $k \in \mathbb{N}$, define

$$\Pi_k := \left\{ \prod_{i=1}^{k} f_i(X_{t_i}) : t_1 < \cdots < t_k \le 1, f_i \text{ is bounded, Borel measurable} \right\}.$$

Then

Corollary

For every k, Π_k is not total in $L^2(\mathbb{P})$.

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Decomposition

Notation

For every $k \in \mathbb{N}$, define

$$K_0 := \Pi_0 = \mathbb{R}, \qquad K_{n+1} = \bar{\Pi}_{n+1} \cap \bar{\Pi}_n^{\perp},$$

where $\perp$ denotes orthogonality relation in $L^2(\mathbb{P})$.

Lemma

$$L^2(\mathbb{P}) = \bigoplus_{n=1}^{\infty} K_n.$$

Remark

K_n is called the nth time-space Wiener chaos.

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Time-Space Wiener Chaos

For every $u \leq 1$, we define the Brownian bridge by

$$X_t^{(u)} := X_t - \int_0^t \frac{X_u - X_s}{u - s}\, ds \quad t < u.$$

Then $(X_t^{(u)})$ is a Brownian motion w.r.t. the enlarged filtration $\mathcal{F}_t \vee \sigma(X_u)$, see, e.g., Yor (1992).

Theorem (Peccati (2001))

$H \in K_n$ if and only if there exists a measurable deterministic function $h(u_1, x_1; \ldots; u_n, x_n)$ such that

$$\int_0^1 \int_0^{u_1} \cdots \int_0^{u_{n-1}} E\left[h^2(u_1, X_{u_1}; \ldots; u_n, X_{u_n})\right] du_n \cdots du_2\, du_1 < \infty, \tag{3}$$

$$H = \int_0^1 \int_0^{u_1} \cdots \int_0^{u_{n-1}} h(u_1, X_{u_1}; \ldots; u_n, X_{u_n})\, dX_{u_n}^{(u_{n-1})} \cdots dX_{u_2}^{(u_1)}\, dX_{u_1}.$$

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Consequences

Corollary

Let F be a real random variable in $L^2(\mathbb{P})$, then there exists a sequence of measurable functions $h_{(F,n)}$ satisfying the integrability condition (3) such that

$$F = E[F] + \sum_{n=1}^{\infty} \int_0^1 \int_0^{u_1} \cdots \int_0^{u_{n-1}} h_{(F,n)}\, dX_{u_n}^{(u_{n-1})} \cdots dX_{u_2}^{(u_1)}\, dX_{u_1}.$$

Comparison (Wiener chaos)

Let F be a real random variable in $L^2(\mathbb{P})$, then there exists a sequence of deterministic square integrable functions $\varphi_{(F,n)} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

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Future Works

1. For $k = 2, 3$, $\langle X \rangle_t = t$?
2. Generalization of weak martingales.
3. The construction of martingale k-marginal property and Markov martingale k-marginal property.
4. The relationship between the weak Brownian motion of order k, the kth Wiener chaos, kth time-space Wiener chaos, and the generalization of the stochastic order.
5. N-complete market, N-mixed trading strategies (Campi (2004)).

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The L^2-cutoff for reversible Markov processes

