SAMPLE PATHS OF FRACTIONAL LÉVY PROCESSES

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Abstract

Recently, Benassi *et al.* (2002,2004) have defined the linear fractional and the real harmonizable fractional processes driven by Lévy processes. The two processes are of stationary increments, yet in general no longer selfsimilar, and Benassi *et al.* study the local and the asymptotic selfsimilarity of the two processes, assuming that the associated Lévy measure is truncated symmetric α-stable. In this report, we establish the different Höler continuity for sample paths of these two fractional Lévy processes, under the assumption that the driving Lévy process is symmetric and is of second moment. Our result, together with an early paper of Billingsley (1974), will imply that the two fractional Lévy processes are *different processes in law*, unless the driving process is a Brownian motion. This corresponds the same question for the case that the driving Lévy is non-Gaussian stable, discussed in Cambanis and Maejima (1989).

1. Introduction

A real-valued stochastic process $X = \{X(t), t \geq 0\}$ is said to be an *infinitely divisible process*, if for every finite many times $t_1, \cdots, t_k$ the law of $(X(t_1), \cdots, X(t_k))$ is a k-dimensional infinite divisible distribution; see for example, Rajput and Rosinski (1989, in which the parameter set can be arbitrary). Under mild assumptions, they proved (in their Theorem 4.11) that the process $X = \{X(t), t \geq 0\}$ admits an

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stochastic integral representation,

$$X(t) = \int_{-\infty}^{\infty} f(t,u)\, dZ(u),$$

where the integrand is a certain kernel function and the integrator $Z = \{Z(u), u \in \mathbb{R}\}$ is an additive process in law (that is, a process of which distributions are of independent increments). The process X is in general not of stationary increments; yet in case Z is a Lévy process in law (i.e. Z is also a process of stationary increments) then X can be stationary or of stationary increments (which depends on the choice of the kernel function); in such case, without loss of generality, we may also assume that Z is of càdlág paths, and it is then a Lévy process in usual sense. We refer the precise definitions on the additive (or Lévy) processes to the intensive book of Sato (1999). There has been long interest to study the following two classes of non-Gaussian infinitely divisible processes; see, for example, the influential book of Samorodnitsky and Taqqu (1994). Let $0 < H < 1$ and $0 < \alpha < 2$,

$$\Delta_{H,1/\alpha}(t) = \int_{-\infty}^{\infty} \left(|t-u|^{H-1/\alpha} - |u|^{H-1/\alpha}\right) dZ_\alpha(u), \qquad \text{if } H \neq \frac{1}{\alpha},$$

$$\Psi_{H,1/\alpha}(t) = Re\left(\int_{-\infty}^{\infty} \frac{e^{itu}-1}{iu} |u|^{1-H-1/\alpha} d\widetilde{Z}_\alpha(u)\right),$$

where $\{Z_\alpha(u), u \in \mathbb{R}\}$ is a real-valued symmetric α-stable Lévy process , and $\{\widetilde{Z}_\alpha(u), u \in \mathbb{R}\}$ is a complex-valued rotationally invariant α-stable Lévy process, both having Lebesgue control measure. $\Delta_{H,1/\alpha}$ is the moving average (MA) type fractional process, and $\Psi_{H,1/\alpha}$ is the real harmonizable (RH) type. These two fractional processes are both selfsimilar with the same selfsimilar parameter H and both of stationary increments; see Samorodnitsky and Taqqu (1994, Chapter 7).

We recall that, a stochastic process $X = \{X(t), t \geq 0\}$ is said to be H-selfsimilar for some $H > 0$ if $\{X(ct), t \geq 0\} \stackrel{\mathrm{d}}{=} \{c^H X(t), t \geq 0\}$ for all $c > 0$, and to have stationary increments if $\{X(t+b) - X(t), t \geq 0\} \stackrel{\mathrm{d}}{=} \{X(t), t \geq 0\}$ for all $b > 0$, where $\stackrel{\mathrm{d}}{=}$ means equality of all finite dimensional distributions. Moreover, a real-valued infinitely divisible process $X = \{X(t), t \geq 0\}$ is a symmetric α-stable process for some $0 < \alpha \leq 2$, if all finite combinations $\sum_{n=1}^{k} a_n X(t_n)$ have characteristic function of the form $\exp\{-c|\theta|^\alpha\}$ for some $c = c(a_1, \cdots, a_k, t_1 \cdots, t_k) > 0$.

