Free Fields Associated with the Relativistic Operator $-(m - \sqrt{m^2 - \Delta})$

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Abstract

The purpose of this article is to consider the construction of free fields associated with the relativistic operator $-(m - \sqrt{m^2 - \Delta})$. The study is based on the viewpoint of pseudo-differential operators, and we present both the Gaussian case and the non-Gaussian infinitely divisible case. We prove that, in the Gaussian case our constructed field is *singular* with respect to the Gaussian free field based on the non-relativistic $m^2 - \Delta$ [15].

1 Introduction

The purpose of this article is to consider the construction of *free fields* associated with the relativistic operator

$$-\Delta_m^{(r)} := -(m - \sqrt{m^2 - \Delta}),$$

where $m > 0$ is the normalized mass of a relativistic particle. On the one hand, the operator $-\Delta_m^{(r)}$ is the so-called free relativistic Hamiltonian, and is key to the physical theory of stability of matter, pioneered by Lieb [7] in 1970's. A seminal paper by Carmona *et al.* [3] investigated the mathematical theory of this operator, and in particular

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its close relation to Lévy processes; see also [12, 1] for the subsequent studies, including a fractional version. A very recent article [4] contains the sharp estimate of the associated heat kernel and also useful references. The operator has interesting multi-scaling property; see [17, 9]. On the other hand, Gaussian free fields (GFFs, for brevity) have been understood as the building block of quantum field theory, and is also key to the recent advent in quantum gravity [5] and the Schramm-Loewner evolution [14], see the inspiring survey of Sheffield [15]. The GFF in [15] is mainly massless, and the massive case mentioned in Section 3.3 there is based on $m^2 - \Delta$ (for which we may say, non-relativistic).

We aim to construct free fields associated with $-\Delta_m^{(r)}$ from the viewpoint of the theory of pseudo-differential operators; see, for example, the book of Wong [21] for a basic treatment of the theory. We present both the Gaussian case and the non-Gaussian infinitely divisible case. We prove that, in the Gaussian case our field constructed is *singular* with respect to the GFF based on non-relativistic $m^2 - \Delta$.

We present our results in Section 2, and all the proofs are given in Section 3.

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2 Main results

Since we treat $-\Delta_m^{(r)}$ as a pseudo-differential operator, as that in [3] and [20]; we work on the whole Euclidean space $\mathbb{R}^n$, $n \geq 2$, and is in the framework of tempered distributions. One may refer to, say, the book [21] for basic notions and properties of pseudo-differential operators and tempered distributions. The so-called *restriction problem* of free fields will be addressed elsewhere. Thus the space of test functions is the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, consisting of all rapidly decreasing functions, equipped with the Schwartz topology for such functions (somewhat different for those C^∞ of compact supports); see [21]. Meanwhile, for an $f \in L^2(\mathbb{R}^n, Leb)$, we use $\hat{f}$ or $\mathcal{F}f$ to denote the Fourier (-Plancherel) transform of f. For an $f \in \mathcal{S}(\mathbb{R}^n)$, define $\Delta_m^{(r)} f$ via the Fourier transform

$$\widehat{\Delta_m^{(r)} f}(\xi) = \theta(\xi)\hat{f}(\xi), \quad \xi \in \mathbb{R}^n,$$

with

$$\theta(\xi) := (m^2 + |\xi|^2)^{1/2} - m > 0, \quad \forall \xi \neq 0.$$

The following proposition is the starting point of our study

Proposition 1. *For $f, g \in \mathcal{S}(\mathbb{R}^n)$, the operator $\Delta_m^{(r)}$ defines an inner product on $\mathbb{R}^n$ by*

$$(f, g)_m^{(r)} := (f, -\Delta_m^{(r)} g) = (-\Delta_m^{(r)} f, g),$$

where $(\cdot, \cdot)$ denotes the usual inner product on $L^2(\mathbb{R}^n, Leb)$.