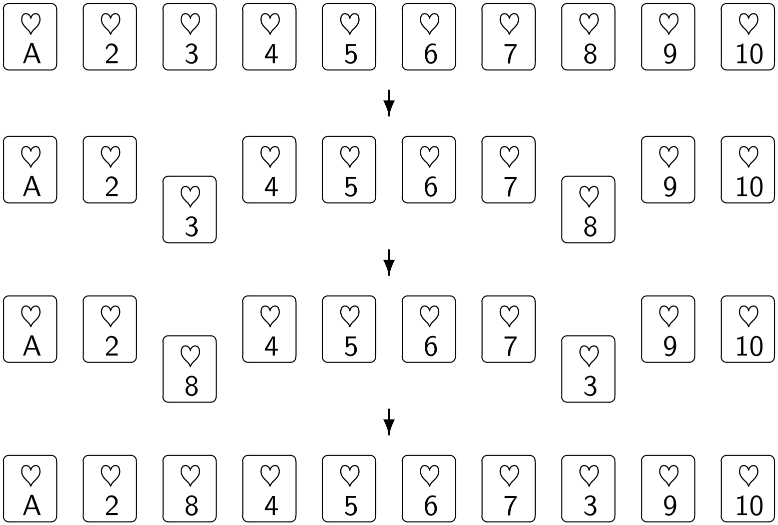
Guan-Yu Chen

July 10, 2008

Outline

- Introduction: Random transpositions
- Markov transition functions
- L^2-distance, L^2-cutoff and L^2-mixing time
- A necessary and sufficient condition for an L^2-cutoff
- Examples

Random transpositions



Random transpositions

For $n = 1, 2, \ldots,$

- S_n: the symmetric group of degree n, $\quad id_n$: the identity of S_n
- p_n^t : the probability on S_n after t shuffles of random transposition starting from the identity. That is,

$$p_n^t(\delta) = \sum_{\sigma \in S_n} p_n^{t-1}(\sigma) p_n(\sigma^{-1}\delta)$$

  where

$$p_n(\sigma) = \begin{cases} \frac{1}{n} & \text{if } \sigma = id_n \\ \frac{2}{n^2} & \text{if } \sigma = (i,j) \text{ with } 1 \leq i < j \leq n \\ 0 & \text{otherwise} \end{cases}$$

- U_n : the uniform probability on S_n

$$\forall B \subset S_n, \quad p_n^t(B) \to U_n(B) \quad \text{as } t \to \infty$$

Total variation and L^2-distance

- **Total variation:** For any two measures μ, ν on a set Ω equipped with a σ-algebra $\mathcal{B}$, their total variation is defined by

$$\|\mu - \nu\|_{\mathsf{TV}} = \sup_{B \in \mathcal{B}} |\mu(B) - \nu(B)|$$

- **$\mathbf{L^2}$-distance:** Let μ, ν be measures on Ω which are absolutely continuous w.r.t. π with Randon-Nikodym derivatives f, g. The $L^2(\Omega, \pi)$-distance between μ and ν is defined to be

$$\|f - g\|_2 = \left(\int_\Omega |f - g|^2 d\pi. \right)^{1/2}.$$

Results on Random transpositions

- Concerning the random transpositions, as a function of the time $t \in \{0, 1, 2, ...\}$, the total variation and the L^2-distance

$$t \mapsto \|p_n^t - U_n\|_{\mathrm{TV}}, \quad t \mapsto \left\| \frac{dp_n^t}{dU_n} - 1 \right\|_2$$

are non-increasing in t and converge to 0 as $t \to \infty$.

- Diaconis & Shahshahoni (1981): Let $t_n = \frac{1}{2} n \log n$.

$$\lim_{n\to\infty} \|p_n^{at_n} - U_n\|_{\mathrm{TV}} = \begin{cases} 1 & \text{for } a \in (0,1) \\ 0 & \text{for } a \in (1,\infty) \end{cases}$$

and

$$\lim_{n\to\infty} \left\| \frac{dp_n^{at_n}}{dU_n} - 1 \right\|_2 = \begin{cases} \infty & \text{for } a \in (0,1) \\ 0 & \text{for } a \in (1,\infty) \end{cases}$$

