INNER PRODUCT FORMULA FOR YOSHIDA LIFTS

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Abstract. We prove an explicit inner product formula for vector-valued Yoshida lifts by an explicit calculation of local zeta integrals in the Rallis inner product formula for O(4) and Sp(4). As a consequence, we obtain the non-vanishing of Yoshida lifts.

Contents

1. Introduction

Explicit formulas for Petersson norms of modular forms play an important role in the study of the connection between congruences among modular forms and special values of L-functions. The aim of this paper is to give an explicit formula for the Petersson norm of Yoshida lifts. Let F be either $\mathbf{Q} \oplus \mathbf{Q}$ or a real quadratic field of $\mathbf{Q}$ and let $\mathfrak{N}$ be an ideal of the ring of integers $\mathcal{O}_F$ of F. Let $\underline{k} = (k_1, k_2)$ be a pair of non-negative integers with $k_1 \geq k_2$. Yoshida lifts are explicit vector-valued Siegel modular forms of genus two and weight $\mathrm{Sym}^{2k_2}(\mathbf{C}^2) \otimes \det^{k_1-k_2+2}$ associated with a holomorphic newform f on $\mathrm{PGL}_2(\mathbf{A}_F)$ of conductor $\mathfrak{N}$ and weight $(2k_1+2, 2k_2+2)$. Note that f is given by a pair (f_1, f_2) of elliptic newforms if $F = \mathbf{Q} \oplus \mathbf{Q}$, and f is a Hilbert modular newform over a real quadratic field if F is a real quadratic field. The scalar-valued Yoshida lifts ($k_2 = 0$) were constructed by H. Yoshida in [Yos80] via theta lifting from SO(4) to Sp(4), and his construction was extended to the vector-valued Yoshida lifts ($k_2 > 0$) by Böcherer and Schulze-Pillot (*cf.* [BSP97, §1] and [HN16, §3]). In the sequel, Yoshida lifts are said to be of type (I) if $F = \mathbf{Q} \oplus \mathbf{Q}$ and of type (II) if F is a real quadratic field. The non-vanishing of Yoshida lifts was also conjectured by Yoshida himself, which was later proved in [BSP97] for Yoshida lifts of type (I). Then our main result is an explicit Rallis inner product formula for Yoshida lifts, which relates the Petersson norm of the Yoshida lift to special values of the Asai L-function $L(\mathrm{As}^+(f), s)$ attached to f. As a consequence of our formula, we prove the non-vanishing of Yoshida lifts of type (I) and (II).

To state our main result precisely, we introduce some notation. Let c be the non-trivial automorphism of F and let Δ_F be the discriminant of F. Denote by f^c the Galois conjugation of f. We assume that f is not Galois self-dual, namely

$$f \neq f^c,$$

and that the conductor $\mathfrak{N}$ of f is a square-free product of prime ideals of $\mathcal{O}_F$ with $(\mathfrak{N}, \Delta_F) = 1$ and is divisible by N^- a square-free product of an odd number of rational primes split in F. Let D_0 be the

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definite quaternion algebra over $\mathbf{Q}$ of absolute discriminant N^- and let $D = D_0 \otimes_{\mathbf{Q}} F$. By our assumption on N^-, D is the totally definite quaternion algebra over F ramified at $N^-\mathcal{O}_F$. Let R denote the Eichler order in D of level $\mathfrak{N}^+$. For $i = 1, 2$, let $\mathcal{W}_{k_i}(\mathbf{C}) := \mathrm{Sym}^{2k_i}(\mathbf{C}^2) \otimes \det^{-k_i}$ be the algebraic representation of $\mathrm{GL}_2(\mathbf{C})$ of the highest weight $(k_i, -k_i)$. By the Jacquet-Langlands-Shimizu correspondence, there exist a vector-valued newform $\mathbf{f} : D^\times \backslash D^\times_{\mathbf{A}} / \widehat{R}^\times \to \mathcal{W}_{k_1}(\mathbf{C}) \boxtimes \mathcal{W}_{k_2}(\mathbf{C})$ unique up to scalar such that $\mathbf{f}$ shares with same Hecke eigenvalues with f at $p \nmid N^-$ (*cf.* [HN16, §3.3]). Let $*$ be the main involution of D and let $V = \{x \in D \mid x^* = x^c\}$ be the four dimensional $\mathbf{Q}$-vector space with the positive definite quadratic form $\mathrm{n}(x) = xx^*$. Following [Yos80, p.196], the group $G' := \{x \in D^\times \mid \mathrm{n}(x) = 1\}$ acts on V via the action $\varrho(a)x = ax(a^c)^*$ and the image $\varrho(G') \subset \mathrm{Aut}(V)$ is the special orthogonal group $\mathrm{SO}(V)$. We can thus view $\mathbf{f}$ as an automorphic form on $\mathrm{SO}(V)(\mathbf{A})$ and consider its theta lifts to $\mathrm{Sp}(4)$. Let $N = \mathfrak{N} \cap \mathbf{Z}$ and let $N_F = N\Delta_F$. Let $\mathfrak{H}_2$ be the Siegel upper half plane of degree two and $\Gamma_0^{(2)}(N_F) \subset \mathrm{Sp}_4(\mathbf{Z})$ be the Siegel parabolic subgroup of level N_F. Let $\mathcal{L}(\mathbf{C}) := \mathrm{Sym}^{2k_2}(\mathbf{C}^2) \det^{k_1-k_2+2}$ be the representation of $\mathrm{GL}_2(\mathbf{C})$ of the highest weight $(k_1 + k_2 + 2, k_1 - k_2 + 2)$. In [HN16, §3.7], we apply the theta lifting from $\mathrm{SO}(V)$ to $\mathrm{Sp}(4)$ to obtain the vector-valued Yoshida lift $\theta^*_{\mathbf{f}} : \mathfrak{H}_2 \to \mathcal{L}(\mathbf{C})$ attached to $\mathbf{f}$ and a distinguished Bruhat-Schwartz funciton $\varphi^\star$ on $V_{\mathbf{A}}^{\oplus 2}$ with value in $\mathcal{W}_{k_1}(\mathbf{C}) \otimes \mathcal{W}_{k_2}(\mathbf{C}) \otimes \mathcal{L}(\mathbf{C})$ (see §4 for more details). In particular, $\theta^*_{\mathbf{f}}$ is exactly the scalar-valued Siegel modular form constructed in [Yos80] when $k_2 = 0$. The Yoshida lift $\theta^*_{\mathbf{f}}$ is a vector-valued Siegel modular form of level $\Gamma_0^{(2)}(N_F)$, which is an eigenfunction of Hecke operators at $p \nmid N_F$, and the associated spin L-function $L(\theta^*_{\mathbf{f}}, s)$ is given by $L(f_1, s-k_2)L(f_2, s-k_1)$ if $F = \mathbf{Q} \oplus \mathbf{Q}$ and $f = (f_1, f_2)$ and by $L(f, s-k_2)$ if F is a real quadratic field. Let $\mathcal{B}_{\mathcal{L}} : \mathcal{L}(\mathbf{C}) \otimes \mathcal{L}(\mathbf{C}) \to \mathbf{C}$ be the positive definite Hermitian pairing defined in (5.15) and define the Petersson norm of $\theta^*_{\mathbf{f}}$ by

$$\langle \theta^*_{\mathbf{f}}, \theta^*_{\mathbf{f}} \rangle_{\mathfrak{H}_2} = \int_{\Gamma_0^{(2)}(N_F) \backslash \mathfrak{H}_2} \mathcal{B}_{\mathcal{L}}(\theta^*_{\mathbf{f}}(Z), \theta^*_{\mathbf{f}}(Z)) (\det Y)^{k_1+2} \frac{\mathrm{d}X \mathrm{d}Y}{(\det Y)^3}.$$

Let $\langle \mathbf{f}, \mathbf{f} \rangle_R$ be the Peterson norm of $\mathbf{f}$ defined in (5.16) of §5.3. Now we state our main result in the simple case $\mathfrak{N} = N\mathcal{O}_F$. For $p \mid N$, denote by $\varepsilon_p \in \{\pm 1\}$ the Atkin-Lehner eigenvalues of $\mathbf{f}$ at p (see (4.4) for the definition).

Theorem A (Theorem 5.7)**.** *Suppose further that $\mathfrak{N} = N\mathcal{O}_F$. Then we have*

$$\begin{aligned}\frac{\langle \theta^*_{\mathbf{f}}, \theta^*_{\mathbf{f}} \rangle_{\mathfrak{H}_2}}{\langle \mathbf{f}, \mathbf{f} \rangle_R} =& L(\mathrm{As}^+(f), k_1 + k_2 + 2) \cdot (4\pi)^{-(2k_1+3)} \Gamma(k_1 + k_2 + 2) \Gamma(k_1 - k_2 + 1) \\ & \times \frac{N \cdot 2^{4r_{F,2} - r_F - 2}}{(2k_1 + 1)(2k_2 + 1)} \cdot \prod_{p \mid N} (1 + \varepsilon_p) \cdot \prod_{p \mid \Delta_F} (1 + p^{-1}),\end{aligned}$$

where r_F is the number of prime factors of Δ_F, and $r_{F,2} = 1$ if $2 \mid \Delta_F$ and 0 otherwise.

If $\mathfrak{N} \neq N\mathcal{O}_F$, then we reply $\mathbf{f}$ with the stablilized newform $\mathbf{f}^\dagger$ defined in (4.6), and Theorem 5.7 provides the formula for the Petersson norm of $\theta^*_{\mathbf{f}^\dagger}$. Note that the left-hand side of the formula is independent of the choice of the newform $\mathbf{f}$ since a newform is unique up to a scalar. In fact, $\mathbf{f}$ can be normalized so that $\mathbf{f}|_{\widehat{D}^\times}$ takes values in the Hecke field $\mathbf{Q}(f)$ of f, namely the field generated by the Fourier coefficients of f over $\mathbf{Q}$, and hence $\theta^*_{\mathbf{f}}$ is defined over $\mathbf{Q}(f)$ in view of the formula for Fourier coefficients of $\theta^*_{\mathbf{f}}$ in the proof of [HN16, Proposition 5.1]. This allows us to the study algebraicity of the special value $L(\mathrm{As}^+(f), k_1 + k_2 + 2)$ by the method in [Sah15].

Remark 1.1. We give some additional comments on the case where $F = \mathbf{Q} \oplus \mathbf{Q}$ and $f = (f_1, f_2)$ is given by a pair of elliptic newforms.

(i) The Asai L-function $L(\mathrm{As}^+(f), s) = L(f_1 \otimes f_2, s)$ is the Rankin-Selberg convolution of f_1 and f_2. In this case, an inner product formula for Yoshida lifts of type (I) was also derived in [BDSP12, Corollary 8.8] by the Rankin-Selberg method. In [AK13, Conjecture 5.19], Agarwal and Klosin formulated a conjecture on an explicit inner product formula of scalar-valued Yoshida lifts of type (I). Our Theorem A confirms their conjecture and further generalizes to the vector-valued Yoshida lifts.

(ii) Given a prime $p > k_1$, under some mild assumptions on the residual p-adic Galois representations attached to f_1 and f_2, it is known that $\mathbf{f}$ can be normalized such that $\theta^*_{\mathbf{f}}$ has Fourier coefficients in the ring of integers of the Hecke field of f_1 and f_2 localized at p and is non-vanishing modulo p (See [HN16, §5]) and that the Petersson norm $\langle \mathbf{f}, \mathbf{f} \rangle$ is given by a product of the congruence numbers of f_1 and f_2 up to a p-adic unit (See [PW11] and [CH16]). In particular, the period ratio $\Omega_{1,2}$ in [AK13, Remark 6.4] is a p-unit in many situations.

Let π_f be the unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ associated with f. Then in terms of automorphic L-functions, we have

$$L(s, \mathrm{As}^+(\pi_f)) = \Gamma_{\mathbf{C}}(s + k_1 + k_2 + 1)\Gamma_{\mathbf{C}}(s + k_1 - k_2)L(\mathrm{As}^+(f), s + k_1 + k_2 + 1).$$

By the non-vanishing of $L(1, \mathrm{As}^+(\pi))$ (*cf.* [GS15, Theorem 4.3]), we obtain the following consequence on the non-vanishing of Yoshida lifts, generalizing the main result in [BSP97] to Yoshida lifts of type (II).

Corollary B. *If $\varepsilon_p = 1$ for every $p \mid N$, then the vector-valued Yoshida lift $\theta^*_{\mathbf{f}}$ is non-zero.*

Remark 1.2. (i) The necessary condition on the Atkin-Lehner eigenvalues for the non-vanishing of Yoshida lifts already appeared in [Yos80].

(ii) Our results are different from those obtained by the representation theoretic method in [Rob01, Theorem 8.3], [Tak09, Theorem 1.1] and [SS13, Proposition 3.1] for these authors prove the non-vanishing of the space generated by some Yoshida lift, while we prove the non-vanishing of a particular Yoshida lift with integral Fourier coefficients.

(iii) Note that a paramodular Yoshida lift of type (II) was constructed in [JLR12], but our Yoshida lift is Siegel parahoric. The local component of the automorphic representation generated by $\theta^*_{\mathbf{f}}$ at a prime $p \mid \Delta_F$ provides an example of generic non-endoscopic supercuspidal representations of $\mathrm{GSp}\,4(\mathbf{Q}_p)$ possessed of a Siegel parahoic fixed vector.

In addition to the application to the non-vanishing of Yoshida lifts, our main motivation for the explicit Petersson norm formula for Yoshida lifts of type (I) originates from the study on the congruences between Hecke eigen-systems of Yoshida lifts and stable forms on GSp(4), the so-called *Yoshida congruence* as well as its application to the Bloch-Kato conjecture for special values of Asai L-functions. The Yoshida congruence was first investigated by the independent works [BDSP12] and [AK13], where the Petersson norm formula was used to relate the congruence primes of Yoshida lifts of type (I) to special values of the Rankin-Selberg L-functions. More precisely, in [BDSP12, Corollary 9.2] and [AK13, Theorem 6.6], the authors proved that if a prime p divides the algebraic part of the L-values $L(f_1 \otimes f_2, k_1 + k_2 + 2)$, then p is a congruence prime for Yoshida lifts attached to a pair of elliptic newforms (f_1, f_2) of weight $(2k_1 + 2, 2k_2 + 2)$ under some restricted hypotheses. It is our hope that this Petersson norm formula together with our previous result on the non-vanishing of the Yoshida lift $\theta^*_{\mathbf{f}}$ modulo a prime in [HN16] serve the first step towards the understanding of Yoshida congruence in a more general setting.

This paper is organized as follows. In §2, we fix the notation and definitions, and in §3, we introduce the Asai L-functions. In §4, we recall the construction of Yoshida lifts in [HN16], which depends on the choice of the particular test function $\varphi^\star = \otimes_p \varphi^\star_p$ given in §4.5. In §5, we realize Yoshida lifts as theta lifts from GO(4) to Sp(4) and then apply the Rallis inner product formula in [GI11] and [GQT14] to reduce the Petersson norm formula to the explicit computation of certain local zeta integrals $\mathcal{I}(\varphi^\star_\infty)$ at the archimedean place and $\mathcal{Z}(\varphi^\star_p, f^\dagger_p)$ at non-archimedean places (see Proposition 5.4). In §6, we carry out the bulk of this paper, the explicit calculation of these local zeta integrals at all places.

2. Notation and definitions

2.1. **Basic notation.** For a number field F, we denote by $\mathcal{O}_F$ (resp. $\Delta_F, \mathfrak{D}_{F/\mathbf{Q}}$) the ring of integers of F (resp. the discriminant of $F/\mathbf{Q}$, the different of $F/\mathbf{Q}$). Let $\mathbf{A}_F$ be the ring of adeles of F. For an element a of $\mathbf{A}_F$ and a place v of F, we denote by a_v the v-component of a.

Let $\Sigma_{\mathbf{Q}}$ be the set of places of the rational number field $\mathbf{Q}$. We write $\mathbf{A}$ for $\mathbf{A}_{\mathbf{Q}}$. Let $\psi = \prod_{v \in \Sigma_{\mathbf{Q}}} \psi_v : \mathbf{A}/\mathbf{Q} \to \mathbf{C}^\times$ be the additive character with $\psi(x_\infty) = \exp(2\pi\sqrt{-1}x_\infty)$ for $x_\infty \in \mathbf{R} = \mathbf{Q}_\infty$.

Let $\widehat{\mathbf{Z}}$ be the profinite completion of $\mathbf{Z}$. If M is an abelian group, let $\widehat{M} = M \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}$. For a place $v \in \Sigma_{\mathbf{Q}}$, $M_v = M \otimes_{\mathbf{Z}} \mathbf{Z}_v$.

For an algebraic group G over $\mathbf{Q}$, let Z_G be the center of G and let $[G]$ be the quotient space $G(\mathbf{Q})\backslash G(\mathbf{A})$.