We observe that, since $\theta(\xi) \simeq |\xi|$ as the latter is large, the Hilbert-space closure of $\mathcal{S}(\mathbb{R}^n)$ under the inner product $(f, g)_m^{(r)}$ is the Sobolev-Bessel space (see, for example, [21], or the more advanced [19]).

$$H^{1,2}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n, Leb) : \quad \mathcal{F}^{-1}(1 + |\xi|^2)^{1/2}(\mathcal{F}f)(\xi) \in L^2(\mathbb{R}^n, Leb)\}.$$

The following notion is essential to our study; it is natural to call the positive quantity $(f, f)_m^{(r)}$ for each nonzero $f \in \mathcal{S}(\mathbb{R}^n)$, or for more general nonzero $f \in H^{1,2}(\mathbb{R}^n)$, to be the *relativistic energy* of f, and denote it by $\mathcal{E}_m^{(r)}(f)$. The notion corresponds to the (traditional) Dirichlet energy, denoted by $\mathcal{E}_\nabla(f)$, based on the Dirichlet product $(f, f)_\nabla$; see [15, Section 2.1]. Now, we present the following definition:

Definition. *Given an underlying complete probability space (Ω, P), the free field associated with $-\Delta_m^{(r)}$, denoted by $X_m^{(r)}$, is the unique linear random functional $(X_m^{(r)}, f)$,* *indexed by $f \in \mathcal{S}(\mathbb{R}^n)$ (the extension to more generally f will be discussed below), taking values in $L^2(\Omega, dP)$ (the Hilbert space of square-integrable random variables defined on Ω), and continuous in the sense that:*

$$f_n \to 0 \ \ in \ \ \mathcal{S}(\mathbb{R}^n) \ \ \Rightarrow \ \ (X_m^{(r)}, f_n) \to 0 \ \ in \ \ L^2(dP).$$

The random variables $(X_m^{(r)}, f)$ are required to be centered (i.e. mean zero) and with the covariance structure

$$E[(X_m^{(r)}, f)(X_m^{(r)}, g)] = (f, g)_m^{(r)};$$

in the above, the notation $E[\ \cdot\ ]$ denotes the expectation, i.e. the mean (the integral), of a random variable with respect to the underlying probability measure P. Moreover, we assume that each $(X_m^{(r)}, f)$ is an infinitely divisible random variable, either Gaussian or non-Gaussian.

We should remark that the above definition is **consistent with** the existing literature, as follows.

Firstly, by Proposition 1, $(f, g)_m^{(r)}$ is a positive-definite bilinear form, it is qualified to be a covariance structure, and therefore the above definition is indeed a kind of generalized random field in the classic paper by Yaglom [22] on the correlation theory (spectral analysis) of L^2 random fields; this paper is mainly in the context of generalized random fields, i.e. linear random functionals, except he used the C^∞ functions with compact supports as test functions. Secondly, in the theory of correlation theory, it is mostly in the framework of the second moment, i.e. for random variables with the $L^2(dP)$ norm; while this article, as we have imposed in the definition, we assume further that each $(X_m^{(r)}, f)$ is an *infinitely divisible* (ID, for short) random variable. Then, due to the linearity assumption, each finite linear combination $a_1(X_m^{(r)}, f_1) + \cdots + a_k(X_m^{(r)}, f_k)$ is also an ID random variable. Therefore, the family of random variables $(X_m^{(r)}, f)$, indexed by f, is then a *centered ID system with second moments*; see Rajput and Rosinski [10] for the intensive study of the structure of such ID systems. Thirdly, and most prominently, in the Gaussian case our proposed definition in above is seen as a one mentioned in [6, Example 1.16, p. 7].

An ID random variable is characterized by its Lévy triplet (see, for example, Sato [13]); the drift (which is zero since our field is centered), the Gaussian part, and the jump-part (which is determined by its Lévy measure $\nu(dx)$). We treat the Gaussian and the non-Gaussian separately. Write, under the ID assumption, the characteristic function (chf) as

$$\psi(s,f) := E[e^{is(X_m^{(r)},f)}] := e^{-k(s,f)}, \quad s \in \mathbb{R},$$

where $k(s,f)$ is the Lévy-Khintchine representation; we treat separately as

$*$ Gaussian: $k(s,f) = c(f)s^2$, where $c(f)$ is a positive constant to be determined.