Markov transition function, invariant measure

- $(\Omega, \mathcal{B})$: a measurable space, T: either $[0, \infty)$ or $\mathbb{N} = \{0, 1, 2, ...\}$.
- A **Markov transition function** is a family $\{p(t, x, \cdot) : t \in T, x \in \Omega\}$ of probability measures on $(\Omega, \mathcal{B})$ satisfying

$$p(0, x, \Omega \setminus \{x\}) = 0$$

and, for $t \in T$, $A \in \mathcal{B}$, $p(t, \cdot, A)$ is a $\mathcal{B}$-measurable function and satisfies

$$p(t + s, x, A) = \int_{\Omega} p(s, y, A) p(t, x, dy).$$

An **invariant** measure π of $p(t, x, \cdot)$ is a measure on Ω satisfying

$$\int_{\Omega} p(t, x, A) \pi(dx) = \pi(A), \quad t > 0, A \in \mathcal{B}.$$

Semigroup and spectral gap

- For $t > 0$, let P_t be an operator defined by
$$P_t f(x) = \int_\Omega f(y)p(t,x,dy),$$
where f is any bounded measurable function.
- For any probability μ on $(\Omega, \mathcal{B})$, let μP_t be a probability defined by
$$\mu P_t(A) = \int_\Omega p(t,x,A)\mu(dx).$$
- Let $p(t,\cdot,\cdot)$ be a Markov transition function with invariant probability π and let P_t be the associated operator as before. The **spectral gap** of P_t is the largest constant c such that
$$\|P_t - E_\pi\|_{L^2(\Omega,\pi)\to L^2(\Omega,\pi)} \le e^{-ct} \quad \forall t > 0,$$
where $E_\pi f = \pi(f)\mathbf{1}$.

The L^2-distance and the L^2-mixing time

- If μP_t has a density $h(t, \mu, \cdot)$ w.r.t. π, then the $\mathbf{L^2}$**-distance** between $p(t, \mu, \cdot)$ and π is defined by

$$D_2(\mu, t) = \|h(t, \mu, \cdot) - 1\|_2 = \left(\int_\Omega h^2(t, \mu, y)\pi(dy) - 1 \right)^{1/2}.$$

  Otherwise, $D_2(\mu, t)$ is set to be infinity.
- Corresponding to the above setting, the ϵ-L^2-mixing time for $p(t, \mu, \cdot)$ is defined by

$$T_2(\mu, \epsilon) = \inf\{t \in T : D_2(\mu, t) \leq \epsilon\}.$$

The L^2-cutoff

Consider a family of Markov transition functions

$$\mathcal{F} = \{p_n(t, x, \cdot), t \in T, x \in \Omega_n : n = 1, 2, ...\}.$$

with initial probabilities μ_n. The family $\mathcal{F}$ is said to present an **$\mathbf{L^2}$-cutoff** if there exists a positive sequence t_n such that

$$\lim_{n\to\infty} D_{n,2}(\mu_n, at_n) = \begin{cases} \infty & \text{for } a \in (0,1) \\ 0 & \text{for } a \in (1,\infty) \end{cases}$$

Here, t_n is called the L^2-cutoff time.

Lemma

The family $\{p_n(t, \mu_n, \cdot) : n = 1, 2, ...\}$ has an L^2-cutoff if and only if

$$\lim_{n\to\infty} \frac{T_{n,2}(\mu_n, \epsilon)}{T_{n,2}(\mu_n, \delta)} = 1 \quad \forall 0 < \epsilon, \delta < \infty.$$

Moreover, $T_{n,2}(\mu_n, \epsilon)$ can be an L^2-cutoff time for any $\epsilon \in (0, \infty)$.

The L^2-distance for random walks on finite groups

- G: a finite group
- Q: a probability on G
- U: the uniform probability on G.

  Then
  $$\left\| \frac{dQ}{dU} - 1 \right\|_2^2 = \sum d_\rho \mathrm{Tr}(\widehat{Q}(\rho)\widehat{Q}(\rho)^*)$$
  where d_ρ is the dimension of ρ, $\widehat{Q}(\rho)$ is the Fourier transformation of Q at ρ defined by
  $$\widehat{Q}(\rho) = \sum_{g \in G} Q(g)\rho(g).$$
  and the summation is over all irreducible representations ρ of G except the trivial one.