For a set S, denote by $\mathbb{I}_S$ the characteristic function of S and by $\#S$ the cardinality of S.

2.2. **Algebraic representations of** GL(2)**.** Let A be an $\mathbf{Z}$-algebra. Let $A[X,Y]_n$ denote the space if two variable homogeneous polynomial of degree n over A. Suppose $n!$ is invertible in A. We define the perfect pairing $\langle \cdot, \cdot \rangle_n : A[X,Y]_n \times A[X,Y]_n \to A$ by

$$\langle X^i Y^{n-i}, X^j Y^{n-j} \rangle_n = \begin{cases} (-1)^i \binom{n}{i}^{-1}, & \text{if } i+j=n, \\ 0, & \text{if } i+j \neq n, \end{cases}$$

where $\binom{a}{b}$ is the binomial coefficient defined by

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(a-b+1)\Gamma(b+1)} \quad (a, b \in \mathbf{Z}).$$

For each polynomial $P \in A[X,Y]_n$ and each $g \in \mathrm{GL}_2(A)$, define the polynomial $g \cdot P$ to be

$$(g \cdot P)(X,Y) = P((X,Y)g).$$

Then, it is well-known that the pairing $\langle \cdot, \cdot \rangle_n$ on $A[X,Y]_n$ satisfies

$$\langle g \cdot P, g \cdot Q \rangle_n = (\det g)^n \cdot \langle P, Q \rangle_n \quad (P, Q \in A[X,Y]_n, g \in \mathrm{GL}_2(A)).$$

For $\kappa = (n+b, b) \in \mathbf{Z}^2$ with $n \in \mathbf{Z}_{\geq 0}$, let $\mathcal{L}_\kappa(A) := A[X,Y]_n$ and let $\rho_\kappa : \mathrm{GL}_2(A) \to \mathrm{Aut}_A \mathcal{L}_\kappa(A)$ be the representation given by

$$\rho_\kappa(g) P(X,Y) = P((X,Y)g) \cdot (\det g)^b.$$

Then $(\rho_\kappa, \mathcal{L}_\kappa(A))$ is the algebraic representation of $\mathrm{GL}_2(A)$ with the highest weight κ. For each non-negative integer k, we put

$$(\tau_k, \mathcal{W}_k(A)) := (\rho_{(k,-k)}, A[X,Y]_{2k}).$$

Then $(\mathcal{W}_k(A), \tau_k)$ is the algebraic representation of $\mathrm{PGL}_2(A) = \mathrm{GL}_2(A)/A^\times$.

2.3. **Representations of** GL(2) **over local fields.** If F is a local field, let $|\cdot|$ be the standard absolute value of F. Denote by $\pi(\mu, \nu)$ the principal series representation of $\mathrm{GL}_2(F)$ with characters $\mu, \nu : F^\times \to \mathbf{C}^\times$ such that $\mu\nu^{-1} \neq |\cdot|^{\pm}$ and by $\mathrm{St} \otimes (\chi \circ \det)$ the special representation of $\mathrm{GL}_2(F)$ attached to a character $\chi : F^\times \to \mathbf{C}^\times$.

The L-functions in this paper are always referred to the *complete* L-function. In particular, the Riemann zeta function $\zeta(s)$ is given by

$$\zeta(s) = \prod_v \zeta_v(s),$$

where $\zeta_\infty(s) = \pi^{-s/2}\Gamma(s/2)$ and $\zeta_p(s) = (1-p^{-s})^{-1}$. For later use, we recall the Γ-factors $\Gamma_{\mathbf{R}}(s)$ and $\Gamma_{\mathbf{C}}(s)$ which are defined as follows:

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2), \quad \Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

2.4. **Siegel modular forms of genus two.** Let GSp_4 be the algebraic group defined by

$$\mathrm{GSp}_4 = \left\{ g \in \mathrm{GL}_4 : g \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix} {}^{\mathrm{t}}g = \nu(g) \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix} \right\}$$

with the similitude character $\nu : \mathrm{GSp}_4 \to \mathbb{G}_m$. For a positive integer N, define

$$\Gamma_0^{(2)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_4(\widehat{\mathbf{Z}}) : C \equiv 0 \bmod N \right\}.$$

Define the automorphy factor $J : \mathrm{GSp}_4(\mathbf{R})^+ \times \mathfrak{H}_2 \to \mathrm{GL}_2(\mathbf{C})$ by

$$J(g, Z) = CZ + D \quad (g \in \begin{pmatrix} A & B \\ C & D \end{pmatrix}).$$

Let $\mathbf{i} := \sqrt{-1} \cdot I_2$. Let η be a quadratic Hecke character of $\mathbf{A}^\times$. A holomorphic Siegel cusp form $F : \mathrm{GSp}_4(\mathbf{A}) \to \mathcal{L}_\kappa(\mathbf{C})$ is said to be of weight κ, level $\Gamma_0^{(2)}(N)$ and type η with the trivial central character if for every $g \in \mathrm{GSp}_4(\mathbf{A})$, we have

$$F(z\gamma g u_\infty u_f) = \rho_\kappa(J(u_\infty, \mathbf{i})^{-1})\eta(\det(D))F(g),$$
$$(\gamma \in \mathrm{GSp}_4(\mathbf{Q}), u_\infty \in \mathrm{U}_2(\mathbf{R}),\, u_f = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(2)}(N)).$$

3. Asai L-functions

3.1. **Local Asai transfer.** If k is a local field, denote by W'_k the Weil-Deligne group of k (*cf.* [Tat79, (4.1.1)]) and by $\mathcal{A}(\mathrm{GL}_2(k))$ the set of isomorphism classes of admissible irreducible representations of $\mathrm{GL}_2(k)$. If $\pi \in \mathcal{A}(\mathrm{GL}_2(k))$, denote by $\varphi_\pi : W'_k \to \mathrm{GL}_2(\mathbf{C})$ a Langlands parameter of π under the local Langlands correspondence. Consider the semi-direct product $\mathcal{G} := (\mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C})) \rtimes \mathbf{Z}/2\mathbf{Z}$, where $\mathbf{Z}/2\mathbf{Z}$ acts by permuting the two factors of $\mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{C})$. Let F be an quadratic étale extension of k. Let π be an irreducible representation of $\mathrm{GL}_2(F)$. Recall that a Langlands parameter $\widetilde{\varphi}_\pi : W'_k \to \mathcal{G}$ attached to the automorphidc induction of π is defined as follows: If $F = k \oplus k$, then $\pi = \pi_1 \oplus \pi_2$ with $\pi_i \in \mathcal{A}(\mathrm{GL}_2(k))$ and define $\widetilde{\varphi}_\pi(\sigma) = (\varphi_{\pi_1}(\sigma), \varphi_{\pi_2}(\sigma), 0)$. If F is a field, then $\pi \in \mathcal{A}(\mathrm{GL}_2(F))$, and fixing a decomposition $W'_k = W'_F \sqcup W'_F c$, define $\widetilde{\varphi}_\pi(\sigma) = (\varphi_\pi(\sigma), \varphi_\pi(c^{-1}\sigma c), 0)$ if $\sigma \in W'_F$ and $\widetilde{\varphi}_\pi(\sigma) = (\varphi_\pi(\sigma c), \varphi_\pi(c^{-1}\sigma), 1)$ for $\sigma \in W'_F c$. Let $r^\pm : \mathcal{G} \to \mathrm{GL}(\mathbf{C}^2 \otimes \mathbf{C}^2)$be the four dimensional representations given by

$$r^\pm(g_1, g_2, 0)(v \otimes w) = g_1 v \otimes g_2 w; \quad r^\pm(\mathbf{1}_2, \mathbf{1}_2, 1)(v \otimes w) = \pm w \otimes v,$$

for each $v, w \in \mathbf{C}^2$. Then the local Asai transfer $\mathrm{As}^\pm(\pi)$ is defined to be the irreducible representation of $\mathrm{GL}_4(k)$ corresponding to the Weil-Deligne representation $r^\pm \circ \widetilde{\varphi}_\pi$ under the local Langlands correspondence ([Kri12, §2]).

3.2. **Asai L-functions.** Let $F/\mathbf{Q}$ be an étale quadratic extension. Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ and factorize it into the restricted tensor product $\pi = \otimes_v \pi_v$, where v runs over all places of $\mathbf{Q}$ and π_v is an irreducible admissible representation of $\mathrm{GL}_2(F_v)$. Define $\mathrm{As}^\pm(\pi) := \otimes_v \mathrm{As}^\pm(\pi_v)$, which is known to be an isobaric automorphic representation of $\mathrm{GL}_4(\mathbf{A})$ ([Kri03, Theorem 6.7]). Note that by definition, we have $\mathrm{As}^-(\pi) = \mathrm{As}^+(\pi) \otimes \tau_{F/\mathbf{Q}}$, where $\tau_{F/\mathbf{Q}}$ is the quadratic character corresponding to $F/\mathbf{Q}$. Let $L(s, \mathrm{As}^\pm(\pi_v))$ be the local L-function attached the Weil-Deligne representation $r^\pm \circ \widetilde{\varphi}_{\pi_v}$ ([Tat79, (4.1.6)]) and let $L(s, \mathrm{As}^\pm(\pi)) = \prod_v L(s, \mathrm{As}^\pm(\pi_v))$ be the automorphic L-function of $\mathrm{As}^\pm(\pi)$. For the convenience in the later application, we give the complete list of local L-functions $L(s, \mathrm{As}^+(\pi_v))$ if $v = \infty$ and if π_v is either a unramified principal series or a special representation.

Definition 3.1. (1) If $v = w\overline{w}$ is split in F, and $\pi_v = \pi_w \otimes \pi_{\overline{w}}$, then

$$L(s, \mathrm{As}^+(\pi_v)) = L(s, \pi_w \otimes \pi_{\overline{w}})$$

is the local tensor product L-function for $\pi_w \otimes \pi_{\overline{w}}$ defined in [GJ78].

(2) If v is non-split in F and $\pi_v = \pi(\mu,\nu)$ is a unramified principal series with two characters $\mu,\nu : F_v^\times \to \mathbf{C}^\times$, then

$$L(s, \mathrm{As}^+(\pi_v)) = L(s, \mu|_{\mathbf{Q}_v^\times})L(s,\mu\nu)L(s,\nu|_{\mathbf{Q}_v^\times}).$$

(3) If v is non-split and $\pi_v = \mathrm{St}\otimes(\chi\circ\det)$ is the special representation twisted by a character $\chi : F_v^\times \to \mathbf{C}^\times$, then

$$L(s, \mathrm{As}^+(\pi_v)) = L(s+1, \chi|_{\mathbf{Q}_v^\times})L(s, \tau_{F_v/\mathbf{Q}_v}\chi|_{\mathbf{Q}_v^\times}).$$

(4) For the archimedean place $v=\infty$ of $\mathbf{Q}$, we define

$$L(s, \mathrm{As}^+(\pi_\infty)) = \Gamma_{\mathbf{C}}(s+k_1+k_2+1)\Gamma_{\mathbf{C}}(s+k_1-k_2).$$

In particular, if $F = \mathbf{Q}\oplus\mathbf{Q}$, then $\pi = \pi_1\oplus\pi_2$ is a direct sum of two automorphic representations of $\mathrm{GL}_2(\mathbf{A})$ and $L(s,\mathrm{As}^+(\pi)) = L(s,\pi_1\otimes\pi_2)$.

Definition 3.2. Let c be the non-trival automorphism of $F/\mathbf{Q}$. We say π is *Galois self-dual* if the contragradient representation $\pi^\vee$ is isomorphic to the Galois conjugate π^c.

The following theorem gives the complete description of the analytic properties of Asai L-functions when π is not Galois self-dual.

Theorem 3.3. *The Asai L-function $L(s,\mathrm{As}^+(\pi))$ is meromorphically continued to the whole $\mathbf{C}$-plane with possible pole at $s=0$ or 1. Furthermore, if π is not Galois self-dual, then $L(s,\mathrm{As}^+(\pi))$ is entire and $L(s,\mathrm{As}^+(\pi))$ is non-zero for $\mathrm{Re}\, s\geq 1$ or $\mathrm{Re}\, s \leq 0$.*

Proof. This is a special case of [GS15, Theorem 4.3]. □

4. Yoshida lifts

In this section, we recall the construction of vector-valued Yoshida lifts in [HN16, §3].

4.1. **Groups.** Let D_0 be a definite quaternion algebra over $\mathbf{Q}$ of discriminant N^- and let F be a quadratic étale algebra over $\mathbf{Q}$. Let $D = D_0\otimes_{\mathbf{Q}} F$. We assume that

(split) every place dividing ∞N^- is split in F.

It follows that F is either $\mathbf{Q}\oplus\mathbf{Q}$ or a real quadratic field over $\mathbf{Q}$, and D is precisely ramified at ∞N^-. Denote by $x\mapsto x^*$ the main involution of D_0 and by $x\mapsto x^c$ the non-trivial automorphism of $F/\mathbf{Q}$, which are extended to automorphisms of D naturally. We define the four dimensional quadratic space (V,n) over $\mathbf{Q}$ by

$$V = \{x\in D : x^* = x^c\}; \quad \mathrm{n}(x) = xx^*.$$

Let H^0 be the algebraic group over $\mathbf{Q}$ given by

$$H^0(\mathbf{Q}) = (D^\times\times\mathbf{Q}^\times)/F^\times,$$

where $F^\times$ sits inside $D^\times\times\mathbf{Q}^\times$ as $(z,\mathrm{N}_{F/\mathbf{Q}}(z))$. Then H^0 acts on V via $\varrho : H^0\to \mathrm{Aut}\, V$ given by

$$\varrho(a,\alpha)(x) = \alpha^{-1}ax(a^c)^* \quad (x\in V, (a,\alpha)\in H^0).$$

This induces an identification $\varrho : H^0\simeq \mathrm{GSO}(V)$ with the similitude map given by

$$\nu(\varrho(a,\alpha)) = \alpha^{-2}\mathrm{N}_{F/\mathbf{Q}}(aa^*).$$

For $a\in D_{\mathbf{A}}^\times$, we write $\varrho(a) = \varrho(a,1)$. Put

$$H_1^0 = \left\{h\in H^0 \mid \nu(\varrho(h)) = 1\right\}\simeq \mathrm{SO}(V).$$

Remark 4.1. If $v = w_1 w_2$ is a place split in F, then $F \otimes_{\mathbf{Q}} \mathbf{Q}_v = \mathbf{Q}_v e_{w_1} \oplus \mathbf{Q}_v e_{w_2}$, where e_{w_1} and e_{w_2} are idempotents corresponding to w_1 and w_2 respectively. Let $D_{0,v} = D \otimes_{\mathbf{Q}} \mathbf{Q}_v$. For a place w_1 lying above v, in the sequel we make the identifications

$$\begin{aligned}(D_{0,v}^\times \times D_{0,v}^\times)/\mathbf{Q}_v^\times &\simeq H^0(\mathbf{Q}_v), \quad (a,d) \mapsto (ae_{w_1} + de_{w_2}, \mathrm{n}(d));\\ D_{0,v} &\simeq V \otimes_{\mathbf{Q}} \mathbf{Q}_v, \quad x \mapsto xe_{w_1} + x^* e_{w_2},\end{aligned} \tag{4.1}$$

where $\mathbf{Q}_v^\times$ sits inside $(D_{0,v}^\times \times D_{0,v}^\times)$ as (z,z). For $(a,d) \in D_{0,v}^\times \times D_{0,v}^\times$, we have $\varrho(a,d)x = axd^{-1}$ for $x \in D_{0,v}$.

4.2. **Notation for quaternion algebras.** We fix an isomorphism $\Phi_p : \mathrm{M}_2(\mathbf{Q}_p) \to D_0 \otimes \mathbf{Q}_p$ for each $p \nmid N^-\infty$ once and for all and we put $\Phi = \prod_{p \nmid N^-\infty} \Phi_p$. Let $\mathcal{O}_{D_0}$ be the maximal order of D_0 such that $\mathcal{O}_{D_0} \otimes \mathbf{Z}_p = \Phi_p(\mathrm{M}_2(\mathbf{Z}_p))$ for all $p \nmid N^-$ and let $R^0 := \mathcal{O}_{D_0} \otimes_{\mathbf{Z}} \mathcal{O}_F$ be a maximal order of D. If $\mathfrak{A}$ is an ideal of $\mathcal{O}_F$ with $(\mathfrak{A}, N^-) = 1$, denote by $R_{\mathfrak{A}}$ the standard Eichler order of D of level $\mathfrak{A}$ contained in R^0.