$**$ Non-Gaussian:

$$k(s,f) = \int_{\mathbb{R}} [e^{isx} - 1]\nu(dx,f),$$

where the Lévy measure ν is to be determined; we remark that, since we have assumed that $(X_m^{(r)}, f)$ has second moment, it is legitimate and convenient to use the integral representation for $k(s,f)$ in this form. We have

Proposition 2. *Let $X_m^{(r)}$ be an ID free field associated with $-\Delta_m^{(r)}$, as defined above, for which the Lévy-Khintchine representation of $(X_m^{(r)}, f)$ is $e^{-k(s,f)}$. Then,*

$*$ *Gaussian: $k(s,f) = c(f)s^2$, with $c(f) = \frac{\mathcal{E}_m^{(r)}(f)}{2}$.*

$**$ *Non-Gaussian: $k(s,f) = \int_{\mathbb{R}} [e^{isx} - 1]\nu(dx,f)$, with Lévy measure $\nu(dx,f)$ satisfying the scaling, for each nonzero a,*

$$\nu(d(x/a), f) = \nu(dx, af).$$

Remark: In a subsequent paper [18], we choose $\nu(dx,f)$ to be with the density $q(x,f)dx$, where

$$q(x,f) = \frac{1}{4\pi} \frac{1}{\sqrt{\mathcal{E}_m^{(r)}(f)}} \int_0^\infty \Big[\frac{1}{\sqrt{u}} \cdot e^{-\frac{x^2}{4u\mathcal{E}_m^{(r)}(f)}}\Big]\Big[\frac{e^{-m^2u}}{u^{\frac{3}{2}}}\Big] du,$$

for which the required scaling in Proposition 2 is satisfied.

In theory of free fields, it is of essential importance to extend $(X_m^{(r)}, f)$ defined for more general f other than $f \in \mathcal{S}(\mathbb{R}^n)$; see [15] for the detailed discussion of the (massless)

GFFs based on the Dirichlet product, $(f,f)_\nabla$ (the massive case also included). We treat the Gaussian and the non-Gaussian separately. We assume Proposition 2 to be valid.

In the Gaussian case,

$$E[e^{is(X_m^{(r)},f)}] := e^{\frac{\mathcal{E}_m^{(r)}(f)}{2}s^2}.$$

Then, $(X_m^{(r)},f)$ is consequently defined for $f \in H^{1,2}(\mathbb{R}^n)$. The latter space is the Hilbert closure of $\mathcal{S}(\mathbb{R}^n)$ under the $(f,g)_m^{(r)}$, as we have observed in above; therefore, we have the *Gaussian-Hilbert space* [6, Definition 1.19, §3, Chapter 1] based on the relativistic energy $\mathcal{E}_m^{(r)}(f)$. One should compare it with the treatment in [15] based on the Dirichlet energy $\mathcal{E}_\nabla(f)$.

The main contribution of this article is to present the following "Green's function representation" for the free field defined above, including both the Gaussian and the non-Gaussian case. We should remark that, the following theorem appears to be new, to our knowledge, even in the Gaussian case.

Theorem 1. *When $n \geq 3$, there is a unique nonnegative function, for which we call the relativistic Green's function $G_m^{(r)}(x,y)$ associated with $\Delta_m^{(r)}$, such that $G_m^{(r)}(x,y) = G_m^{(r)}(y,x)$ and that, for $f,g \in \mathcal{S}(\mathbb{R}^n)$,*

$$(f,g)_m^{(r)} = \int\int f(x)G_m^{(r)}(x,y)g(y)dxdy = \int\int f(y)G_m^{(r)}(y,x)g(x)dydx. \tag{2.1}$$

For $n = 2$, it has to be confined to $x,y \in D(0,R)$, an open disc with center at origin and with radius R (R can be arbitrarily large, yet needs to be finite); equivalently, it has to assume that f,g are supported on $D(0,R)$.

Symbolically, we may say that $G_m^{(r)}(x,y)$ solves, in the sense of tempered distributions,

$$\Delta_m^{(r)} G_m^{(r)}(x,\cdot) = -\delta_x(\cdot);$$

we confer this notion to the Green's function for the classical Laplacian in, say, [8, p. 148].