The L^2-distance for reversible Markov processes

- $(\Omega, \mathcal{B})$: a measurable space, T: $[0, \infty)$ or $\{0, 1, 2, ...\}$
- $p(t, \cdot, \cdot)$, $t \in T$: a Markov transition function on Ω with invariant probability π
- P_t: the operator on $L^2(\Omega, \pi)$ associated with $p(t, \cdot, \cdot)$
- λ: The spectral gap of P_t

Assume that

- P_t is self-adjoint for $t \in [0, 1] \cap T$ (or equivalently, π is reversible);
- P_t is strongly continuous if $T = [0, \infty)$ with infinitesimal generator A;
- $\lambda > 0$;
- μ is a probability whose density f w.r.t. π is in $L^2(\Omega, \pi)$.

Let $\{E_B | B \in \mathcal{B}(\mathbb{R})\}$ be the resolution of the identity for P_1 (resp. $-A$) if $T = \mathbb{N}$ (resp. $T = [0, \infty)$). Then, for $t \in T$,

$$D_2(\mu, t)^2 = \begin{cases} \int_{[\lambda,\infty)} e^{-2t\gamma} d\langle E_\gamma f, f\rangle_\pi & \text{if } T = [0, \infty) \\ \int_{[\lambda,\infty)} \gamma^{2t} d\langle E_\gamma f, f\rangle_\pi & \text{if } T = \{0, 1, 2, ...\} \end{cases}$$

The L^2-distance for reversible finite Markov chains

- Ω: a finite set, $\quad T$: $[0,\infty)$ or $\mathbb{N}$
- $p(t,\cdot,\cdot)$: a Markov transition functions on Ω. For $T=\mathbb{N}$,

$$p(1,x,y)=K(x,y),\;\; p(t,x,y)=\sum_{z\in\Omega}K^{t-1}(x,z)K(z,y).$$

and, for $T=[0,\infty)$,

$$p(t,x,y)=e^{-t(I-K)}(x,y)=e^{-t}\sum_{n=0}^{\infty}\frac{t^n}{n!}K^n(x,y).$$

- Assume that π is a reversible probability for $p(t,\cdot,\cdot)$ and K has eigenvalues $\lambda_0=1,\lambda_1,...,\lambda_{|\Omega|-1}$ and normalized eigenvectors $\psi_0\equiv 1,\psi_1,...,\psi_{|\Omega|-1}$ in $L^2(\Omega,\pi)$. Then, for $t\in T$,

$$D_2(\mu,t)^2=\begin{cases}\sum_{i=1}^{|\Omega|-1}|\mu(\psi_i)|^2e^{-2(1-\lambda_i)t} & \text{if } T=[0,\infty)\\ \sum_{i=1}^{|\Omega|-1}|\mu(\psi_i)|^2\lambda_i^{2t} & \text{if } T=\{0,1,2,...\}\end{cases}$$

A sufficient condition on the L^2-cutoff

Theorem (Chen & Saloff-Coste, 2008)

Consider a family $\mathcal{F}$ of Markov transition functions $p_n(t, x, \cdot)$, $t \in T$ and $x \in \Omega_n$, with invariant probability π_n and spectral gap $\lambda_n > 0$. For $n \geq 1$, let μ_n be a probability on $(\Omega_n, \mathcal{B}_n)$ and set $t_n(\epsilon) = T_{n,2}(\mu_n, \epsilon)$.
(i) For $T = [0, \infty)$, if there exists $\epsilon > 0$ such that

$$\lambda_n t_n(\epsilon) \to \infty,$$

then $\mathcal{F}$ has an L^2-cutoff with cutoff time $t_n(\epsilon)$.
(ii) For $T = \mathbb{N}$, let $\gamma_n = \min\{1, \lambda_n\}$. If there exists $\epsilon > 0$ such that

$$\gamma_n t_n(\epsilon) \to \infty,$$

then $\mathcal{F}$ has an L^2-cutoff with cutoff time $t_n(\epsilon)$.

An equivalent condition on the L^2-cutoff

Theorem (Continuous-time finite Markov chain)

For $n \geq 1$,

- K_n is a Markov kernel on a finite set Ω_n with invariant probability π_n;
- $p_n(t, \cdot, \cdot) = e^{-t(I-K_n)}$;
- $x_n \in \Omega_n$.