For any ring A, the main involution $*$ on $\mathrm{M}_2(A)$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Let $\mathbb{H}$ be the Hamilton's quaternion algebra given by

$$\mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \in \mathrm{M}_2(\mathbf{C}) \right\}.$$

The main involution $* : \mathbb{H} \to \mathbb{H}$ is given by $x \mapsto {}^{\mathrm{t}}\overline{x}$. Fix an identification $\Phi_\infty : D_{0,\infty} \cong \mathbb{H}$ such that $\Phi_\infty(x^*) = \Phi_\infty(x)^*$, which induces an embedding $\Phi_\infty : D_{0,\infty}^\times \cong \mathbb{H}^\times \hookrightarrow \mathrm{GL}_2(\mathbf{C})$.

4.3. **Weil representation on** $\mathrm{O}(V) \times \mathrm{Sp}(4)$**.** Let $(\cdot,\cdot) : V \times V \to \mathbf{Q}$ be the bilinear form defined by $(x,y) = \mathrm{n}(x+y) - \mathrm{n}(x) - \mathrm{n}(y)$. Denote by $\mathrm{GO}(V)$ the orthogonal similitude group with the similitude morphism $\nu : \mathrm{GO}(V) \to \mathbb{G}_m$. Let $\mathbf{X} = V \oplus V$. For a place v of $\mathbf{Q}$, let $V_v = V \otimes_{\mathbf{Q}} \mathbf{Q}_v$ and $\mathbf{X}_v = \mathbf{X} \otimes_{\mathbf{Q}} \mathbf{Q}_v$. Denote by $\mathcal{S}(\mathbf{X}_v)$ the space of $\mathbf{C}$-valued Bruhat-Schwartz functions on $\mathbf{X}_v$. Let $\mathcal{B}_{\omega_v} : \mathcal{S}(\mathbf{X}_v) \otimes \mathcal{S}(\mathbf{X}_v) \to \mathbf{C}$ be the Hermitian pairing given by

$$\mathcal{B}_{\omega_v}(\varphi_{1,v}, \varphi_{2,v}) = \int_{\mathbf{X}_v} \varphi_{1,v}(x_v)\overline{\varphi_{2,v}(x_v)} \mathrm{d}x_v$$

for $\varphi_{1,v}, \varphi_{2,v} \in \mathcal{S}(\mathbf{X}_v)$. Here $\mathrm{d}x_v$ is the self-dual measure on $\mathbf{X}_v$ with respect to the Fourier transform determined by ψ_v. Throughout, we consider the standard Schrödinger realization of the Weil representation $\omega_{V_v} : \mathrm{Sp}_4(\mathbf{Q}_v) \to \mathrm{Aut}_{\mathbf{C}}(\mathbf{X}_v)$, which is explicitly given in [HN16, Section 3.4]. Let $\mathcal{R}(\mathrm{GO}(V) \times \mathrm{GSp}_4)$ be the R-group

$$\mathcal{R}(\mathrm{GO}(V) \times \mathrm{GSp}_4) = \{(h,g) \in \mathrm{GO}(V) \times \mathrm{GSp}_4 \mid \nu(h) = \nu(g)\}.$$

Then the Weil representation can be extended to the R-group by

$$\begin{aligned}\omega_v : &\mathcal{R}(\mathrm{GO}(V_v) \times \mathrm{GSp}_4(\mathbf{Q}_v)) \to \mathrm{Aut}_{\mathbf{C}}\mathcal{S}(\mathbf{X}_v),\\ \omega_v(h,g)\varphi(x) = &|\nu(h)|_v^{-2} (\omega_{V_v}(g_1)\varphi)(h^{-1}x) \quad (g_1 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & \nu(g)^{-1}\mathbf{1}_2 \end{pmatrix} g).\end{aligned}$$

Let $\mathcal{S}(\mathbf{X}_{\mathbf{A}}) = \otimes'_v \mathcal{S}(\mathbf{X}_v)$ and let $\omega = \otimes_v \omega_v : \mathcal{R}(\mathrm{GO}(V)_{\mathbf{A}} \times \mathrm{GSp}_4(\mathbf{A})) \to \mathrm{Aut}_{\mathbf{C}}\mathcal{S}(\mathbf{X}_{\mathbf{A}})$.

4.4. **Representations of** $H^0(\mathbf{A})$**.** We fix $\underline{k} = (k_1, k_2)$ a pair of non-negative integers with $k_1 \geq k_2$ and let $\mathfrak{N}^+$ be a square-free product of primes ideal of $\mathcal{O}_F$ with

$$(\mathfrak{N}^+, N^- \Delta_F) = 1.$$

When $F = \mathbf{Q} \oplus \mathbf{Q}$, $\mathfrak{N}^+$ is given by a pair of square-free positive integers (N_1^+, N_2^+). Let N^+ be the square-free integer such that $N^+\mathbf{Z} = \mathfrak{N}^+ \cap \mathbf{Z}$ (so $N^+ = \mathrm{l.c.m}(N_1^+, N_2^+)$ if $F = \mathbf{Q} \oplus \mathbf{Q}$). Let $\mathfrak{N} = \mathfrak{N}^+ N^-$. Let f^{new} be a *newform* on $\mathrm{PGL}_2(\mathbf{A}_F)$ of weight $2\underline{k}+2 = (2k_1+2, 2k_2+2)$ and level $\mathfrak{N}$. Namely, f^{new} is a pair of elliptic modular newforms (f_1, f_2) of level $(\Gamma_0(N_1^+ N^-), \Gamma_0(N_2^+ N^-))$ and weight $(2k_1+2, 2k_2+2)$ if $F = \mathbf{Q} \oplus \mathbf{Q}$, while f^{new} is a Hilbert modular newform of level $\Gamma_0(\mathfrak{N})$ and weight $2\underline{k}+2$ if F is a real quadratic field. Let π be the cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbf{A}_F)$ generated by the newform f^{new}.

Let $(\tau_{\underline{k}}, \mathcal{W}_{\underline{k}}(\mathbf{C})) := (\tau_{k_1} \otimes \tau_{k_2}, \mathcal{W}_{k_1}(\mathbf{C}) \otimes_{\mathbf{C}} \mathcal{W}_{k_2}(\mathbf{C}))$ be an algebraic representation of $D^\times$ via Φ_∞ and let $\langle\, ,\,\rangle_{\mathcal{W}}$ be the pairing on $\mathcal{W}_{\underline{k}}(\mathbf{C})$ given by $\langle v_1 \otimes v_2, v_1' \otimes v_2' \rangle_{\mathcal{W}} = \langle v_1, v_1' \rangle_{2k_1} \langle v_2, v_2' \rangle_{2k_2}$, where $\langle \cdot, \cdot \rangle_{2k_i} (i = 1, 2)$ is the pairing introduced in Section 2.2. Let $D_{\mathbf{A}} = D \otimes_{\mathbf{Q}} \mathbf{A}$. For an ideal $\mathfrak{A}$ of $\mathcal{O}_F$ with $(\mathfrak{A}, N^-) = 1$, denote by $\mathcal{M}_{\underline{k}}(D_{\mathbf{A}}^\times, \mathfrak{A})$ the space of $\mathcal{W}_{\underline{k}}(\mathbf{C})$-valued modular forms on $D_{\mathbf{A}}^\times$, consisting of functions $\mathbf{f} : D_{\mathbf{A}}^\times \to \mathcal{W}_{\underline{k}}(\mathbf{C})$ such that

$$\mathbf{f}(z\gamma hu) = \tau_{\underline{k}}(h_\infty^{-1})\mathbf{f}(h_f),$$
$$(h = (h_\infty, h_f) \in D_{\mathbf{A}}^\times,\ (z, \gamma, u) \in F_{\mathbf{A}}^\times \times D^\times \times \widehat{R}_{\mathfrak{A}}^\times).$$

Hereafter, we shall view $\mathbf{f}$ as a $\mathcal{W}_{\underline{k}}(\mathbf{C})$-valued function on $Z_{H^0}(\mathbf{A})\backslash H^0(\mathbf{A})$ by the rule $\mathbf{f}(a, \alpha) := \mathbf{f}(a)$. For $u \in \mathcal{W}_{\underline{k}}(\mathbf{C})$, let $\mathbf{f}_u(h) := \langle \mathbf{f}(h), u \rangle_{\mathcal{W}}$. Then $\mathbf{f}_u$ is an automorphic form on $H^0(\mathbf{A})$.

Let π^D be the irreducible automorphic representation of $D_{\mathbf{A}}^\times$ obtained via the Jacquet-Langlands transfer of π and let $\mathcal{A}_{\pi^D}$ be the corresponding space of π^D. Then we have an identification $i : \mathcal{M}_{\underline{k}}(D_{\mathbf{A}}^\times, \mathfrak{A}) \simeq \bigoplus_\pi \mathrm{Hom}_{D_\infty^\times}(\mathcal{W}_{\underline{k}}(\mathbf{C}), \mathcal{A}_{\pi^D}^{\widehat{R}_{\mathfrak{A}}^\times})$ given by $i(\mathbf{f})(u) = \mathbf{f}_u$. In addition, to the newform f^{new}, we can associate a $\mathcal{W}_{\underline{k}}(\mathbf{C})$-valued modular form on $\mathbf{f}^\circ \in \mathcal{M}_{\underline{k}}(D_{\mathbf{A}}^\times, \mathfrak{N}^+)$, which is characterized by the property that $\mathbf{f}^\circ$ shares the same Hecke eigenvalues of f^{new} at primes not dividing $\mathfrak{N}$. Moreover, $\mathbf{f}^\circ$ is unique up to a scalar by strong multiplicity one for GL(2) and local theory of newforms.

We recall the local Atkin-Lehner involutions. If $p \nmid N^-$, then $H^0(\mathbf{Q}_p) = (\mathrm{GL}_2(F_p) \times \mathbf{Q}_p^\times)/F_p^\times$, and for each prime $\mathfrak{p}$ of $\mathcal{O}_F$ lying above p, let $\varpi_{\mathfrak{p}}$ be a uniformizer of $\mathfrak{p}$ and put

$$\eta_{\mathfrak{p}} := (\begin{pmatrix} 0 & 1 \\ -\varpi_{\mathfrak{p}} & 0 \end{pmatrix}, 1) \in H^0(\mathbf{Q}_p). \tag{4.2}$$

If $p \mid N^-$, then $D_{0,p}$ is the division algebra over $\mathbf{Q}_p$ and $p = \mathfrak{p}\mathfrak{p}^c$ is split in F. Let $\varpi_{\mathfrak{p}}^D \in D_p^\times$ such that $\mathrm{n}(\varpi_{\mathfrak{p}}^D) \in F_{\mathfrak{p}}^\times$ is a uniformizer of $\mathfrak{p}$. Put

$$\eta_{\mathfrak{p}} := (\varpi_{\mathfrak{p}}^D, 1), \quad \eta_{\mathfrak{p}^c} = (\varpi_{\mathfrak{p}^c}^D, 1) \in H^0(\mathbf{Q}_p). \tag{4.3}$$

It is well known that if $\mathfrak{p} \mid \mathfrak{N}^+$, then $\mathbf{f}^\circ$ is an eigenfunction of the right translation of $\eta_{\mathfrak{p}}$ (Atkin-Lehner involution at $\mathfrak{p}$). In other words, we have

$$\mathbf{f}^\circ(h\eta_{\mathfrak{p}}) = \varepsilon_{\mathfrak{p}} \cdot \mathbf{f}^\circ(h) \text{ for } \mathfrak{p} \mid \mathfrak{N}^+. \tag{4.4}$$

We call $\varepsilon_{\mathfrak{p}} \in \{\pm 1\}$ the Atkin-Lehner eigenvalue of $\mathbf{f}^\circ$ at $\mathfrak{p}$. Moreover, if we denote by $\varepsilon(\pi_{\mathfrak{p}})$ the local root number of the local component $\pi_{\mathfrak{p}}$ of π at $\mathfrak{p}$, then $\varepsilon_{\mathfrak{p}} = \varepsilon(\pi_{\mathfrak{p}})$ for $\mathfrak{p} \nmid N^+$ and $\varepsilon_{\mathfrak{p}} = -\varepsilon(\pi_{\mathfrak{p}})$ for $\mathfrak{p} \mid N^-$.

We next introduce the modular form $\mathbf{f}^\dagger$ obtained by applying certain level-raising operators to the newform $\mathbf{f}^\circ$. Let $\mathcal{P}$ be a finite subset of finite places of $\mathbf{Q}$ given by

$$\begin{aligned} \mathcal{P} &= \left\{\text{rational primes } p \mid p = \mathfrak{p}\mathfrak{p}^c \text{ is split in } F \text{ with } \mathfrak{p} \nmid \mathfrak{N}^+ \text{ and } \mathfrak{p}^c \mid \mathfrak{N}^+\right\} \\ &= \left\{\text{prime factors } p \text{ of } N^+ \mid p \nmid \mathfrak{N}^+\right\}. \end{aligned} \tag{4.5}$$

Define the level raising operator $\mathscr{V}_p : \mathcal{M}_{\underline{k}}(D_{\mathbf{A}}^\times, \mathfrak{N}^+) \to \mathcal{M}_{\underline{k}}(D_{\mathbf{A}}^\times, \mathfrak{N}^+\mathfrak{p})$ for each $p \in \mathcal{P}$ by

$$\mathscr{V}_p(\mathbf{f})(h) = \mathbf{f}(h) + \varepsilon_{\mathfrak{p}^c} \cdot \mathbf{f}(h\eta_{\mathfrak{p}}).$$

Let $\mathbf{f}^\dagger \in \mathcal{M}_{\underline{k}}(D_{\mathbf{A}}^\times, N^+\mathcal{O}_F)$ be the *$\mathcal{P}$-stabilized newform* defined by

$$\mathbf{f}^\dagger = \mathscr{V}_{\mathcal{P}}(\mathbf{f}^\circ), \quad \mathscr{V}_{\mathcal{P}} := \prod_{p \in \mathcal{P}} \mathscr{V}_p. \tag{4.6}$$

By definition, $\mathbf{f}^\dagger = \mathbf{f}^\circ$ is the newform if $\mathfrak{N}^+ = N^+\mathcal{O}_F (\iff \mathcal{P} = \emptyset)$.

4.5. **The test function $\varphi^\star$.** We recall a distinguished Bruhat-Schwartz function $\varphi^\star \in \mathcal{S}(\mathbf{X}_\mathbf{A})\otimes\mathcal{W}_{\underline{k}}(\mathbf{C})\otimes\mathcal{L}_\kappa(\mathbf{C})$ introduced in [HN16, Section 3.6]. Let

$$R := R_{N^+\mathcal{O}_F}$$

be the standard Eichler order of level $N^+\mathcal{O}_F$ and let

$$L := R\cap V$$

be the lattice of V determined by R. At each finite place p, $L_p = L\otimes_\mathbf{Z}\mathbf{Z}_p$ and define $\varphi^\star_p \in \mathcal{S}(\mathbf{X}_p)$ by

$$\varphi^\star_p = \mathbb{I}_{L_p\oplus L_p} \text{ the characteristic function of } L_p\oplus L_p. \tag{4.7}$$

Note that $\varphi^\star_p$ is invariant by $R_p^\times\times\mathbf{Z}_p^\times$ under the Weil representation ω_p. At the archimedean place ∞, we have identified $H^0(\mathbf{R})$ with $(\mathbb{H}^\times\times\mathbb{H}^\times)/\mathbf{R}^\times$ and V_∞ with $\mathbb{H}$ in (4.1) with respect the inclusion $F\hookrightarrow\mathbf{R}$. For $0\le\alpha\le 2k_2$, define the function $P^\alpha_{\underline{k}} : \mathbf{X}_\infty = \mathbb{H}^{\oplus 2}\to\mathcal{W}_{\underline{k}}(\mathbf{C}) = \mathbf{C}[X_1,Y_1]_{2k_1}\otimes_\mathbf{C}\mathbf{C}[X_2,Y_2]_{2k_2}$ by

$$\begin{aligned}
&P^\alpha_{\underline{k}}(\begin{pmatrix} z_1 & w_1\\ -\overline{w}_1 & \overline{z}_1\end{pmatrix},\begin{pmatrix} z_2 & w_2\\ -\overline{w}_2 & \overline{z}_2\end{pmatrix})\\
=&((z_1\overline{z}_2 + w_1\overline{w}_2 - \overline{w}_1 w_2 - \overline{z}_1 z_2)X_1Y_1 + (z_1w_2 - w_1z_2)X_1^2 + (\overline{z}_1\overline{w}_2 - \overline{z}_2\overline{w}_1)Y_1^2)^{k_1-k_2}\\
&\times(\overline{z}_1Y_1\otimes X_2 + w_1X_1\otimes X_2 - z_1X_1\otimes Y_2 + \overline{w}_1Y_1\otimes Y_2)^\alpha\\
&\times(\overline{z}_2Y_1\otimes X_2 + w_2X_1\otimes X_2 - z_2X_1\otimes Y_2 + \overline{w}_2Y_1\otimes Y_2)^{2k_2-\alpha}.
\end{aligned}\tag{4.8}$$

Then the archimedean Bruhat-Schwartz function $\varphi^\star_\infty : \mathbf{X}_\infty = \mathbb{H}^{\oplus 2}\to\mathbf{C}[X_1,Y_1]_{2k_1}\otimes_\mathbf{C}\mathbf{C}[X_2,Y_2]_{2k_2}\otimes_\mathbf{C}\mathbf{C}[X,Y]_{2k_2}$ is defined by

$$\varphi^\star_\infty(x) = e^{-2\pi(\mathrm{n}(x_1)+\mathrm{n}(x_2))}\sum_{\alpha=0}^{2k_2}P^\alpha(x_1,x_2)\cdot\binom{2k_2}{\alpha}X^\alpha Y^{2k_2-\alpha}\quad(x=(x_1,x_2)\in\mathbf{X}_\infty). \tag{4.9}$$

We note that the following identity holds:

$$(\omega_\infty(h,u)\varphi^\star_\infty)(x) = \tau_{\underline{k}}(h^{-1})\otimes\rho_\kappa({}^{\mathrm{t}}u)(\varphi^\star_\infty(x)).$$

for each $(h,u)\in H^0_1(\mathbf{R})\times\mathrm{U}_2(\mathbf{R})$ by [HN16, Lemma 3.5].