It was shown firstly in Cambanis and Maejima (1989) that, if $1 < \alpha < 2$, then the law of $\Delta_{H,1/\alpha}$ is distinct from that of $\Psi_{H,1/\alpha}$ for any given (H, α). However, it was not proved for the case $0 < \alpha \leq 1$ in Cambanis and Maejima (1989), since the existence of the first moments of the processes was required in the proof there. Remark that when $\alpha = 2$, both $\Delta_{H,1/\alpha}$ and $\Psi_{H,1/\alpha}$ are the same in law as the fractional Brownian motion up to a scaling constant. Later, in Samorodnitsky and Taqqu (1994, Chapter 7, p. 358), they show that the two processes are essentially different(that is, they are not equivalent up to a scaling factor), for all range $0 < \alpha < 2$.

As for the sample path regularity of these two fractional stable motions, standard Kolmogorov criterion may give some Hölder continuity of the processes. Yet there have been proved more precise uniform modulus of continuity, for the linear fractional case, by Takashima (1989), and for the real harmonizable fractional case by Kôno and Maejima (1991).

Recently Benassi *et al.* (2002,2004) have defined the the linear fractional and the real harmonizable fractional processes driven by Lévy processes (indeed they consider the multi-parameter case). The definitions of these two fractional Lévy processes, abbreviated respectively as MAFLP and RHFLP, are respectively (we consider only the one-parameter case),

$$\Delta_H(t) = \int_{-\infty}^{\infty} \left(|t-u|^{H-1/2} - |u|^{H-1/2}\right) dZ(u),$$
$$\Psi_H(t) = Re\left(\int_{-\infty}^{\infty} \frac{e^{itu}-1}{iu} |u|^{1-(H+1/2)} d\widetilde{Z}(u)\right),$$

where $\{Z(u), u \in \mathbb{R}\}$ is a real-valued Lévy process which is centered, without Gaussian part, and with all the $p \geq 1$ moments. The $\{\widetilde{Z}(u), u \in \mathbb{R}\}$ is a complex-valued rotationally invariant Lévy process, with the real part of $\widetilde{Z}(u)$ being the same as $Z(u)$ (we may, for convenience, say that two driving process are the "same"). Here, the rotational invariance happens, if the polar decomposition of the Lévy measure of $\widetilde{Z}(u)$, when it is regarded as a 2-dimensional Lévy process, in Barndorff-Nielsen *et al.* (2006, Lemma 2.1, with $d = 2$ there) has the uniform measure as their measure $\lambda(d\xi)$ and their ν_ξ is independent on ξ.

The two processes are of stationary increments, yet in general no longer selfsimilar, and Benassi *et al.* (2002,2004) study the local and the asymptotic selfsimilarity of the two processes, assuming that the associated Lévy measure is truncated symmetric α-stable.

In this report, we establish the different Höler continuity for sample paths of these two fractional Lévy processes, under the assumption that the driving Lévy process is symmetric and is of the second moment. Our result, together with an early paper of Billingsley (1974), will imply that the two fractional Lévy processes are *different* *processes in law*, unless the driving process is a Brownian motion. This corresponds the same question for the case that the driving Lévy is non-Gaussian stable, discussed in Cambanis and Maejima (1989).

We should mention that, such an idea of using sample path property to distinguish the classes of the processes are unknown in previous literatures, to our knowledge. We also mention that, under the second moment condition on Z, as we will impose in this paper, it is natural to examine the covariance function; however, it is pointed in Benassi *et al.* (2004, p. 358) that the both covariance functions are in the same form as that of fractional BM with parameter H. Namely,

$$E\big[|X(t) - X(s)|^2\big] = const \cdot |t-s|^{2H}$$

(the constant may be different).

Remark: The covariance function can be used successfully to distinguish the two processes with finite second moments; for one recent example, we may see a paper by Cheridito *et al.* (2003) for the two processes, Ornstein-Uhlenbeck process and Lamperti process, driven by a fractional Brownian motion (fBM).

In the followings, we always assume the range $1/2 < H < 1$, and the symmetry and the second moment conditions on $\{Z(u), u \in \mathbb{R}\}$.

2. Uniform Hölder continuity of sample paths

We firstly state the following uniform Hölder continuity for the process Δ_H.