Assume that K_n is irreducible and reversible w.r.t. π_n with eigenvalues

$$1 > \lambda_{n,1} \geq \lambda_{n,2} \geq \cdots \geq \lambda_{n,|\Omega_n|-1}$$

and eigenvectors (normalized in $L^2(\Omega_n, \pi_n)$)

$$\mathbf{1}, \psi_{n,1}, \psi_{n,2}, ..., \psi_{n,|\Omega_n|-1}.$$

An equivalent condition for the L^2-cutoff

Theorem (Continuous-time finite Markov chain)

For $C > 0$, set

$$j_n = j_n(C) = \min\left\{ j \geq 1 : \sum_{i=1}^{j} |\psi_{n,i}(x_n)|^2 > C \right\}$$
$$\tau_n = \tau_n(C) = \sup_{j \geq j_n} \left\{ \frac{\log(\sum_{i=0}^{j} |\psi_{n,i}(x_n)|^2)}{2(1-\lambda_{n,j})} \right\}$$

Then, the family $\{p_n(t, x_n, \cdot) | n = 1, 2, ...\}$ has an L^2-cutoff if and only if, for some $C > 0$ and $\epsilon > 0$,

$$\lim_{n\to\infty} \tau_n(1 - \lambda_{n,j_n}) = \infty, \quad \lim_{n\to\infty} \sum_{i=1}^{j_n - 1} |\psi_{n,i}(x_n)|^2 e^{-\epsilon(1-\lambda_{n,i})\tau_n} = 0. \qquad (1)$$

Furthermore, if the above limit holds, then τ_n is an L^2-cutoff time.

Remarks on the main theorem

(a) If (1) holds true for some $C > 0$ and $\epsilon > 0$, then it must be true for all $C > 0$ and $\epsilon > 0$.

(b) The theorem remains true if τ_n is replaced by the L^2-mixing time $T_{n,2}(x_n, \epsilon)$.

(c) Consider the case that $p_n(t, \cdot, \cdot)$ is invariant under transitive group action, that is, some compact group G_n acts transitively on Ω_n with

$$p_n(t, gx, gy) = p_n(t, x, y) \quad \forall x, y \in \Omega_n, g \in G_n.$$

Then, $\psi_{n,i}(x_n)$ can be replaced by 1.

Random walks on hypercubes

- Ω_n: the finite group $(\mathbb{Z}_2)^n$
- K_n: a Markov kernel on Ω_n given by

$$K_n(x,y) = \begin{cases} \frac{1}{n+1} & \text{if } y = x \text{ or } y = x + e_{n,i} \\ 0 & \text{otherwise} \end{cases}$$

where

$$e_{n,i} = \overbrace{0\cdots0}^{i-1}1\overbrace{0\cdots0}^{n-i} \quad \forall i = 1,2,...,n.$$

For $n = 4$,

$$\begin{tikzcd}
 & & 0\ 0\ 1\ 0 \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
1\ 0\ 1\ 0 & 0\ 1\ 1\ 0 & 0\ 0\ 1\ 0 & 0\ 0\ 0\ 0 & 0\ 0\ 1\ 1
\end{tikzcd}$$

- $p_n(t,\cdot,\cdot)$: the Markov transition function $e^{-t(I-K_n)}$.

L^2-cutoff for random walks on hypercubes

- K_n has eigenvectors $\psi_{n,x}$, $x \in \Omega_n$, where

$$\psi_{n,x}(y) = (-1)^{x \cdot y}, \quad x \cdot y = x_1 y_1 + \cdots + x_n y_n.$$

and eigenvalues

$$1 - \tfrac{2|x|}{n+1}, \quad |x| = x_1 + x_2 + \cdots + x_n.$$

- For $C = 1/2$, $j_n(C) = 1$, $1 - \lambda_{n,j_n(C)} = \frac{2}{n+1}$ and

$$\tau_n(C) = \max_{1 \le i < 2^n} \left(\frac{\log(i+1)}{2\lambda_{n,i}} \right) = \frac{(n+1)\log(n+1)}{4}$$

Using the main theorem, the family has an L^2-cutoff with cutoff time $\frac{1}{4} n \log n$.