4.6. **Theta lifts from** $\mathrm{GSO}(V)$ **to** GSp_4**.** Let $\kappa := (k_1+k_2+2, k_1-k_2+2)$. For each vector-valued Bruhat-Schwartz function $\varphi\in\mathcal{S}(\mathbf{X}_\mathbf{A})\otimes\mathcal{W}_{\underline{k}}(\mathbf{C})\otimes\mathcal{L}_\kappa(\mathbf{C})$, define the theta kernel $\theta(-,-;\varphi):\mathcal{R}(\mathrm{GO}(V)_\mathbf{A}\times\mathrm{GSp}_4(\mathbf{A}))\to\mathcal{W}_{\underline{k}}(\mathbf{C})\otimes\mathcal{L}_\kappa(\mathbf{C})$ by

$$\theta(h,g;\varphi) = \sum_{x\in\mathbf{X}}\omega(h,g)\varphi(x).$$

Let GSp_4^+ be the group of elements $g\in\mathrm{GSp}_4$ with $\nu(g)\in\nu(\mathrm{GO}(V))$. Let $U_R = \prod_v U_{R_v}$ be the open-compact subgroup of $H^0(\mathbf{A})$ given by

$$\begin{aligned}
U_{R_v} =& H^0(\mathbf{R}) \text{ if } v=\infty;\quad U_{R_v} = (R_v^\times\times\mathbf{Z}_v^\times)/\mathcal{O}_{F_v}^\times \text{ if } v<\infty,\\
\mathcal{U} :=& H^0_1(\mathbf{A})\cap U_R.
\end{aligned}\tag{4.10}$$

For a vector-valued function $\mathbf{f}: H^0(\mathbf{Q})\backslash H^0(\mathbf{A})\to\mathcal{W}_{\underline{k}}(\mathbf{C})$, define the theta lift $\theta(\mathbf{f},\varphi):\mathrm{GSp}_4^+(\mathbf{Q})\backslash\mathrm{GSp}_4^+(\mathbf{A})\to\mathcal{L}_\kappa(\mathbf{C})$ by

$$\theta(\varphi,\mathbf{f})(g) = \int_{[H^0_1]}\langle\theta(hh',g;\varphi),\mathbf{f}(hh')\rangle_\mathcal{W}\mathrm{d}h\quad(\nu(h')=\nu(g)).$$

Here $\mathrm{d}h := \prod_v\mathrm{d}h_v$ is the Tamagawa measure on $H^0_1(\mathbf{A})$.We extend uniquely $\theta(\varphi,\mathbf{f})$ to a function on $\mathrm{GSp}_4(\mathbf{Q})\backslash\mathrm{GSp}_4(\mathbf{A})$ by defining $\theta(\varphi,\mathbf{f})(g)=0$ for $g\notin\mathrm{GSp}_4(\mathbf{Q})\,\mathrm{GSp}_4^+(\mathbf{A})$.

Definition 4.2. The Yoshida lift is the theta lift $\theta(\varphi^\star,\mathbf{f}^\dagger)$ attached to the Bruhat-Schwartz function $\varphi^\star := \bigotimes_v\varphi^\star_v\in\mathcal{S}(\mathbf{X}_\mathbf{A})\otimes\mathcal{W}_{\underline{k}}(\mathbf{C})\otimes\mathcal{L}_\kappa(\mathbf{C})$ and the $\mathcal{P}$-stabilized newform $\mathbf{f}^\dagger$ attached to the cuspidal automorphic representation π of $\mathrm{PGL}_2(\mathbf{A}_F)$. When $k_2=0$, $\theta(\varphi^\star,\mathbf{f}^\dagger)$ is the scalar valued Siegel modular form constructed by Yoshida [Yos80].

Proposition 4.3. *Let $N_F = N^+N^-\Delta_F$. The Yoshida lift $\theta(\varphi^\star, \mathbf{f}^\dagger)$ is a Siegel modular form of weight κ, level $\Gamma_0^{(2)}(N_F)$ and of type $\eta_{F/\mathbf{Q}}$ with the trivial central character. Moreover, $\theta(\varphi^\star, \mathbf{f}^\dagger)$ is a cusp form if π is not Galois self-dual.*

Proof. This follows directly from [HN16, §3.7, Lemma 3.2, 3.3 and 3.4]. □

5. Rallis Inner product formula of Yoshida lifts

In this section, we realize Yoshida lifts as theta lifts from $\mathrm{GO}(V)$ to GSp_4 and apply the Rallis inner product formula to calculate the inner product of the Yoshida lift $\theta(\varphi^\star, \mathbf{f}^\dagger)$.

5.1. Automorphic forms on $\mathrm{GO}(V)$.

In this subsection, we will retain the notation in §4.4. Let $H = \mathrm{GO}(V)$. Let $\mathbf{t}$ be the order two element in $H(\mathbf{Q})$ with the action $x \mapsto x^*$, $x \in V$. Let $\boldsymbol{\mu}_2 = \{1, \mathbf{t}\}$ and we regard $\boldsymbol{\mu}_2$ as the multiplicative group scheme of order 2 defined over $\mathbf{Q}$. For each $v \in \Sigma_{\mathbf{Q}}$, let $\mathbf{t}_v$ be the image of $\mathbf{t}$ in $H(\mathbf{Q}_v)$. If $\mathcal{R}$ is a subset of $\Sigma_{\mathbf{Q}}$, denote by $\mathbf{t}_{\mathcal{R}} \in \boldsymbol{\mu}_2(\mathbf{A})$ the element such that $(\mathbf{t}_{\mathcal{R}})_v = \mathbf{t}_v$ for $v \in \mathcal{R}$ and $(\mathbf{t}_{\mathcal{R}})_v = 1$ if $v \notin \mathcal{R}$. Then we have $H(\mathbf{A}) = H^0(\mathbf{A})\boldsymbol{\mu}_2(\mathbf{A})$. For $h = \varrho(a, \alpha) \in H^0(\mathbf{A}) = (D_{\mathbf{A}}^\times \times \mathbf{A}^\times)/\mathbf{A}_F^\times$, put $h^c = \varrho(a^c, \alpha)$. One verifies easily that $\mathbf{t}h\mathbf{t} = h^c$.

5.1.1. *From* $\mathrm{GSO}(V)$ *to* $\mathrm{GO}(V)$. Recall that π^D is the Jacquet-Langlands transfer of π. We have $(\pi^D, \mathcal{A}_{\pi^D}) \simeq \otimes_v(\pi_v^D, \mathcal{V}_v)$, where π_v^D is an irreducible admissible representation of $D_v^\times$ on the space $\mathcal{V}_v$. Let $\sigma = \pi^D \boxtimes \mathbf{1}$ be an automorphic representation of $H^0(\mathbf{A})$ with the space $\mathcal{A}_\sigma = \mathcal{A}_{\pi^D}$. Here $\mathbf{1}$ is the trivial representation of $\mathbf{A}^\times$. We have $\sigma \simeq \otimes_v \sigma_v$, where $\sigma_v = \pi_v^D \boxtimes \mathbf{1}$ with the same space $\mathcal{V}_v$. Let $R = R_{\mathfrak{N}^+}$ is the Eichler order of D of level $\mathfrak{N}^+$. If v is a finite place of $\mathbf{Q}$, viewing $R_v^\times = (R \otimes \mathbf{Z}_v)^\times$ as a subgroup of $H^0(\mathbf{Z}_v)$, the $R_v^\times$-invariant subspace of $\mathcal{V}_v$ is one-dimensional by the theory of newforms of irreducible representations of $D_v^\times$. We fix a non-zero vector f_v^0 in $\mathcal{V}_v^{R_v^\times}$.

Let $\sigma_v^\sharp := \mathrm{Ind}_{H^0(\mathbf{Q}_v)}^{H(\mathbf{Q}_v)} \sigma_v$ be the induced representation of $H(\mathbf{Q}_v)$. Namely, the space of $\sigma_v^\sharp$ is $\mathcal{V}_v^\sharp := \mathcal{V}_v \oplus \mathcal{V}_v$, on which $h \in H^0(\mathbf{Q}_v)$ acts by $\sigma_v^\sharp(h)(x, y) = (\sigma_v(h)x, \sigma_v(h^c)y)$ and $\sigma_v^\sharp(\mathbf{t}_v)(x, y) = (y, x)$ for all $x, y \in \mathcal{V}_v$. We define the sub-representation $\widetilde{\sigma}_v \subset \sigma_v^\sharp$ of $H(\mathbf{Q}_v)$ with the space $\widetilde{\mathcal{V}}_v \subset \mathcal{V}_v^\sharp$ as follows. Let

$$\mathfrak{S} = \{v \in \Sigma_{\mathbf{Q}} \mid \sigma_v \simeq \sigma_v^c\}.$$

(1) $v \notin \mathfrak{S}$: in this case, $\sigma_v^\sharp$ is irreducible, and we set $(\widetilde{\sigma}_v, \widetilde{\mathcal{V}}_v) := (\sigma_v^\sharp, \mathcal{V}_v^\sharp)$.
(2) $v \in \mathfrak{S}$: in this case, there exist two linear maps $\xi^\pm : \mathcal{V}_v \to \mathcal{V}_v$ such that $\xi^\pm \circ \sigma_v(h) = \sigma_v(h^c) \circ \xi^\pm$ for all $h \in H^0(\mathbf{Q}_v)$, $(\xi^\pm)^2 = \mathrm{Id}$ and $\xi^+ = (-1) \cdot \xi^-$. If v is finite, let ξ^+ be chosen so that $\xi^+(f_v^0) = f_v^0$ (this is possible as $R_v^c = R_v$ for $v \in \mathfrak{S}$), and if v is archimedean, then $\mathcal{V}_\infty = \mathcal{W}_{k_1}(\mathbf{C}) \otimes \mathcal{W}_{k_2}(\mathbf{C})$ with $k_1 = k_2$ since $\sigma_\infty \simeq \sigma_\infty^c$, let ξ^+ be the map $u_1 \otimes u_2 \mapsto u_2 \otimes u_1$. Let $\sigma_v^\pm$ be the sub-representation of $\sigma_v^\sharp$ with the space given by

$$\mathcal{V}_v^\pm = \left\{(x, \xi^\pm(x)) \in \mathcal{V}_v^\sharp \mid x \in \mathcal{V}_v\right\}.$$

Then $\sigma_v^\pm \simeq \sigma_v$ with the action $\mathbf{t}_v$ via $\xi^\pm$. We define

$$(\widetilde{\sigma}_\infty, \widetilde{\mathcal{V}}_\infty) = (\sigma_\infty^\sharp, \mathcal{V}_\infty^\sharp); \quad (\widetilde{\sigma}_v, \widetilde{\mathcal{V}}_v) = (\sigma_v^+, \mathcal{V}_v^+) \text{ if } v \text{ is finite.}$$

Let $\widehat{\sigma}$ be the automorphic representation of $H(\mathbf{A})$ whose space $\mathcal{A}_{\widehat{\sigma}}$ consisting of automorphic forms f on $H(\mathbf{A})$ such that $f|_{H^0(\mathbf{A})} \in \mathcal{A}_\sigma$. Suppose that $\sigma \not\simeq \sigma^c$. It is well known that

$$\widehat{\sigma} \simeq \bigoplus_{\delta} \left(\bigotimes_{v \in \mathfrak{S}} \sigma_v^{\delta(v)} \bigotimes_{v \notin \mathfrak{S}} \sigma_v^\sharp \right),$$

where δ runs over all maps from $\mathfrak{S}$ to $\{\pm 1\}$such that $\delta(v) = +1$ for all but finitely $v \in \mathfrak{S}$ (*cf.* [Tak09, Proposition 5.4]). In particular, there exists a unique constitute $\widetilde{\sigma} \subset \widehat{\sigma}$ with the space $\mathcal{A}_{\widetilde{\sigma}} \subset \mathcal{A}_{\widehat{\sigma}}$ such that $\widetilde{\sigma} \simeq \otimes_v \widetilde{\sigma}_v$. Let $\widetilde{\sigma}^+$ be a unique irreducible constitute of $\widetilde{\sigma}$ with the space $\mathcal{A}_{\widetilde{\sigma}^+} \subset \mathcal{A}_{\widetilde{\sigma}}$ given as follows:

(1) $\infty \notin \mathfrak{S}$: let $(\widetilde{\sigma}^+, \mathcal{A}_{\widetilde{\sigma}^+}) := (\widetilde{\sigma}, \mathcal{A}_{\widetilde{\sigma}})$;

(2) $\infty \in \mathfrak{S}$: then $(\widetilde{\sigma}, \mathcal{A}_{\widetilde{\sigma}}) = (\widetilde{\sigma}^+, \mathcal{A}_{\widetilde{\sigma}^+}) \oplus (\widetilde{\sigma}^-, \mathcal{A}_{\widetilde{\sigma}^-})$ is reducible, where

$$(\widetilde{\sigma}^\pm, \mathcal{A}_{\widetilde{\sigma}^\pm}) \simeq (\sigma_\infty^\pm \bigotimes_{v\neq\infty} \widetilde{\sigma}_v, \mathcal{V}_\infty^\pm \bigotimes_{v\neq\infty} \widetilde{\mathcal{V}}_v).$$

Remark 5.1. When $v \in \mathfrak{S}$, our choices of σ_v^+ agree with those in [Tak11, §6.1]. By [Tak11, Proposition 6.5], the local theta lifts $\theta(\sigma_v^+)$ to $\mathrm{GSp}_4(\mathbf{Q}_v)$ is non-zero, and if $v \in \mathfrak{S}$ is split in F, then $\theta(\sigma_v^-)$ is zero. In particular, the global theta lift $\theta(\widetilde{\sigma}^-)$ is zero if $\infty \in \mathfrak{S}$.