Theorem 2.1. *There exists a version $\widetilde{\Delta}_H$ of Δ_H whose sample paths are continuous such that, for any positive continuous function $\phi(t)$ defined for $t > 0$,*

$$\lim_{\delta \downarrow 0} \sup_{\substack{t,s \in [0,1] \\ |t-s| < \delta}} \frac{|\widetilde{\Delta}_H(t) - \widetilde{\Delta}_H(s)|}{|t-s|^{H-1/2} \cdot \phi(|t-s|)} = 0; \tag{2.1}$$

$$\lim_{\delta \downarrow 0} \sup_{\substack{t,s \in [0,1] \\ |t-s| < \delta}} \frac{|\widetilde{\Delta}_H(t) - \widetilde{\Delta}_H(s)|}{|t-s|^{H-1/2} \cdot \phi(|t-s|)} = \infty; \tag{2.2}$$

depending on $\lim_{t \downarrow 0} \phi(t)$ equals to ∞ or to 0.

Next, we state the following uniform Hölder continuity for the process Ψ_H, which is rather different from the above result for Δ_H.

Theorem 2.2. *There exists a version* $\widetilde{\Psi}_H$ *of* Ψ_H *whose sample paths are continuous such that, for any* $\varepsilon > 0$*,*

$$\lim_{\delta \downarrow 0} \sup_{\substack{t,s \in [0,1] \\ |t-s|<\delta}} \frac{|\widetilde{\Psi}_H(t) - \widetilde{\Psi}_H(s)|}{|t-s|^H| \log |t-s||^{1+\varepsilon}} = 0 \tag{2.3}$$

These two theorems are derived from useful alternate representations of the two processes. For the Δ_H, we have

Lemma 2.3. *Let*

$$f(t,u) := |t-u|^{H-1/2} - |u|^{H-1/2}.$$

Then,

$$Y_{MA}(t) := \int_{-\infty}^{\infty} Z(u) \frac{-\partial f(t,u)}{\partial u} du, \quad t > 0, \quad Y_{MA}(0) := 0, \tag{2.4}$$

defines essentially a version of Δ_H*.*

Proof. Marquardt (2006, Theorem 3.4) pointed that, under the second moment condition on Z, the linear fractional integral with respect to Z can be written pathwise as a Riemann-type integral, and the by-parts formula holds. □

For Ψ_H, we need some notations as follows; they are adapted from Rosinski (1989, Proposition 2). Let $\nu(dx)$ be the Lévy measure of the driving Lévy process Z, and $R(u), u > 0$ be the right continuous inverse of the tail distribution function $x \to \nu(x, \infty), x > 0$; that is, $R(u) = \inf\{x > 0 : \nu(x, \infty) \leq u\}$. Let $\varphi(x), x \in \mathbb{R}$, be an everywhere positive probability density function on $\mathbb{R}$. Let ξ_n be iid random variables with common distribution φ. Let g_n be iid $\mathbb{C}$-valued standard normal random sequence(i.e. the real part and the imaginary part of the random vector are independent and are $N(0, 1/2)$ distributed). Let Γ_n be a sequence of Poisson arrival times with unit rate. Suppose that ξ_n, g_n, Γ_n are totally independent.

Lemma 2.4. *Let*

$$f(t,u) := \frac{e^{itu-1}}{iu} |u|^{-1-(H+1/2)}.$$

Then,

$$Y_{RH}(t) := \sum_{n=1}^{\infty} g_n R(\Gamma_n \varphi(\xi_n)) f(t, \xi_n) \tag{2.5}$$

defines essentially a version of Ψ_H*.*

Proof. This can be regarded as a special case of Rosinski (1989, Proposition 2, with the $q(s, dx)$ there being independent on s). Indeed, the author used symmetric Bernoulli sequence as g_n, and then remarked (in p. 79) that the Gaussian sequence can also be used in a parallel way. $\square$

Now, with the above two lemmas we can prove our different uniform Hölder continuity for the two fractional Lévy processes. The proofs are basically along the same line as those in Takashima (1989) and Kôno and Maejima (1991) for the symmetric α-stable driven case; therefore here we only sketch them.

Firstly, we observe that, since $E\big[|X(t)-X(s)|^2\big] = const \cdot |t-s|^{2H}$, and $H > 1/2$, by Kolmogorov Theorem, the both processes have continuous versions.