Ehrenfest processes

- $\Omega_n = \{0, 1, ..., n\}$, K_n is a Markov kernel on Ω_n given by

$$K_n(i, i+1) = 1 - \frac{i}{n}, \quad K_n(i+1, i) = \frac{i+1}{n}, \quad \forall 0 \le i < n.$$

- Let K_n' be the simple random walk on $(\mathbb{Z}_2)^n$, that is,

$$K_n'(x, x + e_{n,i}) = \frac{1}{n}, \quad \forall x \in (\mathbb{Z}_2)^n, 1 \le i \le n.$$

  Let $X_i = \{x \in (\mathbb{Z}_2)^n : |x| = i\}$ for $0 \le i \le n$ and set

$$K_n''(X_i, X_j) = K_n'(x_i, x_j)$$

  where $x_i \in X_i$ and $x_j \in X_j$. Then,

$$K_n(i, j) = K_n''(X_i, X_j).$$

Spectral information for Ehrenfest processes

Theorem

Let K_n be the Markov kernel of the Ehrenfest chain on $\{0, 1, \dots, n\}$. Then, K_n has invariant probability $\pi_n(i) = \binom{n}{i} 2^{-n}$, eigenvalues

$$\lambda_{n,i} = 1 - \frac{2i}{n}, \quad 0 \leq i \leq n,$$

and eigenvectors

$$\psi_{n,i}(x) = \binom{n}{i}^{-1/2} \sum_{k=0}^{i} (-1)^k \binom{x}{k} \binom{n-x}{i-k}, \quad 0 \leq i, x \leq n$$

which are normalized in $L^2(\pi_n)$.

Note that the vectors $\psi_{n,i}$ are in fact the Krawtchouk polynomials.

L^2-cutoff for Ehrenfest processes

Theorem

For $n \geq 1$, let K_n be the Markov kernel of the Ehrenfest chain on $\{0, 1, ..., n\}$ and $p_n(t, \cdot, \cdot) = e^{-t(I-K_n)}$. Let $0 \leq x_n \leq n$ be a sequence of starting states. Then the family $p_n(t, x_n, \cdot)$ has an L^2-cutoff if and only if

$$\lim_{n\to\infty} \frac{|n - 2x_n|}{\sqrt{n}} = \infty.$$

Moreover, if there is an L^2-cutoff, then the cutoff time can be

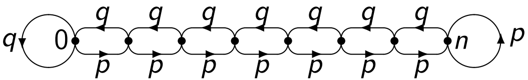
$$t_n = \frac{n}{2} \log \frac{|n - 2x_n|}{\sqrt{n}}.$$

Constant rate birth-and-death processes

- $\Omega_n = \{0, 1, ..., n\}$
- K_n: a Markov kernel on Ω_n given by

$$\begin{cases} K_n(x, x+1) = K_n(n, n) = p \\ K_n(x+1, x) = K_n(0, 0) = q = 1 - p \end{cases} \qquad \forall 0 \leq x < n$$

with $p < 1/2$.



- $p_n(t, \cdot, \cdot) = e^{-t(I - K_n)}$
- $x_n \in \{0, ..., n\}$: the initial state

Constant rate birth-and-death processes

- K_n has invariant probability π_n given by

$$\pi_n(x) = c_n(p/q)^x, \quad c_n = \frac{1 - p/q}{1 - (p/q)^{n+1}}$$

and eigenvalues

$$\lambda_{n,0} = 1, \quad \lambda_{n,j} = 2\sqrt{pq}\cos\frac{j\pi}{n+1}, \quad 1 \le j \le n,$$

and normalized eigenvectors $\psi_{n,0} \equiv 1$,

$$\psi_{n,j}(x) = C_{n,j}\left\{\left(\frac{q}{p}\right)^{(x+1)/2}\sin\frac{j(x+1)\pi}{n+1} - \left(\frac{q}{p}\right)^{(x+2)/2}\sin\frac{jx\pi}{n+1}\right\}$$

for $1 \le j \le n$, where $C_{n,j} = \left(c_n(n+1)q(1-\lambda_{n,j})/(2p^2)\right)^{-1/2}$.