5.1.2. *Automorphic forms.* Let $\mathbf{f} \in \mathcal{M}_{\underline{k}}(D_{\mathbf{A}}^\times, \mathfrak{A})$. For $h \in H^0(\mathbf{A})$ and $u \in \mathcal{V}_\infty = \mathcal{W}_{\underline{k}}(\mathbf{C})$, put

$$\mathbf{f}_u(h) = \langle \mathbf{f}(h), u \rangle_{\mathcal{W}}.$$

For $h \in H^0(\mathbf{A})$ and $\mathbf{t}_{\mathcal{R}} \in \boldsymbol{\mu}_2(\mathbf{A})$, put

$$\widetilde{\mathbf{f}}_u(h\mathbf{t}_{\mathcal{R}}) = \begin{cases} \mathbf{f}_u(h), & \infty \notin \mathcal{R}, \\ \mathbf{f}_u(h^c), & \infty \in \mathcal{R}. \end{cases} \tag{5.1}$$

Then $\mathbf{f}_u \in \mathcal{A}_\sigma$ and $\widetilde{\mathbf{f}}_u \in \mathcal{A}_{\widetilde{\sigma}}$. For each finite place v and $f_v \in \mathcal{V}_v$, we put $\widetilde{f}_v = (f_v, f_v) \in \mathcal{V}_v^\sharp$ if $v \notin \mathfrak{S}$, and $\widetilde{f}_v = f_v \in \mathcal{V}_v^+$ if $v \in \mathfrak{S}$. We shall fix an isomorphism $j : \otimes_v \widetilde{\sigma}_v \simeq \widetilde{\sigma}$ such that

$$j((u_1, u_2) \bigotimes_{v<\infty} \widetilde{f}_v^0) = \widetilde{\mathbf{f}}^\circ_{(u_1,u_2)} := \widetilde{\mathbf{f}}^\circ_{u_1} + \widetilde{\sigma}(\mathbf{t}_\infty)\widetilde{\mathbf{f}}^\circ_{u_2} \in \mathcal{A}_{\widetilde{\sigma}}. \tag{5.2}$$

Note that if $\infty \in \mathfrak{S}$, then

$$\widetilde{\mathbf{f}}^\circ_{(u,\pm\mathbf{t}_\infty u)} \in \mathcal{A}_{\widetilde{\sigma}^\pm}, \text{ and } j(\otimes_v \mathcal{V}_v^\pm) = \mathcal{A}_{\widetilde{\sigma}^\pm}.$$

For each finite place p, put

$$f_p^\dagger := \begin{cases} f_p^0 & \text{if } p \notin \mathcal{P}, \\ f_p^0 + \varepsilon_{\mathfrak{p}^c} \cdot \sigma_p(\eta_p) f_p^0 & \text{if } p \in \mathcal{P}. \end{cases} \tag{5.3}$$

By the definition of $\mathcal{P}$-stabilized newform $\mathbf{f}^\dagger$ (4.6), we have

$$j((u_1, u_2) \bigotimes_p \widetilde{f}_p^\dagger) = \widetilde{\mathbf{f}}^\dagger_{u_1} + \widetilde{\sigma}(\mathbf{t}_\infty)\widetilde{\mathbf{f}}^\dagger_{u_2}.$$

5.1.3. *Hermitian pairings.* If v is a finite place, let $\mathcal{B}_{\sigma_v} : \mathcal{V}_v \otimes \overline{\mathcal{V}}_v \to \mathbf{C}$ be the $H^0(\mathbf{Q}_v)$-invariant pairing such that $\mathcal{B}_{\sigma_v}(f_v^0, f_v^0) = 1$. If $v = \infty$, put

$$\mathcal{J} = (\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \in \mathbb{H}^\times \times \mathbb{H}^\times / \mathbf{R}^\times = H^0(\mathbf{R}), \tag{5.4}$$

and let $\mathcal{B}_{\sigma_\infty} : \mathcal{W}_{\underline{k}}(\mathbf{C}) \otimes \mathcal{W}_{\underline{k}}(\mathbf{C}) \to \mathbf{C}$ be the pairing given by

$$\mathcal{B}_{\sigma_\infty}(u_1, u_2) = \langle u_1, \tau_{\underline{k}}(\mathcal{J})\overline{u_2} \rangle_{\mathcal{W}}.$$

Let $\mathcal{B}_{\sigma_v^\sharp} : \mathcal{V}_v^\sharp \otimes \overline{\mathcal{V}}_v^\sharp \to \mathbf{C}$ be the pairing given by

$$\mathcal{B}_{\sigma_v^\sharp}((u_1, w_1), (u_2, w_2)) := \frac{1}{2}(\mathcal{B}_{\sigma_v}(u_1, u_2) + \mathcal{B}_{\sigma_v}(w_1, w_2)).$$

Let $\mathcal{B}_{\sigma_v^\pm} := \mathcal{B}_{\sigma_v^\sharp}|_{\mathcal{V}_v^\pm}$ if $v \in \mathfrak{S}$. By definition, if v is finite, we have $\mathcal{B}_{\widetilde{\sigma}_v}(\widetilde{f}_v^0, \widetilde{f}_v^0) = 1$.

Let $\mathrm{d}\widetilde{h}$ (resp. $\mathrm{d}h_0$) be the Tamagawa measure on $Z_H(\mathbf{A})\backslash H(\mathbf{A})$ (resp. $Z_{H^0}(\mathbf{A})\backslash H^0(\mathbf{A})$). Let $\mathrm{d}\epsilon_v$ be the Haar measure on $\mu_2(\mathbf{Q}_v)$, which satisfies $\mathrm{vol}(\mu_2(\mathbf{Q}_v), \mathrm{d}\epsilon_v) = 1$ and let $\mathrm{d}\epsilon$ be the product measure $\prod_v \mathrm{d}\epsilon_v$ on $\mu_2(\mathbf{A})$. Then for each $f \in L^1(Z_H(\mathbf{A})H(\mathbf{Q})\backslash H(\mathbf{A}))$,

$$\int_{Z_H(\mathbf{A})H(\mathbf{Q})\backslash H(\mathbf{A})} f(\widetilde{h})\mathrm{d}\widetilde{h} = \int_{\mu_2(\mathbf{Q})\backslash\mu_2(\mathbf{A})} \int_{Z_{H^0}(\mathbf{A})H^0(\mathbf{Q})\backslash H^0(\mathbf{A})} f(h_0\epsilon)\mathrm{d}h_0\mathrm{d}\epsilon.$$

Define the Petersson pairing $\mathcal{B}_{\widetilde{\sigma}} : \mathcal{A}_{\widetilde{\sigma}} \otimes \overline{\mathcal{A}}_{\widetilde{\sigma}} \to \mathbf{C}$ by

$$\mathcal{B}_{\widetilde{\sigma}}(f_1, f_2) = \int_{Z_H(\mathbf{A})H(\mathbf{Q})\backslash H(\mathbf{A})} f_1(\widetilde{h})\overline{f_2(\widetilde{h})}\mathrm{d}\widetilde{h}.$$

Let $\langle\,,\,\rangle_{H^0}$ be the Hermitian pairing on $\mathcal{M}_{\underline{k}}(D_{\mathbf{A}}^\times, \mathfrak{A})$ given by

$$\langle \mathbf{f}_1, \mathbf{f}_2\rangle_{H^0} = \int_{Z_{H^0}(\mathbf{A})H^0(\mathbf{Q})\backslash H^0(\mathbf{A})} \langle \mathbf{f}_1(h_0), \tau_{\underline{k}}(\mathcal{J})\overline{\mathbf{f}_2(h_0)}\rangle_{\mathcal{W}} \mathrm{d}h_0.$$

Lemma 5.2. *We have*

(1) $\langle \mathbf{f}^\dagger, \mathbf{f}^\dagger\rangle_{H^0} = \langle \mathbf{f}^\circ, \mathbf{f}^\circ\rangle_{H^0} \cdot \prod_{p\in\mathcal{P}} \mathcal{B}_{\sigma_p}(f_p^\dagger, f_p^\dagger)$;

(2) if $\sigma \not\simeq \sigma^c$, *then*

$$\mathcal{B}_{\widetilde{\sigma}} = \frac{\langle \mathbf{f}^\circ, \mathbf{f}^\circ\rangle_{H^0}}{\dim \mathcal{W}_{\underline{k}}(\mathbf{C})} \cdot \prod_v \mathcal{B}_{\widetilde{\sigma}_v}$$

under the isomorphism $\widetilde{\sigma} \simeq \otimes_v \widetilde{\sigma}_v$ *in* (5.2).

Proof. From the Schur orthogonality relations, we see that for $u_1, u_2 \in \mathcal{V}_\infty = \mathcal{W}_{\underline{k}}(\mathbf{C})$,

$$\mathcal{B}_\sigma(\mathbf{f}^\circ_{u_1}, \mathbf{f}^\circ_{u_2}) = \frac{\langle \mathbf{f}^\circ, \mathbf{f}^\circ\rangle_{H^0}}{\dim \mathcal{W}_{\underline{k}}(\mathbf{C})} \cdot \langle u_1, \tau_{\underline{k}}(\mathcal{J})\overline{u_2}\rangle_{\mathcal{W}} = \frac{\langle \mathbf{f}^\circ, \mathbf{f}^\circ\rangle_{H^0}}{\dim \mathcal{W}_{\underline{k}}(\mathbf{C})} \cdot \mathcal{B}_{\sigma_\infty}(u_1, u_2).$$

On the other hand, note that for $\widetilde{f}_1, \widetilde{f}_2 \in \mathcal{A}_{\widetilde{\sigma}}$,

$$\mathcal{B}_\sigma(\widetilde{f}_1|_{H^0(\mathbf{A})}, \widetilde{f}_2|_{H^0(\mathbf{A})}) = 2\mathcal{B}_{\widetilde{\sigma}}(\widetilde{f}_1, \widetilde{f}_2)$$

by [GI11, Lemma 2.1] and that for $f_1, f_2 \in \mathcal{A}_\sigma$,

$$\int_{Z_{H^0}(\mathbf{A})H^0(\mathbf{Q})\backslash H^0(\mathbf{A})} f_1(h_0)\overline{f_2(h_0^c)}\mathrm{d}h_0 = 0$$

since $\sigma^\vee = \sigma \not\simeq \sigma^c$. We thus find that

$$\begin{aligned}\mathcal{B}_{\widetilde{\sigma}}(\widetilde{\mathbf{f}}^\circ_{(u_1,w_1)}, \widetilde{\mathbf{f}}^\circ_{(u_2,w_2)}) =& \frac{1}{2}\int_{Z_{H^0}(\mathbf{A})H^0(\mathbf{Q})\backslash H^0(\mathbf{A})} (\mathbf{f}_{u_1}(h_0) + \mathbf{f}^\circ_{w_1}(h_0^c))(\mathbf{f}^\circ_{u_2}(h_0) + \mathbf{f}^\circ_{w_2}(h_0^c))\mathrm{d}h_0 \\ =& \frac{1}{2}(\mathcal{B}_\sigma(\mathbf{f}^\circ_{u_1}, \mathbf{f}^\circ_{u_2}) + \mathcal{B}_\sigma(\mathbf{f}^\circ_{w_1}, \mathbf{f}^\circ_{w_2})) \\ =& \frac{\langle \mathbf{f}^\circ, \mathbf{f}^\circ\rangle_{H^0}}{\dim \mathcal{W}_{\underline{k}}(\mathbf{C})} \cdot \mathcal{B}_{\widetilde{\sigma}_\infty}((u_1, w_1), (u_2, w_2)).\end{aligned} \tag{5.5}$$

If $\infty \notin \mathfrak{S}$, then $\widetilde{\sigma}$ is irreducible, and we can write $\mathcal{B}_{\widetilde{\sigma}} = C_0 \cdot \prod_v \mathcal{B}_{\widetilde{\sigma}_v}$ for some constant C_0, while if $\infty \in \mathfrak{S}$, then $\widetilde{\sigma} = \widetilde{\sigma}^+ \oplus \widetilde{\sigma}^-$ is reducible and

$$\mathcal{B}_{\widetilde{\sigma}} = (C_+\mathcal{B}_{\sigma_\infty^+} + C_-\mathcal{B}_{\sigma_\infty^-}) \prod_{v<\infty} \mathcal{B}_{\widetilde{\sigma}_v}$$

for some constants $C_\pm$. In either of the cases, (5.5) implies that $C_0 = C_\pm = \frac{\langle \mathbf{f}^\circ, \mathbf{f}^\circ\rangle_{H^0}}{\dim \mathcal{W}_{\underline{k}}(\mathbf{C})}$, and the lemma follows. □

5.2. **Rallis inner product formula.** From here to the end of this paper, we always assume that the automorphic representation π of $\mathrm{PGL}_2(\mathbf{A}_F)$ introduced in Section 4.4 is not Galois self-dual.

Let $G = \mathrm{GSp}_4$. For $g \in G(\mathbf{A})^+$, $\varphi \in \mathcal{S}(\mathbf{X}_\mathbf{A})$ and $f \in \mathcal{A}_{\widetilde{\sigma}}$, choose $h \in H(\mathbf{A})$ such that $\nu(g) = \nu(h)$ and put

$$\theta(\varphi, f)(g) = \int_{[H_1]} \theta(h_1 h, g; \varphi) f(h_1 h)\mathrm{d}h_1,$$

where $H_1 = \mathrm{O}(V)$ and $\mathrm{d}h_1 = \prod_v \mathrm{d}h_{1,v}$ is the Tamagawa measure of $H_1(\mathbf{A})$ such that $f \in L^1(H_1(\mathbf{Q}_v))$, we have

$$\int_{H_1(\mathbf{Q}_v)} f(h_{1,v})\mathrm{d}h_{1,v} = \int_{\boldsymbol{\mu}_2(\mathbf{Q}_v)}\int_{H_1^0(\mathbf{Q}_v)} f(h_v\epsilon_v)\mathrm{d}h_v\mathrm{d}\epsilon_v. \tag{5.6}$$

Proposition 5.3 (Rallis inner product formula)**.** *Let* $\varphi_1 = \otimes_v \varphi_{1,v}, \varphi_2 = \otimes\varphi_{2,v} \in \mathcal{S}(\mathbf{X}_{\mathbf{A}}) = \otimes_v \mathcal{S}(\mathbf{X}_v)$ *and* $f_1 = \otimes_v f_{1,v},\ f_2 = \otimes_v f_{2,v} \in \mathcal{A}_{\widetilde{\sigma}^+} \simeq \otimes_v \widetilde{\mathcal{V}}_v^+$*. Then*

$$\begin{aligned}\langle \theta(\varphi_1, f_1), \theta(\varphi_2, f_2)\rangle :=& \int_{Z_G(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} \theta(\varphi_1, f_1)(g)\overline{\theta(\varphi_2, f_2)(g)}\mathrm{d}g \\ =& \frac{\langle \mathbf{f}^\circ, \mathbf{f}^\circ\rangle_{H^0}}{\dim \mathcal{W}_{\underline{k}}(\mathbf{C})} \cdot \frac{L(1, \mathrm{As}^+(\pi))}{\zeta(2)\zeta(4)} \prod_v \mathcal{Z}_v^*(\varphi_{1,v}, \varphi_{2,v}, f_{1,v}, f_{2,v}),\end{aligned}$$

where

$$\mathcal{Z}_v^*(\varphi_{1,v}, \varphi_{2,v}, f_{1,v}, f_{2,v}) = \frac{\zeta_v(2)\zeta_v(4)}{L(1, \mathrm{As}^+(\pi_v))} \int_{H_1(\mathbf{Q}_v)} \mathcal{B}_{\omega_v}(\omega_v(h_{1,v})\varphi_{1,v}, \varphi_{2,v})\mathcal{B}_{\widetilde{\sigma}_v}(\widetilde{\sigma}_v(h_{1,v})f_{1,v}, f_{2,v})\mathrm{d}h_{1,v}.$$

Proof. This is a special case of the Rallis inner product formula proved in [GQT14, Proposition 11.2, Theorem 11.3] (*cf.* [GI11, Lemma 7.11]) for $H(V_r) = \mathrm{Sp}_4$ and $G(U_n) = \mathrm{O}(V)$. Apply $n = m = 4, r = 2, \epsilon_0 = -1$ in the notation [GQT14]. The non-vanishing of $L(1, \mathrm{As}^+(\pi))$ follows from Theorem 3.3. The local integrals are absolutely convergent by [GI11, Lemma 7.7]. □

Define $\langle\,,\,\rangle_{\mathcal{L}} : \mathcal{L}_\kappa(\mathbf{C})\otimes\mathcal{L}_\kappa(\mathbf{C}) \to \mathbf{C}$ to be the pairing $\langle\,,\,\rangle_{2k_2}$ introduced in Section 2.2. For vector-valued Siegel cusp forms $F_1, F_2 : G(\mathbf{A}) \to \mathcal{L}_\kappa(\mathbf{C})$, we define the Hermitian pairing

$$\begin{aligned}(F_1, F_2)_G =& \int_{Z_G(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} \langle F_1(g), \overline{F_2(g)}\rangle_{\mathcal{L}}\mathrm{d}g \\ =& \int_{Z_G(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} \langle\!\langle F_1(g), F_2(g)\rangle\!\rangle \mathrm{d}g,\end{aligned}$$

where $\langle\!\langle\,,\,\rangle\!\rangle : \mathcal{L}_\kappa(\mathbf{C}) \otimes \mathcal{L}_\kappa(\mathbf{C}) \to \mathbf{C}$ is the $\mathrm{SU}_2(\mathbf{R})$-invariant Hermitian pairing given by

$$\langle\!\langle v_1, v_2\rangle\!\rangle := \int_{\mathrm{SU}_2(\mathbf{R})} \langle \rho_\lambda(u)v_1, \overline{\rho_\lambda(u)v_2}\rangle_{\mathcal{L}}\mathrm{d}^*u, \tag{5.7}$$

where d^*u is the Haar measure on $\mathrm{SU}_2(\mathbf{R})$ with $\mathrm{vol}(\mathrm{SU}_2(\mathbf{R})) = 1$. Denote the pairing $\langle\,,\,\rangle_{\mathcal{W}} \otimes \langle\,,\,\rangle_{\mathcal{L}}$ on $\mathcal{W}(\mathbf{C}) \otimes \mathcal{L}(\mathbf{C})$ by $\langle\,,\,\rangle_{\mathcal{W}\otimes\mathcal{L}}$. To apply Rallis inner product formula to our case, we define local zeta integrals $\mathcal{I}(\varphi^\star_\infty)$ and $\mathcal{Z}_p(\varphi^\star_p, f^\dagger_p)$ by

$$\mathcal{I}(\varphi^\star_\infty) = \int_{\mathbf{X}_\infty} \langle \varphi^\star_\infty(x), \varphi^\star_\infty(x)\rangle_{\mathcal{W}\otimes\mathcal{L}}\,\mathrm{d}x, \tag{5.8}$$

$$\mathcal{Z}_p(\varphi^\star_p, f^\dagger_p) = \int_{H^0_1(\mathbf{Q}_p)} \mathcal{B}_{\omega_p}(\omega_p(h_p)\varphi^\star_p, \varphi^\star_p)\mathcal{B}_{\sigma_p}(\sigma_p(h_p)f^\dagger_p, f^\dagger_p)\,\mathrm{d}h_p. \tag{5.9}$$