Here, the *dimensionality* n is crucial: for $n \geq 3$, $G_m^{(r)}(x,y)$ can be, as we will do in the following context, defined for all $x,y \in \mathbb{R}^n$; however, for $n = 2$, we need f,g to be supported on $inD(0, R)$, an open disc with center at origin and with radius R (R can be arbitrarily large, yet needs to be finite). The proof of the above Theorem 1 (given in Section 3) is mainly based on the probabilistic viewpoint, in which the dimension $n = 2$ has to be separated. The author is not able, it would be very interesting, to give an wholly analytic proof.

Remark: It is here an occasion to mention a terminology in [17]. In that article, the notation $G_t(x, y)$ is also referred as a Green's function, although it is better known as the heat kernel associate with $\Delta_m^{(r)}$.

We extend, as a consequence, $(X_m^{(r)}, f)$ to $f \in B_+(\mathbb{R}^n)$, the space of bounded nonnegative functions, and the equation (2.1) in Theorem 1 thus holds for all $f, g \in B_+(\mathbb{R}^n)$; in the $n = 2$, we need to confine f, g being supported on $D(0, R)$.

We remark that the Green's function representation is crucial to employ GFFs to both the quantum gravity [5] and the Schramm-Loewner evolution [14]. We mention the Gaussian-Hilbert setting in the Gaussian case, mainly for the following

Theorem 2. *The Gaussian-Hilbert space based on the relativistic energy $\mathcal{E}_m^{(r)}(f)$ and that one based on the Dirichlet energy $\mathcal{E}_\nabla(f)$ are mutually singular.*

We will prove this "natural" result by the Gaussian dichotomy, see (for example) [2, Chapter 2], and by an early result of Shepp [16] (there are a lot of refinements on the Gaussian dichotomy in vast literature, admittedly).

Remark: In general, the existence of "Green's function" in other related models is a core issue; see a very recent article on Kahane's theory of multiplicative chaos [11].

3 Proofs

Proof of Proposition 1. Define, for each $t > 0$, a kernel $K_t(x), x \in \mathbb{R}^n$, via its Fourier transform by

$$\hat{K}_t(\xi) = \Big(\frac{1}{\sqrt{2\pi}}\Big)^n e^{-t\theta(\xi)}.$$

The kernel $K_t(x)$ is symmetric $K_t(x) = K_t(-x)$, and is radial $K_t(x) = K_t(||x||)$. We then define a semigroup of operators on $L^2(\mathbb{R}^n)$ by

$$T_t(f) = (K_t * f)(x) = \int K_t(x-y)f(y)dy, \quad T_0 f = f.$$

We claim that $\{T_t\}$ is a contractive semigroup: $\|T_t f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^2(\mathbb{R}^n)}$; by the Fourier transform for convolutions and the Plancherel identity, we have,

$$\begin{aligned}\int_x (T_t f)^2(x)\ dx &= \int_\xi |\hat{K}_t(\xi)\hat{f}(\xi)|^2\ d\xi \\ &= \int_\xi e^{-t\theta(\xi)}\hat{f}(\xi)|^2 \\ &\leq \int_\xi |\hat{f}(\xi)|^2\ d\xi = \int_x f^2(x)\ dx.\end{aligned}$$

Now, we prove that, for each $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\|\frac{(T_t - I)\phi}{t} + \Delta_m^{(r)}\phi\|_{L^2(\mathbb{R}^n)} \to 0, \quad \text{as} \quad t \downarrow 0,$$

which means that, *c.f.* [20, Theorem 4.1], the infinitesimal generator of the L^2 semigroup $\{T_t\}$ is $-\Delta_m^{(r)}$, at least acting on $\mathcal{S}(\mathbb{R}^n)$. To prove the above display, write $D_t(\phi) = \frac{(T_t-I)\phi}{t} + \Delta_m^{(r)}\phi$, and observe that

$$\begin{aligned}\int (D_t\phi)^2(x)dx &= \int |\widehat{D_t\phi}|^2(\xi)d\xi \\ &= \int \Big[\frac{e^{-t\theta(\xi)} - 1}{t} + \theta(\xi)\Big]^2 \hat{\phi}(\xi)\xi.\end{aligned}$$

For each ξ, the term in the above bracket tends to 0 as $t \downarrow 0$; hence the whole integral in the above must also tend to 0, since $\hat{\phi}(\xi)$ is rapidly decreasing in ξ.