L^2-cutoff for constant rate birth-and-death processes

Theorem

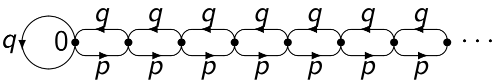
For $n \geq 1$, let $p_n(t,\cdot,\cdot)$ be the continuous-time (p,q)-random walks on $\Omega_n = \{0,1,...,n\}$ with $p \in (0,1/2)$ and let $x_n \in \Omega_n$. Then, the family $p_n(t,x_n,\cdot)$ has an L^2-cutoff if and only if $x_n \to \infty$. Moreover, if there exists an L^2-cutoff, then the cutoff time can be

$$t_n = \frac{\log q - \log p}{2(1-2\sqrt{pq})} x_n.$$

(p, q)-random walks on nonnegative integers

- $\Omega = \{0, 1, 2, ...\}$, $\quad K$: the Markov kernel on Ω given by

$$\begin{cases} K(x, x+1) = p \\ K(x+1, x) = K(0,0) = q = 1-p \end{cases} \qquad \forall x \geq 0.$$



- $p(t, \cdot, \cdot) = e^{-t(I-K)}$
- For $p < 1/2$, K has an invariant probability π given by

$$\pi(x) = (1 - p/q)(p/q)^x, \quad x \geq 0.$$

(p, q)-random walks on nonnegative integers

Theorem

Let $p(t, \cdot, \cdot)$ be the Markov transition function for the (p, q)-random walk on $\{0, 1, ...\}$ with $p \in (0, 1/2)$ and $q = 1 - p$. Then, the family $\{p(t, x_n, \cdot) | n = 1, 2, ...\}$ has an L^2-cutoff if and only if

$$x_n \to \infty.$$

Furthermore, if the L^2-cutoff exists, then the cutoff time can be

$$t_n = \frac{\log q - \log p}{2(1 - 2\sqrt{pq})} x_n.$$

Sketch of the proof for the main theorem

$\tau_n(1/2)(1-\lambda_{n,1}) \to \infty \Rightarrow L^2$-cutoff.

Here, we prove the case that $p_n(t,\cdot,\cdot)$ is invariant under transitive group action. If $D_{n,2}(x,t)$ is the L^2-distance between $p_n(t,x,\cdot)$ and π_n, then

$$D_{n,2}(x,t)^2 = \sum_{i\geq 1} e^{-2(1-\lambda_{n,i})t}.$$

Let $j \geq 1$ be such that $\tau_n = \frac{\log(j+1)}{2(1-\lambda_{n,j})}$. Then, $D_{n,2}(x_n, \tau_n(1/2)) \geq 1/2$ and, hence, $\tau_n(1/2) \leq T_{n,2}(x_n, 1/2)$.

$$\begin{aligned} \tau_n(1/2)(1-\lambda_{n,1}) \to \infty \quad &\Rightarrow \quad T_{n,2}(x_n,1/2)(1-\lambda_{n,1}) \to \infty \\ &\Rightarrow \quad L^2\text{-cutoff} \end{aligned}$$

$\square$

Sketch of the proof for the main theorem

L^2-cutoff $\Rightarrow \tau_n(1/2)(1-\lambda_{n,1}) \to \infty$.