Proposition 5.4. *We have*

$$\frac{(\theta(\varphi^\star, \mathbf{f}^\dagger), \theta(\varphi^\star, \mathbf{f}^\dagger))_G}{\langle \mathbf{f}^\circ, \mathbf{f}^\circ\rangle_{H^0}} = \frac{\mathrm{vol}(H^0_1(\mathbf{R}))}{(\dim \mathcal{W}_{\underline{k}}(\mathbf{C}))^2} \cdot \frac{L(1, \mathrm{As}^+(\pi))}{\zeta(2)\zeta(4)} \cdot \mathcal{I}^*(\varphi^\star_\infty) \cdot \prod_{p<\infty} \mathcal{Z}^*_p(\varphi^\star_p, f^\dagger_p),$$

where

$$\mathcal{I}^*(\varphi^\star_\infty) = \frac{\zeta_\infty(2)\zeta_\infty(4)}{L(1, \mathrm{As}^+(\pi_\infty))} \cdot \mathcal{I}(\varphi^\star_\infty); \quad \mathcal{Z}^*_p(\varphi^\star_p, f^\dagger_p) = \frac{\zeta_p(2)\zeta_p(4)}{L(1, \mathrm{As}^+(\pi_p))} \cdot \mathcal{Z}_p(\varphi^\star_p, f^\dagger_p).$$

Proof. We begin with some notation. Define the set

$$\mathbf{B} := \{(i, j) \mid 0 \le i \le 2k_1,\ 0 \le j \le 2k_2\}.$$

Let $\{\mathbf{v}_I\}_{I\in\mathbf{B}}$ be the standard basis of $\mathcal{W}_{\underline{k}}(\mathbf{C})$ given by

$$\mathbf{v}_I := X_1^i Y_1^{2k_1-i} \otimes X_2^j Y_2^{2k_2-j} \text{ if } I = (i, j).$$

Recall the pairing $\langle\,,\,\rangle_{\mathcal{W}}$ on $\mathcal{W}_{\underline{k}}(\mathbf{C})$ is introduced in Section 4.4. Then the corresponding dual basis $\{\mathbf{v}_I^*\}_{I\in\mathbf{B}}$ with respect to $\langle\,,\,\rangle_{\mathcal{W}}$ is given by $\mathbf{v}_I^* := \mathbf{v}_{2\underline{k}-I}\cdot\binom{2k_1}{i}\binom{2k_2}{j}(-1)^{i+j}$. Write

$$\varphi_\infty^\star(x) = \sum_{\alpha=0}^{2k_2}\varphi_\infty^\alpha(x)\binom{2k_2}{\alpha}X^\alpha Y^{2k_2-\alpha},$$
$$\varphi_\infty^\alpha(x) = \sum_{I\in\mathbf{B}}\varphi_{I,\infty}^\alpha(x)\mathbf{v}_I.$$

Then $\varphi_{I,\infty}^\alpha(x) = \langle\varphi_\infty^\alpha(x),\mathbf{v}_I^*\rangle_{\mathcal{W}} = (-1)^\alpha\cdot\langle\varphi_\infty^\star(x),\mathbf{v}_I^*\otimes X^{2k_2-\alpha}Y^\alpha\rangle_{\mathcal{W}\otimes\mathcal{L}}$. Put

$$\varphi_I^\alpha = \varphi_{I,\infty}^\alpha\bigotimes_{v<\infty}\varphi_v^\star \in \mathcal{S}(\mathbf{X}_{\mathbf{A}}).$$

For each $I\in\mathbf{B}$, put

$$\mathscr{F}_{\mathbf{v}_I} = \prod_{p\in\mathcal{P}}(1+\varepsilon_{\mathfrak{p}^c}\cdot\sigma_p(\eta_p))\widetilde{\mathbf{f}}^\circ_{\mathbf{v}_I}\in\mathcal{A}_{\widetilde{\sigma}},$$

where $\widetilde{\mathbf{f}}^\circ_{\mathbf{v}_I}\in\mathcal{A}_{\widetilde{\sigma}}$ is defined as in (5.1). Then one checks that (i) $\widetilde{\sigma}(\mathbf{t}_v)\mathscr{F}_{\mathbf{v}_I} = \mathscr{F}_{\mathbf{v}_I}$ for any finite place v by (5.1) and (ii) $\mathscr{F}_{\mathbf{v}_I}|_{H^0(\mathbf{A})} = \mathbf{f}^\dagger_{\mathbf{v}_I}(=\langle\mathbf{f}^\dagger,\mathbf{v}_I\rangle_{\mathcal{L}})$. From (5.6) and the fact that $\omega_v(\mathbf{t}_v)\varphi_v^\star = \varphi_v^\star$ for every finite place v, we can deduce that

$$\theta(\varphi_I^\alpha,\mathscr{F}_{\mathbf{v}_I}) = 2^{-1}\theta(\varphi_I^\alpha,\mathscr{F}_{\mathbf{v}_I}|_{H^0(\mathbf{A})}) = 2^{-1}\theta(\varphi_I^\alpha,\mathbf{f}^\dagger_{\mathbf{v}_I}). \tag{5.10}$$

It follows that

$$\theta(\varphi^\star,\mathbf{f}^\dagger) = 2\cdot\sum_{\alpha=0}^{2k_2}\theta(\varphi_I^\alpha,\mathscr{F}_{\mathbf{v}_I})X^\alpha Y^{2k_2-\alpha}\binom{2k_2}{\alpha},$$

and hence, for the pairing $\langle\;,\;\rangle$ in Proposition 5.3, we have

$$(\theta(\varphi^\star,\mathbf{f}^\dagger),\theta(\varphi^\star,\mathbf{f}^\dagger))_G = 4\cdot\sum_{\alpha=0}^{2k_2}\sum_{I,J\in\mathbf{B}}\langle\theta(\varphi_I^\alpha,\mathscr{F}_{\mathbf{v}_I}),\theta(\varphi_J^{2k_2-\alpha},\mathscr{F}_{\mathbf{v}_J})\rangle(-1)^\alpha\binom{2k_2}{\alpha}. \tag{5.11}$$

In the case $\infty\notin\mathfrak{S}$, the local vector in $\widetilde{\sigma}_\infty$ corresponding to $\mathscr{F}_{\mathbf{v}_I}$ is $(\mathbf{v}_I,0)$ by the fixed isomorphism $j:\otimes_v\widetilde{\sigma}_v\simeq\widetilde{\sigma}$ given in (5.2). To emphasis this correspondence, we also write $\mathscr{F}_{\mathbf{v}_I}$ as $\mathscr{F}_{(\mathbf{v}_I,0)}$ according to the notion in (5.2).

In the case $\infty\in\mathfrak{S}$, we can decompose $\mathscr{F}_{\mathbf{v}_I} = \mathscr{F}^+_{\mathbf{v}_I}+\mathscr{F}^-_{\mathbf{v}_I}$, where

$$\mathscr{F}^\pm_{\mathbf{v}_I} = \frac{1}{2}(\mathscr{F}_{\mathbf{v}_I}\pm\widetilde{\sigma}(\mathbf{t}_\infty)\mathscr{F}_{\mathbf{v}_{I^{\rm sw}}})\in\widetilde{\sigma}^\pm.$$

Here $I^{\rm sw}=(j,i)$ for $I=(i,j)$. The global lift $\theta(\widetilde{\sigma}^-)=0$ by Remark 5.1, so we have

$$\theta(\varphi_I^\alpha,\mathscr{F}_{\mathbf{v}_I}) = \theta(\varphi_I^\alpha,\mathscr{F}^+_{\mathbf{v}_I}).$$

The fixed isomorphism $j:\otimes_v\widetilde{\sigma}_v\simeq\widetilde{\sigma}$ given in (5.2) shows that the local vector in $\widetilde{\sigma}_\infty$ corresponding $\mathscr{F}^+_{\mathbf{v}_I}$ is given by $2^{-1}(\mathbf{v}_I,\mathbf{v}_{I^{\rm sw}})$.

Given $0\le\alpha,\beta\le 2k_2$, we consider

$$\langle\theta(\varphi_I^\alpha,\mathscr{F}_{\mathbf{v}_I}),\theta(\varphi_J^\beta,\mathscr{F}_{\mathbf{v}_J})\rangle = \begin{cases}\langle\theta(\varphi_I^\alpha,\mathscr{F}_{(\mathbf{v}_I,0)}),\theta(\varphi_J^\beta,\mathscr{F}_{(\mathbf{v}_J,0)})\rangle & \text{if }\infty\notin\mathfrak{S},\\ \langle\theta(\varphi_I^\alpha,\mathscr{F}^+_{\mathbf{v}_I}),\theta(\varphi_J^\beta,\mathscr{F}^+_{\mathbf{v}_J})\rangle & \text{if }\infty\in\mathfrak{S}.\end{cases}$$

By Rallis inner product formula (Proposition 5.3), we have

$$\langle\theta(\varphi_I^\alpha,\mathscr{F}_{\mathbf{v}_I}),\theta(\varphi_J^\beta,\mathscr{F}_{\mathbf{v}_J})\rangle = \frac{\langle\mathbf{f}^\circ,\mathbf{f}^\circ\rangle_{H^0}}{\dim\mathcal{W}_{\underline{k}}(\mathbf{C})}\cdot\frac{L(1,\mathrm{As}^+(\pi))}{\zeta(2)\zeta(4)}\cdot\left(\frac{\zeta_\infty(2)\zeta_\infty(4)}{L(1,\mathrm{As}^+(\pi_\infty))}\widetilde{\mathcal{Z}}_{I,J}\right)\prod_{p<\infty}\widetilde{\mathcal{Z}}_p^*, \tag{5.12}$$

where $\widetilde{\mathcal{Z}}_{I,J}$ and $\widetilde{\mathcal{Z}}_p$ are local zeta integrals defined by

(5.13)

$$\widetilde{\mathcal{Z}}_{I,J}=\begin{cases}\int_{H_1(\mathbf{R})}\mathcal{B}_{\omega_\infty}(\omega_\infty(h_{1,\infty})\varphi_I^\alpha,\varphi_J^\beta)\mathcal{B}_{\widetilde{\sigma}_\infty}(\widetilde{\sigma}_\infty(h_{1,\infty})(\mathbf{v}_I,0),(\mathbf{v}_J,0))\mathrm{d}h_{1,\infty} & \text{if } \infty\notin\mathfrak{S},\\ 4^{-1}\int_{H_1(\mathbf{R})}\mathcal{B}_{\omega_\infty}(\omega_\infty(h_{1,\infty})\varphi_I^\alpha,\varphi_J^\beta)\mathcal{B}_{\sigma_\infty^+}(\sigma_\infty^+(h_{1,\infty})(\mathbf{v}_I,\mathbf{v}_{I^{\rm sw}}),(\mathbf{v}_J,\mathbf{v}_{J^{\rm sw}}))\mathrm{d}h_{1,\infty} & \text{if } \infty\in\mathfrak{S},\end{cases}$$

$$\widetilde{\mathcal{Z}}_p^* =\frac{\zeta_p(2)\zeta_p(4)}{L(1,\mathrm{As}^+(\pi_p))}\int_{H_1(\mathbf{Q}_p)}\mathcal{B}_{\omega_p}(\omega(h_{1,p})\varphi_p^\star,\varphi_p^\star)\mathcal{B}_{\widetilde{\sigma}_p}(\widetilde{\sigma}_p(h_{1,p})\widetilde{f}_p^\dagger,\widetilde{f}_p^\dagger)\mathrm{d}h_{1,p}\quad \text{if } p<\infty.$$

For any finite place p, we have

$$\omega_p(\mathbf{t}_p)\varphi_p^\star=\varphi_p^\star,\quad \widetilde{\sigma}_p(\mathbf{t}_p)\widetilde{f}_p^\dagger=\widetilde{f}_p^\dagger,$$

and hence the local zeta integral $\widetilde{\mathcal{Z}}_p$ equals

(5.14)

$$\begin{aligned}&\frac{1}{2}\int_{H_1^0(\mathbf{Q}_p)}\mathcal{B}_{\omega_p}(\omega_p(h_p)\varphi_p^\star,\varphi_p^\star)\mathcal{B}_{\widetilde{\sigma}_p}(\widetilde{\sigma}_p(h_p)\widetilde{f}_p^\dagger,\widetilde{f}_p^\dagger)+\mathcal{B}_{\omega_p}(\omega_p(h_p\mathbf{t}_p)\varphi_p^\star,\varphi_p^\star)\cdot\mathcal{B}_{\widetilde{\sigma}_p}(\widetilde{\sigma}_p(h_p\mathbf{t}_p)\widetilde{f}_p^\dagger,\widetilde{f}_p^\dagger)\mathrm{d}h_p\\ =&\int_{H_1^0(\mathbf{Q}_p)}\mathcal{B}_{\omega_p}(\omega_v(h_p)\varphi_p^\star,\varphi_p^\star)\mathcal{B}_{\sigma_p}(\sigma_v(h_p)f_p^\dagger,f_p^\dagger)\mathrm{d}h_v=\mathcal{Z}_p.\end{aligned}$$

To compute the archimedean local zeta integral $\widetilde{\mathcal{Z}}_{I,J}$, we put

$$\mathcal{Z}_{I,J}:=\int_{H_1^0(\mathbf{R})}\mathcal{B}_{\omega_\infty}(\omega_\infty(h_\infty)\varphi_I^\alpha,\varphi_J^\beta)\mathcal{B}_{\sigma_\infty}(\sigma_\infty(h_\infty)\mathbf{v}_I,\mathbf{v}_J)\mathrm{d}h_\infty.$$

Assume that $\infty\notin\mathfrak{S}$. By the definition of $\mathcal{B}_{\widetilde{\sigma}_\infty}$, we find that

$$\begin{aligned}\mathcal{B}_{\widetilde{\sigma}_\infty}(\widetilde{\sigma}_\infty(h_\infty)(\mathbf{v}_I,0),(\mathbf{v}_J,0)) =&\frac{1}{2}\mathcal{B}_{\sigma_\infty}(\sigma_\infty(h_\infty)\mathbf{v}_I,\mathbf{v}_J),\\ \mathcal{B}_{\widetilde{\sigma}_\infty}(\widetilde{\sigma}_\infty(h_\infty\mathbf{t}_\infty)(\mathbf{v}_I,0),(\mathbf{v}_J,0)) =&\mathcal{B}_{\widetilde{\sigma}_\infty}(\widetilde{\sigma}_\infty(h_\infty)(0,\mathbf{v}_I),(\mathbf{v}_J,0))=0\end{aligned}$$

for each $h_\infty\in H_1^0(\mathbf{R})$. Since $\mathrm{vol}(\mu_2(\mathbf{R}),\mathrm{d}\epsilon_\infty)=1$, we obtain

$$\widetilde{\mathcal{Z}}_{I,J}=\frac{1}{2}\int_{H_1^0(\mathbf{R})}\mathcal{B}_{\omega_\infty}(\omega_\infty(h_\infty)\varphi_I^\alpha,\varphi_J^\beta)\frac{1}{2}\mathcal{B}_{\sigma_\infty}(\sigma_\infty(h_\infty)\mathbf{v}_I,\mathbf{v}_J)\mathrm{d}h_\infty=4^{-1}\mathcal{Z}_{I,J}.$$

If $\infty\in\mathfrak{S}$, then $k_1=k_2$ and $\omega_\infty(\mathbf{t}_\infty)\varphi_I^\alpha=\varphi_{I^{\rm sw}}^\alpha$. By the definition of $\mathcal{B}_{\sigma_\infty^+}$, we have

$$\mathcal{B}_{\sigma_\infty^+}((\mathbf{v}_I,\mathbf{v}_{I^{\rm sw}}),(\mathbf{v}_J,\mathbf{v}_{J^{\rm sw}}))=\frac{1}{2}\left\{\mathcal{B}_{\sigma_\infty}(\mathbf{v}_I,\mathbf{v}_J)+\mathcal{B}_{\sigma_\infty}(\mathbf{v}_{I^{\rm sw}},\mathbf{v}_{J^{\rm sw}})\right\}=\mathcal{B}_{\sigma_\infty}(\mathbf{v}_I,\mathbf{v}_J).$$