For each $t > 0$, we have the symmetric (the self-adjoint) property of T_t on $\mathcal{S}(\mathbb{R}^n)$: $(f, T_t g) = (T_t f, g)$, which follows from the symmetry of the kernel $K_t(x)$. Thus, so is for the operator $-\Delta_m^{(r)}$, i.e. $(f, -\Delta_m^{(r)} g) = (-\Delta_m^{(r)} f, g)$. Moreover, for each $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\int (-\Delta_m^{(r)} f)(x) \cdot f(x)dx = \int \widehat{\Delta_m^{(r)} f}(\xi) \cdot \overline{\hat{f}(\xi)}d\xi = \int \theta(\xi)|\hat{f}(\xi)|^2 d\xi \geq 0,$$

since $\theta(\xi) > 0, \forall \xi \neq 0$; the above equals to 0 only when $f = 0$. Therefore, the $(f, g)_m^{(r)}$ indeed defines an inner product on $\mathcal{S}(\mathbb{R}^n)$. □

Proof of Proposition 2. Observe that the chf $\psi(s,f) := E[e^{is(X_m^{(r)},f)}] := e^{-k(s,f)}$ must satisfy

$$\psi'(s,f)|_{s=0} = iE[(X_m^{(r)},f)], \quad \psi''(s,f)|_{s=0} = -E[(X_m^{(r)},f)^2].$$

Thus, for the Gaussian case, in which $k(s,f) = c(f)s^2$, it follows that $c(f) = E[(X_m^{(r)},f)^2]/2 = \mathcal{E}_m^{(r)}(f)/2$.

As for the non-Gaussian case, the $k(s,f) = \int_{\mathbb{R}}[e^{isx} - 1]\nu(dx,f)$, with Lévy measure $\nu(dx,f)$, gives, *c.f.* [13, p.163],

$$\int_{\mathbb{R}} x\nu(dx,f) = 0, \quad \int_{\mathbb{R}} x^2\nu(dx,f) = E[(X_m^{(r)},f)^2].$$

The scaling property of the Lévy measure then follow from the linear requirement of $X_m^{(r)}$; namely $a(X_m^{(r)},f) = (X_m^{(r)},af)$. □

Proof of Theorem 1. The following construction is already noted in [3]; see also [12, 4]. Let $B(t)$ be the n-dimensional standard Brownian motion, and let the so-called relativistic $\frac{1}{2}$-subordinator be defined as a Lévy process $\mathcal{T}_t$ with increasing sample paths and with the Laplace function determined by

$$E\Big[e^{-u\mathcal{T}_t}\Big] = e^{-t(\sqrt{m^2+u}-m)}, \quad u > 0.$$

Assume that the process $\mathcal{T}_t$ and the Brownian motion B_t are totally independent, then the subordinated process $X_t = B_{\mathcal{T}_t}$ is a Lévy process in $\mathbb{R}^n$, for which the characteristic function is given by,

$$E\Big[e^{i<\xi,X_t>}\Big] = e^{-t(\sqrt{m^2+|\xi|^2}-m)}, \quad \xi \in \mathbb{R}^n.$$

The transition density function $p(t,x,y)$ of the process X_t is then existent, jointly continuous in (t,x,y), and $p(t,x,y) = p(t,0,y-x)$. The estimate of $p(t,x,y)$ given in [3], see also the latest sharp one in [4] (with $\alpha = 1$ there), tells that, for $n \geq 3$, the "probabilistic Green's function"

$$G(x,y) := \int_0^\infty p(t,x,y)dt,$$

is defined, finite-valued and symmetric in $x, y \in \mathbb{R}^n$.