- Fix $\epsilon > 0$. Let $N_{n,j} = |\{i \geq 1 : 1-\lambda_{n,i} \leq (1-\lambda_{n,1})(1+\epsilon)^{j+1}\}|$.

$$\Rightarrow \begin{cases} \log N_{n,j} \leq 2(1-\lambda_{n,1})\tau_n(1/2)(1+\epsilon)^{j+1} \\ D_{n,2}(x_n,t)^2 \leq \sum_{j\geq 0} N_{n,j}\exp\{-2t(1-\lambda_{n,1})(1+\epsilon)^j\} \end{cases}$$

$$\Rightarrow D_{n,2}(x_n,(1+\epsilon)^2\tau_n(1/2))^2 \leq \frac{1}{1-\exp\{-2\epsilon^2(1-\lambda_{n,1})\tau_n(1/2)\}}$$

- Set $\epsilon = \epsilon_n = ((1-\lambda_{n,1})\tau_n(1/2))^{-1/2}$. Then,

$$D_{n,2}(x_n,(1+\epsilon_n)^2\tau_n(1/2))^2 \leq (1-1/e^2)^{-1} \leq 2$$

- Since $\epsilon_n \leq \sqrt{2/\log 2} \leq 2$,

$$\Rightarrow 9(1-\lambda_{n,1})\tau_n(1/2) \geq (1-\lambda_{n,1})T_{n,2}(x_n,2) \to \infty$$

$\square$

Sketch of the proof for the main theorem

L^2-cutoff time.

- As a consequence of the above proof, if $\epsilon_n = ((1-\lambda_{n,1})\tau_n(1/2))^{-1/2}$, then

$$(1+\epsilon_n)^{-2} T_{n,2}(x_n, 2) \le \tau_n(1/2) \le T_{n,2}(x_n, 1/2).$$

- Recall that

$$\begin{aligned} L^2\text{-cutoff} \quad &\Leftrightarrow \quad (1-\lambda_{n,1}) T_{n,2}(x_n,\epsilon) \to \infty \quad \text{for some } \epsilon > 0 \\ &\Leftrightarrow \quad T_{n,2}(x_n,\epsilon)/T_{n,2}(x_n,\delta) \to 1 \quad \forall \epsilon, \delta > 0 \\ &\Leftrightarrow \quad (1-\lambda_{n,1})\tau_n(1/2) \to \infty \end{aligned}$$

- Thus, $\epsilon_n \to 0$ and

$$\tau_n(1/2)/T_{n,2}(x_n,\epsilon) \to 1 \quad \forall \epsilon > 0.$$

$\square$

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Some Stochastic Analysis of ORNSTEIN-UHLENBECK-type Processes

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Abstract

In this talk, we report some recent works on the expected maximum increments and the correlation decay of the exponential process determined by an OU-type process. As an application, the works show that a scheme for large-deviation-based multifractal spectra proposed by Mannersalo *etal* can be carried out for such processes. These are joint works with Vo Anh(Brisbane), Nikolai Leonenko(Cardiff), and Muneya Matsui(Yokohama).

1. OU Process and its generalization

The (unique) stationary process X which solves the SDE

$$dX(t) = -\lambda X(t)dt + dB(t),$$

or in a mean-reverting form

$$dX(t) = -\theta(X(t) - \mu)dt + dB(t).$$

To generalize, we may consider

1. the background driving process can be a Lévy process, or a fractional BM.

2. the more general mean-reverting process as the solution of SDE

$$dX(t) = -\theta(X(t) - \mu)dt + \sqrt{v(X(t))}dB(t).$$

2. Exponential Process

Given a process X with exp moment, let $c(t)$ be the normalizing factor so that

$$Y(t) := e^{X(t)-c(t)}$$

is a positive-valued mean 1 process.

When X is stationary, $c(t)$ is a constant in t.

The object of the works:

1. the estimate of correlation decay $E(Y(t+s)Y(t))$.

2. the estimate of the expectation $E[\max_{0\le s\le h}|Y(t+s)-Y(t)|]$.

3. An application

The papers of J.P. Kahane (1985,1987,1989,2000) on positive T-martingales and multiplicative chaos lead to a certain multi-scale fractional analysis of some random clustering phenomena. In particular, the MF products of stationary stochastic processes by Mannersalo, Norros and Riedi (AAP 2002).

The above two are crucial steps toward to the scheme.