Proof of Theorem 2.1. It suffices to prove the theorem for the process Y_{MA} defined in Lemma 2.3. Let $\beta := H - 1/2$. Since Y_{MA} is a path-wise defined process, real analytic argument in Takashima (1989, p. 182-184) shows that the following holds a.s. (which is the assertion of his Lemma 4.7)

$$\lim_{h\downarrow 0}\{\sup_{0<s<t\leq 1, t-s\leq h} |Y_{MA}(t) - Y_{MA}(s)||t-s|^{-\beta}\} = \max_{-\infty<u<\infty} |f(1,u)| \cdot \sup_{0\leq u\leq 1} |\Delta_Z(u)|.$$

On the right-handed side, the first term is 1, and the second term is a finitely positive quantity. Therefore, the required assertion of the theorem follows from the multiplication on the both sides of the above display by a the factor $(\phi(|t-s|))^{-1}$. $\square$

Proof of Theorem 2.2. It suffices to prove the theorem for the process Y_{RH} defined in Lemma 2.4. We observe that, the second moment assumption on Z, which means $\int_{|x|>1} x^2\nu(dx) < \infty$, enforces that the decay of tail distribution $\nu(x,\infty)$ is at least $O(x^{-2})$ as $x \to \infty$. Therefore the right-continuous inverse function $R(u) = O(u^{-1/2})$ as $u \to \infty$. We consider the expectation of $|Y_{RH}(t) - Y_{RH}(s)|^2$ with respect to $\{g_n\}$, and write it as $a^2(|t-s|)$. Let $b(r) := r^H\big|\log|r|\big|^{(1+\varepsilon)/2}, \quad r > 0$. Using the decay $R(u) = O(u^{-1/2})$ as $u \to \infty$ and the arguments in Kôno and Maejima (1991, p. 96-97), we have that

$$\lim_{r\downarrow 0} \frac{a(r)}{b(r)} = 0 \quad a.s. \quad (\{\xi_n\}, \{\Gamma_n\}),$$

and thus for small $r > 0$,

$$a(r) \leq r^H\big|\log|r|\big|^{(1+\varepsilon)/2} \quad a.s. \quad (\{\xi_n\}, \{\Gamma_n\}).$$

Therefore, we have, the expectation of $|Y_{RH}(t) - Y_{RH}(s)|^2$ with respect to $\{g_n\}$,

$$E_{\{g_n\}}\big[|Y_{RH}(t) - Y_{RH}(s)|^2\big] \leq \sigma^2(|t-s|),$$

with $\sigma^2(r) := Cr^{2H}\big|\log|r|\big|^{1+\varepsilon}$.

Since, for $a.s.$ $(\{\xi_n\}, \{\Gamma_n\})$, $\{Y_{RH}(t)\}$ is a Gaussian process defined by the iid Gaussian sequence $\{g_n\}$, the assertion of the theorem follows from the Lemma 2 in Kôno and Maejima (1991, p. 95). □

3. MAFLP and RHFLP are different

We now use the different behavior of modulus of continuity of MAFLP and RHFLP, together with an early result of Billingsley (1974) to prove the following theorem

Theorem 3.1. *Under the assumptions in Section 3, the MAFLP Δ_H and the RHFLP Ψ_H are essentially not equivalent in the law.*

The result of Billingsley (1974) which we are going to use is the following. Suppose L is a subset of the space $\mathbb{R}^T$ of all real-valued functions on $T = [0,1]$. Property ρ is defined as that if there exists a version with sample paths in L a.s., then every separable version has its sample paths in L a.s. The problem is which L has Property ρ. For a countable dense subset D of T, let S_D be the set of functions x in $\mathbb{R}^T$ that are separable with respect to D, namely, $x \in S_D$ if and only if for each $t \in T$ there exists s sequence $\{t_n\} \subset D$ such that $t_n \to t$ and $x(t_n) \to x(t)$. Let $\mathcal{B}^T$ be the σ-field in $\mathbb{R}^T$ generated by the sets of the form $\{x : x(t) \le a\}$. Let $\mathcal{L}$ consist of those L in $\mathbb{R}^T$ such that, for each countable dense subset $D \subset T$, there exists a set $\bar{L}_D$ in $\mathbb{R}^T$ such that

$$\bar{L}_D \in \mathcal{B}^T, \quad \bar{L}_D \supset L, \quad \bar{L}_D - L \subset \mathbb{R}^T - S_D. \tag{3.1}$$

Billingsley (1974) proved the following.