We prove that this "probabilistic Green's function" $G(x, y)$ satisfies (2.1). Firstly, the defining property of $G(x, y)$ asserts that $G(x, y) = G(y, x) = G(0, x-y) = G(0, y-x)$, since $p(t, x, y)$ has such spatial homogeneity for each t. Next, comparing the characteristic function of X_t given in above and the Fourier transform of the kernel K_t in the proof of Proposition 1, we see that $K_t(x) = c_n \cdot p(t, 0, x)$, where c_n is an absolute constant depending only on the dimension n. Therefore, the symmetric (the self-adjoint) property of T_t on $\mathcal{S}(\mathbb{R}^n)$: $(f, T_t g) = (T_t f, g)$, which is mentioned in the proof of Proposition 1, can be re-written as, for $t > 0$ and for $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\int\int f(x)p(t,x,y)g(y)dxdy = \int\int f(y)p(t,y,x)g(x)dydx.$$

Integrating the above over $t \in (0, \infty)$, we have

$$\int\int f(x)G(x,y)g(y)dxdy = \int\int f(y)G(y,x)g(x)dydx;$$

we remark that it is legitimate to change the order of the integration, since $G(x, y)$ has been defined as a nonnegative finite-valued symmetric function in $x, y \in \mathbb{R}^n$. To see the above is equal to the $(f, g)_m^{(r)}$, and thus have the equation (2.1), with $G_m^{(r)}(x, y)$ being this "probabilistic Green's function" $G(x, y)$, we use the same argument as that in [8, p. 166] for the Laplacian Δ and the Gaussian heat kernel; it is also parallel to some argument in the proof of [20, Theorem 4.1]. Indeed, $-\Delta_m^{(r)}$ is shown to be the infinitesimal generator of the semigroup $\{T_t\}$ in the proof of Proposition 1, and thus $T_t g = e^{-t\Delta_m^{(r)}} g$, at least for $g \in \mathcal{S}(\mathbb{R}^n)$. Thus, we have

$$-\Delta_m^{(r)} T_t g = \frac{d}{dt} T_t g; \tag{3.1}$$

confer to [8, p. 166]. We recall, from the proof of Proposition 1 and the relation of $K_t(x)$& $p(t, x, y)$ in above, that

$$T_t g(x) = (K_t * g)(x) = \int K_t(x-y)g(y)dy = \int p(t,x,y)g(y)dy, \quad T_0 g = g.$$

We then have

$$(f, e^{-t\Delta_m^{(r)}} g)_{Leb(dx)} = (f, T_t g)_{Leb(dx)} = \int\int f(x)p(t,x,y)g(y)dxdy.$$

Integrating the above over $t \in (0, \infty)$, we have

$$\int\int f(x)G(x,y)g(y)dxdy = (f, (\Delta_m^{(r)})^{-1} g)_{Leb(dx)} = (-\Delta_m^{(r)} f, g)_{Leb(dx)},$$

which is the left-handed side of (2.1). In the second term of the above, the inverse of the operator $\Delta_m^{(r)}$ is implied by (3.1). This completes the proof of the theorem for $n \geq 3$.

For the $n = 2$, as in [12, 4], we need to consider, not the whole process X_t but the "subprocess" killed upon exiting $D(0, R)$, i.e. $X_t^R := X_t, \quad t < \tau_R = \inf\{t \geq 0; \quad X_t \notin D(0,R)\}, \ = \partial, \ t \geq \tau_R$, and consider the residual $p_R(t,x,y)$ which is $p(t,x,y) - r_R(t,x,y)$ (r_R is the hitting probability density, see, for example, [12, p. 9]); the corresponding Green's function is then

$$G(x,y) := G_R(x,y) := \int_0^\infty p_R(t,x,y)dt.$$

The killing is mandate for $n = 2$, due to the standard fact that the planar Brownian motion is recurrent; while it is transient if the dimension is higher. Therefore, the (2.1) in the dimension $n = 2$ must need f, g to be supported on $D(0, R)$. $\square$

Remark: The proof in above, as we may have seen, blends the analysis and probability, and it hinges on that the probabilistic method in [12, 4] gives the proper estimate to assert the integrability of $K_t(x) = c_n \cdot p(t, 0, x)$ over $t \in (0, \infty)$.