By using $\mathrm{vol}(\mu_2(\mathbf{R}),\mathrm{d}\epsilon_\infty)=1$ again, we obtain

$$\begin{aligned}\widetilde{\mathcal{Z}}_{I,J} =&8^{-1}\int_{H_1^0(\mathbf{R})}\mathcal{B}_{\omega_\infty}(\omega_\infty(h_\infty)\varphi_I^\alpha,\varphi_J^\beta)\mathcal{B}_{\sigma_\infty^+}(\sigma_\infty^+(h_\infty)(\mathbf{v}_I,\mathbf{v}_{I^{\rm sw}}),(\mathbf{v}_J,\mathbf{v}_{J^{\rm sw}}))\\ &+\mathcal{B}_{\omega_\infty}(\omega_\infty(h_\infty\mathbf{t}_\infty)\varphi_I^\alpha,\varphi_J^\beta)\mathcal{B}_{\sigma_\infty^+}(\sigma_\infty^+(h_\infty\mathbf{t}_\infty)(\mathbf{v}_I,\mathbf{v}_{I^{\rm sw}}),(\mathbf{v}_J,\mathbf{v}_{J^{\rm sw}}))\mathrm{d}h_\infty\\ =&8^{-1}\int_{H_1^0(\mathbf{R})}\mathcal{B}_{\omega_\infty}(\omega_\infty(h_\infty)\varphi_I^\alpha,\varphi_J^\beta)\mathcal{B}_{\sigma_\infty^+}(\sigma_\infty^+(h_\infty)(\mathbf{v}_I,\mathbf{v}_{I^{\rm sw}}),(\mathbf{v}_J,\mathbf{v}_{J^{\rm sw}}))\\ &+\mathcal{B}_{\omega_\infty}(\omega_\infty(h_\infty)\varphi_{I^{\rm sw}}^\alpha,\varphi_J^\beta)\mathcal{B}_{\sigma_\infty^+}(\sigma_\infty^+(h_\infty)(\mathbf{v}_{I^{\rm sw}},\mathbf{v}_I),(\mathbf{v}_J,\mathbf{v}_{J^{\rm sw}}))\mathrm{d}h_\infty\\ =&8^{-1}\int_{H_1^0(\mathbf{R})}\mathcal{B}_{\omega_\infty}(\omega_\infty(h_\infty)\varphi_I^\alpha,\varphi_J^\beta)\mathcal{B}_{\sigma_\infty}(\sigma_\infty(h_\infty)\mathbf{v}_I,\mathbf{v}_J)\\ &+\mathcal{B}_{\omega_\infty}(\omega_\infty(h_\infty)\varphi_{I^{\rm sw}}^\alpha,\varphi_J^\beta)\mathcal{B}_{\sigma_\infty}(\sigma_\infty(h_\infty)\mathbf{v}_{I^{\rm sw}},\mathbf{v}_J)\mathrm{d}h_\infty\\ =&8^{-1}(\mathcal{Z}_{I,J}+\mathcal{Z}_{I^{\rm sw},J}).\end{aligned}$$

To simply $\mathcal{Z}_{I,J}$, we note that by the definition of $\varphi_\infty^\star$ we have

$$\overline{\varphi_\infty^\star(x)}=\tau_{\underline{k}}(\mathcal{J})\varphi_\infty^\star(x).$$

This implies that $\overline{\varphi^{\alpha}_{I,\infty}(x)}=(-1)^I\varphi^{\alpha}_{2\underline{k}-I,\infty}(x)$. We have

$$\begin{aligned}\mathcal{Z}_{I,J}&=\int_{H_1^0(\mathbf{R})}\int_{\mathbf{X}_\infty}\langle\varphi^{\alpha}_{\infty}(x),\tau_{\underline{k}}(h_\infty)\mathbf{v}_I^*\rangle_{\mathcal{W}}\cdot\overline{\varphi^{\beta}_{J,\infty}(x)}\cdot\langle\tau_{\underline{k}}(h_\infty)\mathbf{v}_I,\mathbf{v}_{2\underline{k}-J}\rangle_{\mathcal{W}}\cdot(-1)^J\mathrm{d}x\mathrm{d}h_\infty\\&=\frac{\langle\mathbf{v}_I^*,\mathbf{v}_I\rangle_{\underline{k}}\operatorname{vol}(H_1^0(\mathbf{R}))}{\dim\mathcal{W}_{\underline{k}}(\mathbf{C})}\cdot\int_{\mathbf{X}_\infty}\langle\varphi^{\alpha}_{\infty}(x),\mathbf{v}_{2\underline{k}-J}\rangle_{\mathcal{W}}\cdot\overline{\varphi^{\beta}_{J,\infty}(x)}(-1)^J\mathrm{d}x\\&=\frac{\operatorname{vol}(H_1(\mathbf{R}))}{\dim\mathcal{W}_{\underline{k}}(\mathbf{C})}\cdot\int_{\mathbf{X}_\infty}\langle\varphi^{\alpha}_{\infty}(x),\mathbf{v}_{2\underline{k}-J}\rangle_{\mathcal{W}}\cdot\varphi^{\beta}_{2\underline{k}-J,\infty}(x)\mathrm{d}x.\end{aligned}$$

In particular, $\mathcal{Z}_{I,J}$ is independent of I. Therefore, we obtain

$$\widetilde{\mathcal{Z}}_{I,J}=4^{-1}\mathcal{Z}_{I,J}=\frac{\operatorname{vol}(H_1(\mathbf{R}))}{4\dim\mathcal{W}_{\underline{k}}(\mathbf{C})}\cdot\int_{\mathbf{X}_\infty}\langle\varphi^{\alpha}_{\infty}(x),\mathbf{v}_{2\underline{k}-J}\rangle_{\mathcal{W}}\cdot\varphi^{\beta}_{2\underline{k}-J,\infty}(x)\mathrm{d}x.$$

$$\begin{aligned}\sum_{\alpha=0}^{2k_2}\sum_{I,J\in\mathbf{B}}\mathcal{Z}_{I,J}\cdot(-1)^{\alpha}\binom{2k_2}{\alpha}&=\sum_{\alpha=0}^{2k_2}\operatorname{vol}(H_1^0(\mathbf{R}))\cdot\int_{\mathbf{X}_\infty}\sum_{J\in\mathbf{B}}\langle\varphi^{\alpha}_{\infty}(x),\mathbf{v}_{2\underline{k}-J}\rangle_{\mathcal{W}}\cdot\varphi^{2k_2-\alpha}_{2\underline{k}-J,\infty}(x)\cdot(-1)^{\alpha}\binom{2k_2}{\alpha}\mathrm{d}x\\&=\operatorname{vol}(H_1^0(\mathbf{R}))\cdot\int_{\mathbf{X}_\infty}\sum_{\alpha=0}^{2k_2}\langle\varphi^{\alpha}_{\infty}(x),\varphi^{2k_2-\alpha}_{\infty}(x)\rangle_{\mathcal{W}}\cdot(-1)^{\alpha}\binom{2k_2}{\alpha}\mathrm{d}x\\&=\operatorname{vol}(H_1^0(\mathbf{R}))\cdot\int_{\mathbf{X}_\infty}\langle\varphi^{\star}_{\infty}(x),\varphi^{\star}_{\infty}(x)\rangle_{\mathcal{W}\otimes\mathcal{L}}\mathrm{d}x\end{aligned}$$

Combined with (5.11), (5.12) and (5.14), the above equation yields the proposition. □

The explicit calculations of local integrals $\mathcal{I}(\varphi^{\star}_{\infty})$ and $\mathcal{Z}_p^0$ will be postponed to the next section.

Corollary 5.5. *Assume that Δ_F and N^+N^- are coprime and that π is not Galois self-dual. For $p\mid\mathfrak{N}$, put*

$$\varepsilon_p=\begin{cases}\varepsilon_{\mathfrak{p}}\varepsilon_{\mathfrak{p}^c} & \text{if } p=\mathfrak{p}\mathfrak{p}^c \text{ is split in } F,\\ \varepsilon_{\mathfrak{p}} & \text{if } p=\mathfrak{p} \text{ is inert in } F.\end{cases}$$

Then we have

$$\begin{aligned}\frac{(\theta(\varphi^{\star},\mathbf{f}^{\dagger}),\theta(\varphi^{\star},\mathbf{f}^{\dagger}))_G}{\langle\mathbf{f}^{\dagger},\mathbf{f}^{\dagger}\rangle_{H^0}}=&\frac{L(1,\mathrm{As}^+(\pi))}{\zeta(2)\zeta(4)}\cdot\frac{(-1)^{k_2}\operatorname{vol}(\mathcal{U},\mathrm{d}h)2^{\#\mathcal{P}}}{2^{2k_1+7}(2k_1+1)(2k_2+1)^2N^2\Delta_F^3}\\&\times\frac{\zeta_{N_F}(4)}{\zeta_{N_F}(1)}\cdot\prod_{p\mid\mathfrak{N}}(1+\varepsilon_p)\cdot\prod_{p\mid\Delta_F}(1+p^{-1}).\end{aligned}$$

Proof. Recall that $L(s,\mathrm{As}^+(\pi_\infty))=\Gamma_{\mathbf{C}}(s+k_1+k_2+1)\cdot\Gamma_{\mathbf{C}}(s+k_1-k_2)$. By Proposition 6.2, we have

$$\mathcal{I}(\varphi^{\star}_{\infty})=\frac{(-1)^{k_2}(2k_1+1)}{2^{2k_1+7}}\cdot\frac{L(1,\mathrm{As}^+(\pi_\infty))}{\zeta_\infty(2)\zeta_\infty(4)}.$$

On the other hand, by the formulas of the local zeta integrals $\mathcal{Z}_p^0(\varphi^{\star}_p,f_p^{\dagger})$ in Proposition 6.3, 6.5, 6.6, and 6.7, we find that

$$\prod_p\mathcal{Z}_p^*(\varphi^{\star}_p,f_p^{\dagger})=\frac{L(1,\mathrm{As}^+(\pi))}{\zeta(2)\zeta^{N_F}(4)\zeta_{N_F}(1)}\cdot\frac{\operatorname{vol}(\mathcal{U},\mathrm{d}h)}{N_F^2\Delta_F}\cdot\prod_{p\in\mathcal{P}}2\mathcal{B}_{\sigma_p}(f_p^{\dagger},f_p^{\dagger})\cdot\prod_{p\mid\mathfrak{N}}(1+\varepsilon_p)\cdot\prod_{p\mid\Delta_F}(1+p^{-1}).$$

The corollary follows from Proposition 5.4 and Lemma 5.2 (1). □

5.3. **The Petersson norm of classical Yoshida lifts.** Note that the pairing $(\ ,\ )_G$ may not be positive definite unless $k_2 = 0$. In this subsection, we introduce a positive definite Hermitian pairing on the space of Siegel modular forms and rephrase Corollary 5.5 in terms of classical Siegel cusp forms of genus two. Define the Hermitian pairing $\mathcal{B}_\mathcal{L} : \mathcal{L}_\kappa(\mathbf{C}) \otimes \mathcal{L}_\kappa(\mathbf{C}) \to \mathbf{C}$ by

$$\mathcal{B}_\mathcal{L}(v_1, v_2) := \langle v_1, \rho_\kappa(w_0)\overline{v_2}\rangle_\mathcal{L}, \quad w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.15}$$

Then it is easy to see that $\mathcal{B}_\mathcal{L}$ is an $\mathrm{SU}_2(\mathbf{R})$-invariant and positive definite Hermitian pairing.

Lemma 5.6. *For $v_1, v_2 \in \mathcal{L}_\kappa(\mathbf{C})$, we have*

$$\langle\langle v_1, v_2\rangle\rangle = \frac{(-1)^{k_2}}{2k_2+1} \cdot \mathcal{B}_\mathcal{L}(v_1, v_2).$$

Proof. Since $\langle\langle\ ,\ \rangle\rangle$ and $\mathcal{B}_\mathcal{L}$ are both $\mathrm{SU}_2(\mathbf{R})$-invariant Hermitian pairing and $\mathcal{L}_\kappa(\mathbf{C})$ is irreducible, we have

$$\langle\langle v_1, v_2\rangle\rangle = C \cdot \mathcal{B}_\mathcal{L}(v_1, v_2) = C \cdot \langle v_1, \rho_\kappa(w_0)\overline{v_2}\rangle_\mathcal{L}$$

for some constant C. Letting $v_1 = v_2 = X^{2k_2}$, we have

$$\begin{aligned} C &= \int_{\mathrm{SU}_2(\mathbf{R})} \langle \rho_\kappa(u)X^{2k_2}, \rho_\kappa(\overline{u})X^{2k_2}\rangle_\mathcal{L} \mathrm{d}^*u \\ &= \int_{\mathrm{SU}_2(\mathbf{R})} \sum_a (-1)^a \binom{2k_2}{a} \alpha^a \overline{\alpha}^{2k_2-a} \beta^a \overline{\beta}^{2k_2-a} \mathrm{d}^*u, \ u = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}_2(\mathbf{R}). \end{aligned}$$

For $u \in \mathrm{SU}_2(\mathbf{R})$, we introduce the coordinates $u = u(\psi, \theta, \varphi)$:

$$u = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \ \alpha = \cos\psi \cdot e^{\sqrt{-1}\theta}, \ \beta = \sin\psi \cdot e^{\sqrt{-1}\varphi}, \ 0 \le \theta, \varphi \le 2\pi, 0 \le \psi \le \pi/2.$$

Then the Haar measure d^*u is given by

$$\mathrm{d}^*u = (4\pi)^{-2} \sin 2\psi \mathrm{d}\psi \mathrm{d}\theta \mathrm{d}\varphi.$$

We thus find that

$$\begin{aligned} C =& (-1)^{k_2} \binom{2k_2}{k_2} 2^{-2k_2} \int_0^{\pi/2} (\sin 2\psi)^{2k_2+1} \mathrm{d}\psi \\ =& (-1)^{k_2} \binom{2k_2}{k_2} 2^{-2k_2} \cdot 2^{2k_2} \frac{(k_2!)^2}{(2k_2+1)!} = \frac{(-1)^{k_2}}{2k_2+1}. \end{aligned}$$

□

Define the classical normalized Yoshida lift $\theta^*_{\mathbf{f}^\dagger} : \mathfrak{H}_2 \to \mathcal{L}_\kappa(\mathbf{C})$ by

$$\begin{aligned} \theta^*_{\mathbf{f}^\dagger}(Z) =& \frac{1}{\mathrm{vol}(\mathcal{U}, \mathrm{d}h)} \rho_\kappa(J(g_\infty, \mathbf{i}))\theta(\varphi^\star, \mathbf{f}^\dagger)(g_\infty) \\ & (g_\infty \in \mathrm{Sp}_4(\mathbf{R}), g_\infty \cdot \mathbf{i} = Z). \end{aligned}$$

Applying the proof of [HN16, Proposition 3.6] verbatim, one can show that $\theta^*_{\mathbf{f}^\dagger}$ is a holomorphic vector-valued Siegel modular from of weight $\mathrm{Sym}^{2k_2}(\mathbf{C}) \otimes \det^{k_1-k_2+2}$ and level $\Gamma_0^{(2)}(N_F)$ and has ℓ-adic integral Fourier coefficients if $\mathbf{f}$ is normalized so that the values of $\mathbf{f}$ on $\widehat{D}^\times$ are all ℓ-adically integral.

Define the Petersson norm of $\theta^*_{\mathbf{f}^\dagger}$ by

$$\langle \theta^*_{\mathbf{f}^\dagger}, \theta^*_{\mathbf{f}^\dagger}\rangle_{\mathfrak{H}_2} = \int_{\Gamma_0^{(2)}(N_F)\backslash\mathfrak{H}_2} \mathcal{B}_\mathcal{L}(\theta^*_{\mathbf{f}^\dagger}(Z), \theta^*_{\mathbf{f}^\dagger}(Z))(\det Y)^{k_1+2} \frac{\mathrm{d}X\mathrm{d}Y}{(\det Y)^3},$$

where $Z = X + \sqrt{-1}Y \in \mathfrak{H}_2$ and $\mathrm{d}X = \prod_{j\le l}\mathrm{d}x_{jl}, \mathrm{d}Y = \prod_{j\le l}\mathrm{d}y_{jl}$ for $X = (x_{jl})$ and $Y = (y_{jl})$. Recall that R is the Eichler order of level $N^+\mathcal{O}_F$ contained in R^0. For $\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{M}_{\underline{k}}(D^\times_{\mathbf{A}}, N^+\mathcal{O}_F^\times)$, put

$$\begin{aligned}\langle \mathbf{f}_1, \mathbf{f}_2\rangle_R :=& \frac{1}{\mathrm{vol}(\overline{U}_R, \mathrm{d}h_0)}\langle \mathbf{f}_1, \mathbf{f}_2\rangle_{H^0} \\ =& \sum_{[a]\in D^\times\backslash \widehat{D}^\times/\widehat{R}^\times} \langle \mathbf{f}_1(a), \overline{\tau_{\underline{k}}(\mathcal{J})\mathbf{f}_2(a)}\rangle_{\mathcal{W}} \cdot \frac{1}{\#\Gamma_a},\end{aligned} \tag{5.16}$$

where $\overline{U}_R$ is the image of U_R in $H^0(\mathbf{A})/Z_{H^0}(\mathbf{A})$ and $\Gamma_a = (a\widehat{R}^\times a^{-1} \cap D^\times)/\{\pm 1\}$.