Proposition 3.2. *Each L in $\mathcal{L}$ has Property ρ. Especially, the class of all continuous functions on T has Property ρ.*

We define

$$L = \left\{ x \in \mathbb{R}^T : \lim_{\delta\downarrow 0} \sup_{\substack{t,s\in T\\ |t-s|<\delta}} \frac{|x(t)-x(s)|}{\varphi(|t-s|)} = 0 \right\}, \tag{3.2}$$

and claim that

Lemma 3.3. *L is in $\mathcal{L}$.*

Proof. The proof is similar to that of 6° in Billingsley (1974). Let C be the class of all continuous functions on T. The natural candidate for $\bar{L}_D$ is

$$\alpha_D(L) = \{x : x \text{ agrees on } D \text{ with some } y \text{ in } L\}.$$

In Billingsley (1974), it is observed that if

$$L \subset C \quad \text{and} \quad \alpha_D(L) \in \mathcal{B}^T,$$

then $\alpha_D(L)$ will satisfy ((3.1)). It is obvious from the definition of L that $L \subset C$. Also we have

$$\alpha_D(L) = \bigcap_{\varepsilon} \bigcup_{\delta} \bigcap_{\substack{t,s \in D \\ |t-s|<\delta}} \left\{ x : \frac{|x(t) - x(s)|}{\varphi(|t-s|)} < \varepsilon \right\},$$

where ε and δ range over the positive rationals. Thus $\alpha_D(L) \in \mathcal{B}^D \subset \mathcal{B}^T$, and we conclude that $L \in \mathcal{L}$. □

Now we can prove the theorem as follows. Suppose $\Delta_H \overset{\mathrm{d}}{=} \Psi_H$. Then any version of Δ_H whose sample paths are continuous is regarded as a separable version of Ψ_H, for which, by Theorem 2.2, must be in the family L defined with $\varphi(t) = |t|^H |\log|t||^{1+\varepsilon}$. However, this contradicts Theorem 2.1, and concludes the theorem. □

A digressive remark: By the same idea of using sample-path behaviors to distinguish the processes, we can also see that the non-Gaussian Lévy-Chensov random field constructed in Shieh (1996), which is $(1/\alpha)$-selfsimilar and symmetric α-stable, $1 < \alpha < 2$, is not law-equivalent to the log-fractional symmetric α-stable field defined in Cambanis and Maejima (1989). Indeed, in that paper, it was proved that such Lévy-Chensov field has a separable version of sample path which is bounded and nowhere continuous on any annulus of the parameter space; while for the log-fractional case, it has a separable version of sample path which is nowhere bounded, see Maejima (1983). Likewise, this non-Gaussian Lévy-Chensov field is not law-equivalent to any field which has nowhere bounded sample paths, as those constructed in Samorodnitsky and Taqqu (1994, p. 402 and p. 453).

Some possible extensions: 1. We may relax the second moment condition of the driving Z by assuming that Z has the moment $E\big[|Z(1)|^\beta\big] < \infty$ for some $1 < \beta \leq 2$. Then we proceed the $H - \frac{1}{\beta}$ instead of $H - \frac{1}{2}$. This can recover the non-Gaussian symmetric α-stable, $1 < \alpha < 2$, case which was discussed in Cambanis and Maejima (1989); we firstly let the β slightly less than α, and then let $\beta \uparrow \alpha$.
2. We may relax the symmetry condition of Z by the usual symmetrization; we

consider the symmetric Lévy process $\overline{Z}(t) := Z_1(t) - Z_2(t)$, where Z_1, Z_2 are two independent copies of Z. One particular interesting case for such concern is the Gamma process or more generally the Lévy process with GGC as its distribution at time 1; see James *et al.* (2008).

3. A recent paper of Marcus and Rosinski (2005) discusses the continuity and boundedness of infinitely divisible processes based on Poisson point process approach. Their results can apply to stochastic integrals of the general form on T, a compact metric space or pseudo-metric space,

$$Y(t) = \int_S g(t,s)M(ds), \quad t \in T.$$

where M is a zero-mean, independently scattered, infinitely divisible random measure without Gaussian component. They give several examples which show that in many cases the conditions obtained are quite sharp. It seems that such sharp estimates can be proceeded for the fractional-type integrand and thus we may obtain the exact modulus of continuity for fractional Lévy processes (the fBM can be obtained from the vast literatures of regularity theory of Gaussian processes).

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