Proof of Theorem 2. From the Gaussian dichotomy, the two $(X_m^{(r)}, f)$ and (X, f) must be either mutually singular or mutually equivalent. Notice that the Gaussian-Hilbert space $(X_m^{(r)}, f)$ is for $f \in H^{1,2}(\mathbb{R}^n)$ while the Gaussian-Hilbert space (X, f) is for $f \in H^{2,2}(\mathbb{R}^n)$ (see the Section 2 for the first, and the [15] for the latter), and that $H^{2,2}(\mathbb{R}^n) \subsetneq H^{1,2}(\mathbb{R}^n)$. For any

$$f \in H^{1,2}(\mathbb{R}^n) \setminus H^{2,2}(\mathbb{R}^n),$$

which means that the relativistic energy $\mathcal{E}_m^{(r)}(f)$ is finite *and* the (traditional) Dirichlet energy $\mathcal{E}_\nabla(f)$ is infinite, it must have, in the sense of probability distributions, $(X, f) \sim$

δ_0 (degenerate) while $(X_m^{(r)}, f) \sim \mathcal{N}(0, \mathcal{E}_m^{(r)}(f))$ (properly Gaussian distributed). This show that the two Gaussian measures cannot be mutually equivalent, and hence must be mutually singular.

We can also prove the mutual singularity of $(X_m^{(r)}, f)$ and (X, f), for f indexed by the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, as follows. We apply Shepp's [16] result in the following form. Let $\{H_l(r),\ l = 0, 1, 2, ...\}$ be the Hermite polynomials, i.e.

$$H_l(x) = (-1)^l e^{\frac{x^2}{2}} \frac{d^l}{dx^l} e^{-\frac{x^2}{2}}, \quad \text{for} \quad l \in \{0, 1, 2, ...\}, x \in \mathbb{R}.$$

Thus, the Hermite functions

$$h_l(x) := \frac{1}{\sqrt[4]{2\pi}} \frac{1}{\sqrt{n!}} e^{-\frac{x^2}{4}} H_l(x), \quad l = 0, 1, 2, ...,$$

are in $\mathcal{S}(\mathbb{R}^1)$, and form an orthonormal basis for $L^2(\mathbb{R}^1, dx)$. Therefore,

$$h_{\bar{k}}(x) := h_{k_1}(x_1) \cdots h_{k_n}(x_n), \quad \bar{k} = (k_1, \cdots, k_n), x = ((x_1, \cdots, x_n),$$

form an approximating family in $\mathcal{S}(\mathbb{R}^n)$. We write, for $l = 0, 1, 2, ...$

$$\pi^l := \{h_{\bar{k}}(x), \quad \bar{k} = (k_1, \cdots, k_n), \quad k_i \le l, \forall i\}.$$

Let $\rho_i, i = 0, 1$, denote the covariance matrix associated, respectively, with $(X, h_{\bar{k}})$ and $(X_m^{(r)}, h_{\bar{k}})$, and ρ be the average matrix $\rho = \frac{\rho_0 + \rho_1}{2}$. The Hellinger functional up to the order l, $\mathcal{H}(\pi^l)$, see [16, p. 169 (1.8)] (remark that we are now in the centered, that is mean zero, and thus there is *no* exponential part in his (1.8)), is then

$$\mathcal{H}(\pi^l) = \frac{(\det \rho_0(\pi^l))^{1/4} (\det \rho_1(\pi^l))^{1/4}}{(\det \rho(\pi^l))^{1/2}}.$$

The reciprocal of its 4th power is the product of l strictly positive numbers, $\prod_j^l \frac{\lambda_j + 1}{2} \cdot \frac{1 + \lambda_j^{-1}}{2}$ [16, p.169 (1.10), p.170 (1.16)]. Each λ_j itself is split as $\lambda_j = \delta_{j,0} \delta_{j,1}^{-1}$, where $\delta_{j,i}$ ($j = 0, 1, 2, \ldots$) is the eigenvalues corresponding to ρ_i. The sequence of eigenvalues of the covariance matrices ρ_i defined above is unbounded, and thus the l products in the above tend to ∞ as $l \uparrow \infty$. Consequently, the Hellinger functional $\mathcal{H}$ of the two Gaussian free fields is zero, and thus they mutually singular. □

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