Theorem 5.7. *Let r_F be the number of primes ramified in F. Put*

$$r_{F,2} = \begin{cases} 1 & \text{if } 2 \mid \Delta_F, \\ 0 & \text{if } 2 \nmid \Delta_F. \end{cases}$$

We have

$$\frac{\langle \theta^*_{\mathbf{f}^\dagger}, \theta^*_{\mathbf{f}^\dagger}\rangle_{\mathfrak{H}_2}}{\langle \mathbf{f}^\dagger, \mathbf{f}^\dagger\rangle_R} = \frac{2^\beta N}{(2k_1+1)(2k_2+1)} \cdot L(1, \mathrm{As}^+(\pi)) \cdot \prod_{p\mid \mathfrak{N}} (1+\varepsilon_p) \cdot \prod_{p\mid \Delta_F}(1+p^{-1}),$$

where

$$\beta = \#\mathcal{P} + 4r_{F,2} - 2k_1 - 7 - r_F$$

and $\mathcal{P}$ is the finite set defined in (4.5).

Proof. We recall some facts:

- the Tamagwa number $\tau(\mathrm{PGSp}_4) = \tau(\mathrm{SO}(3,2)) = 2$,
- $\mathrm{vol}(\mathrm{Sp}_4(\mathbf{Z})\backslash\mathfrak{H}_2, \frac{\mathrm{d}X\mathrm{d}Y}{(\det Y)^3}) = 2\zeta(2)\zeta(4)$ ([Sie43, Theorem 11]),
- $[\mathrm{Sp}_4(\mathbf{Z}) : \Gamma_0(N_F)] = N_F^3 \prod_{p\mid N_F} \frac{1-p^{-4}}{1-p^{-1}}$ ([Kli59, p114, (1)]).

The above combined with Lemma 5.6 yield

$$\frac{1}{\mathrm{vol}(\mathcal{U}, \mathrm{d}h)^2} \cdot (\theta(\varphi^\star, \mathbf{f}^\dagger), \theta(\varphi^\star, \mathbf{f}^\dagger))_G = \frac{(-1)^{k_2}}{2k_2+1} \cdot N_F^{-3} \prod_{p\mid N_F} \frac{1-p^{-1}}{1-p^{-4}} \cdot \frac{\langle \theta^*_{\mathbf{f}^\dagger}, \theta^*_{\mathbf{f}^\dagger}\rangle_{\mathfrak{H}_2}}{\zeta(2)\zeta(4)}. \tag{5.17}$$

Then we have

$$\langle \mathbf{f}^\dagger, \mathbf{f}^\dagger\rangle_{H^0} = \langle \mathbf{f}^\dagger, \mathbf{f}^\dagger\rangle_R \cdot \mathrm{vol}(\overline{U}_R, \mathrm{d}h_0),$$

so by Corollary 5.5, we find that

$$\frac{\langle \theta^*_{\mathbf{f}^\dagger}, \theta^*_{\mathbf{f}^\dagger}\rangle_{\mathfrak{H}_2}}{\langle \mathbf{f}^\dagger, \mathbf{f}^\dagger\rangle_R} \cdot N_F^{-3} = \frac{\mathrm{vol}(\overline{U}_R, \mathrm{d}h_0)\cdot 2^{\#\mathcal{P}} \cdot L(1, \mathrm{As}^+(\pi))}{\mathrm{vol}(\mathcal{U}, \mathrm{d}h)2^{2k_1+7}(2k_1+1)(2k_2+1)N_F^2\Delta_F 2^{-4r_{F,2}}} \cdot \prod_{p\mid\mathfrak{N}}(1+\varepsilon_p) \cdot \prod_{p\mid\Delta_F}(1+p^{-1}).$$

Therefore, it remains to show that

$$\mathrm{vol}(\overline{U}_R, \mathrm{d}h_0) = 2^{-r_F} \cdot \mathrm{vol}(\mathcal{U}, \mathrm{d}h)$$

for Tamagawa measures $\mathrm{d}h_0$ and $\mathrm{d}h$.

Following [GI11, §8, p.279], let ω_{H^0}be a rational invariant differential top form on H^0/Z_{H^0} and $\omega_{H^0_1}$ be the pull-back of ω_{H^0} by the natural isogeny $H^0_1 \to H^0/Z_{H^0}$. For each place $v \in \Sigma_{\mathbf{Q}}$, let $\mathrm{d}^t h_{0,v}$ and $\mathrm{d}^t h_v$ be the measures on $H^0(\mathbf{Q}_v)$ and $H^0_1(\mathbf{Q}_v)$ induced by ω_{H^0} and $\omega_{H^0_1}$. Then $\mathrm{d}h_0 = \prod_v \mathrm{d}^t h_{0,v}$ and $\mathrm{d}h = \prod_v \mathrm{d}^t h_v$ are Tamagawa measures on H^0 and H^0_1. We have

$$\frac{\mathrm{vol}(\overline{U}_{R_v}, \mathrm{d}h^t_{0,v})}{\mathrm{vol}(\mathcal{U}_v, \mathrm{d}h^t_v)} = 2^{-1}[\mathrm{N}_{F/\mathbf{Q}}(\mathrm{n}(R_v^\times)) : (\mathbf{Z}_v^\times)^2] = \begin{cases} 1/2 & \text{if } v \nmid 2,\ v \mid \infty\Delta_F, \\ 2 & \text{if } v = 2 \nmid \Delta_F, \\ 1 & \text{otherwise.} \end{cases}$$

We see that $\mathrm{vol}(\overline{U}_R, \mathrm{d}h_0) = \mathrm{vol}(\mathcal{U}, \mathrm{d}h) \cdot 2^{-r_F}$. ∎

6. The calculations of the local integrals

6.1. **The local integral at the infinite place.** In this subsection, we evaluate the integral $\mathcal{I}(\varphi_\infty^\star)$ in (5.8). Recall that if we define $P_{\underline{k}} : \mathbb{H}^{\oplus 2} \to \mathcal{W}_{\underline{k}}(\mathbf{C}) \otimes \mathcal{L}_\kappa(\mathbf{C})$ by

$$P_{\underline{k}}(x_1, x_2) = \sum_{\alpha=0}^{2k_2} P_{\underline{k}}^\alpha(x_1, x_2) \binom{2k_2}{\alpha} X^\alpha Y^{2k_2-\alpha},$$

where $P_{\underline{k}}^\alpha$ is the polynomial introduced in (4.8), then

$$\varphi_\infty^\star(x_1, x_2) = e^{-2\pi(\mathrm{n}(x_1)+\mathrm{n}(x_2))} \cdot P_{\underline{k}}(x_1, x_2).$$

Lemma 6.1. *We have*

$$\int_{\mathrm{SU}_2(\mathbf{R})^2} \langle P_{\underline{k}}(u), P_{\underline{k}}(u)\rangle_{\mathcal{W}\otimes\mathcal{L}} \,\mathrm{d}^*u =(-1)^{k_2}(2k_1+1) \cdot \frac{\Gamma(k_1+k_2+2)\Gamma(k_1-k_2+1)}{\Gamma(k_1+2)^2},$$

where d^*u *is the Haar measure on* $\mathrm{SU}_2(\mathbf{R})^2$ *with* $\mathrm{vol}(\mathrm{SU}_2(\mathbf{R})^2) = 1$.

Proof. Set $\Psi(u) := \langle P_{\underline{k}}(u), P_{\underline{k}}(u)\rangle_{\mathcal{W}\otimes\mathcal{L}}$. We have

$$\int_{\mathrm{SU}_2(\mathbf{R})^2} \Psi(u)\mathrm{d}^*u = \int_{\mathrm{SU}_2(\mathbf{R})} \int_{\mathrm{SU}_2(\mathbf{R})} \Psi(u_1, u_2)\mathrm{d}u_1\mathrm{d}u_2,$$

where $\mathrm{d}u_1, \mathrm{d}u_2$ are the Haar measure on $\mathrm{SU}_2(\mathbf{R})$ with $\mathrm{vol}(\mathrm{SU}_2(\mathbf{R})) = 1$. By [HN16, Lemma 3.2], we find that

$$\Psi(u_1, u_2) = \Psi(u_2^{-1}u_1, \mathbf{1}_2), \quad \Psi(u_1, \mathbf{1}_2) = \Psi(u_2^{-1}u_1u_2, \mathbf{1}_2).$$

It follows that

$$\begin{aligned}\int_{\mathrm{SU}_2(\mathbf{R})^2} \Psi(u)\mathrm{d}^*u &= \int_{\mathrm{SU}_2(\mathbf{R})} \int_{\mathrm{SU}_2(\mathbf{R})} \Psi(u_2^{-1}u_1, \mathbf{1}_2)\mathrm{d}u_1\mathrm{d}u_2 \\ &= \int_{\mathrm{SU}_2(\mathbf{R})} \Psi(u_1, 1)\mathrm{d}u_1.\end{aligned}$$

Moreover, the function $u_1 \mapsto \Psi(u_1, \mathbf{1}_2)$ is a class function on $\mathrm{SU}_2(\mathbf{R})$, so by Weyl's integral formula, we obtain

$$\int_{\mathrm{SU}_2(\mathbf{R})^2} \Psi(u)\mathrm{d}^*u = \frac{1}{4\pi}\int_0^{2\pi} \left|e^{\sqrt{-1}\theta} - e^{-\sqrt{-1}\theta}\right|^2 \Psi(\begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta}\end{pmatrix}, \mathbf{1}_2)\mathrm{d}\theta.$$

By definition (4.8), we have

$$\begin{aligned}&P_{\underline{k}}^\alpha(\begin{pmatrix} e^{\sqrt{-1}\theta} & \\ & e^{-\sqrt{-1}\theta}\end{pmatrix}, \mathbf{1}_2) \\
=&((e^{\sqrt{-1}\theta} - e^{-\sqrt{-1}\theta})X_1Y_1)^{k_1-k_2} \cdot (e^{-\sqrt{-1}\theta}Y_1 \otimes X_2 - e^{\sqrt{-1}\theta}X_1 \otimes Y_2)^\alpha (Y_1 \otimes X_2 - X_1 \otimes Y_2)^{2k_2-\alpha} \\
=&(e^{\sqrt{-1}\theta} - e^{-\sqrt{-1}\theta})^{k_1-k_2}(X_1Y_1)^{k_1-k_2} \cdot \sum_{a=0}^{\alpha}\binom{\alpha}{a}(-1)^a e^{\sqrt{-1}\theta(a-\alpha+a)} X_1^aY_1^{\alpha-a} \otimes X_2^{\alpha-a}Y_2^a \\
&\times \sum_{b=0}^{2k_2-\alpha}\binom{2k_2-\alpha}{b}(-1)^b X_1^bY_1^{2k_2-\alpha-b} \otimes X_2^{2k_2-\alpha-b}Y_2^b \\
=&(e^{\sqrt{-1}\theta} - e^{-\sqrt{-1}\theta})^{k_1-k_2}\sum_{a=0}^{\alpha}\sum_{b=0}^{2k_2-\alpha}\binom{\alpha}{a}\binom{2k_2-\alpha}{b}(-1)^{a+b}e^{\sqrt{-1}\theta(2a-\alpha)}X_1^{k_1-k_2+a+b}Y_1^{k_1+k_2-(a+b)} \otimes X_2^{2k_2-(a+b)}Y_2^{a+b}.\end{aligned}$$

From the above equation, we see that

$$\begin{aligned}
&\Psi(\begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta}\end{pmatrix}, \mathbf{1}_2) \\
=&(e^{\sqrt{-1}\theta}-e^{-\sqrt{-1}\theta})^{2k_1-2k_2}\sum_{\alpha=0}^{2k_2}(-1)^{\alpha}\binom{2k_2}{\alpha}\sum_{a=0}^{\alpha}\sum_{b=0}^{2k_2-\alpha}\binom{\alpha}{a}\binom{2k_2-\alpha}{b}(-1)^{a+b}e^{\sqrt{-1}\theta(2a-\alpha)} \\
&\times\sum_{c+d=2k_2-a-b}\binom{2k_2-\alpha}{c}\binom{\alpha}{d}(-1)^{c+d}e^{\sqrt{-1}\theta(2c-(2k_2-\alpha))}\cdot\frac{(-1)^{k_1-k_2+a+b}}{\binom{2k_1}{k_1-k_2+a+b}}\frac{(-1)^{a+b}}{\binom{2k_2}{a+b}} \\
=&(-1)^{k_1-k_2}(e^{\sqrt{-1}\theta}-e^{-\sqrt{-1}\theta})^{2k_1-2k_2}\times\sum_{\alpha=0}^{2k_2}(-1)^{\alpha}\binom{2k_2}{\alpha} \\
&\times\sum_{a,b,c}\binom{\alpha}{a}\binom{2k_2-\alpha}{b}\binom{2k_2-\alpha}{c}\binom{\alpha}{2k_2-a-b-c}\binom{2k_1}{k_1-k_2+a+b}^{-1}\binom{2k_2}{a+b}^{-1}\cdot e^{\sqrt{-1}\theta(2a+2c-2k_2)}.
\end{aligned}$$

By the formula

$$\int_0^{2\pi}\left|e^{\sqrt{-1}\theta}-e^{-\sqrt{-1}\theta}\right|^2(e^{\sqrt{-1}\theta}-e^{-\sqrt{-1}\theta})^{K}e^{\sqrt{-1}\theta A}\mathrm{d}\theta=2\pi(-1)^{\frac{K+A}{2}}\binom{K+2}{\frac{K+A+2}{2}}$$

for even integers K and A, we find that

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\left|e^{\sqrt{-1}\theta}-e^{-\sqrt{-1}\theta}\right|^2\Psi(\begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta}\end{pmatrix}, \mathbf{1}_2)\mathrm{d}\theta \\
=&(-1)^{k_1-k_2}\sum_{\alpha=0}^{2k_2}(-1)^{\alpha}\binom{2k_2}{\alpha}\sum_{a,b,c}\binom{\alpha}{a}\binom{2k_2-\alpha}{b}\binom{2k_2-\alpha}{c}\binom{\alpha}{2k_2-a-b-c} \\
&\times\binom{2k_1}{k_1-k_2+a+b}^{-1}\binom{2k_2}{a+b}^{-1}(-1)^{k_1+a+c}\binom{2k_1-2k_2+2}{k_1-2k_2+a+c+1} \\
\overset{b\mapsto b-a+k_2}{=}&(-1)^{k_1-k_2}\sum_{\alpha=0}^{2k_2}(-1)^{\alpha}\binom{2k_2}{\alpha}\sum_{b=-k_2}^{k_2}\sum_{a,c}(-1)^{a+c}\binom{\alpha}{a}\binom{2k_2-\alpha}{k_2+b-a}\binom{2k_2-\alpha}{c}\binom{\alpha}{k_2-b-c} \\
&\times\binom{2k_1}{k_1+b}^{-1}\binom{2k_2}{k_2+b}^{-1}\binom{2k_1-2k_2+2}{k_1-2k_2+a+c+1} \\
=&(-1)^{k_2}\sum_{b=-k_2}^{k_2}\binom{2k_1}{k_1+b}^{-1}\binom{2k_2}{k_2+b}^{-1}\sum_{\alpha=0}^{2k_2}(-1)^{\alpha}\binom{2k_2}{\alpha}T_b^{\alpha},
\end{aligned}$$

where

$$T_b^{\alpha}=\sum_{a,c}(-1)^{a+c}\binom{\alpha}{a}\binom{2k_2-\alpha}{k_2+b-a}\binom{2k_2-\alpha}{c}\binom{\alpha}{k_2-b-c}\binom{2k_1-2k_2+2}{k_1-2k_2+a+c+1}.$$

Note that T_b^{α} is equal to the coefficient of $X^{k_1+1}Y^{k_2-b}Z^{k_2+b}$ of the following polynomial

$$F^{\alpha}(X,Y,Z):=(1-Z)^{\alpha}(1+XZ)^{2k_2-\alpha}(1-Y)^{2k_2-\alpha}(1+XY)^{\alpha}(1+X)^{2k_1-2k_2+2},$$

so we obtain the identity

$$\sum_{\alpha=0}^{2k_2}(-1)^{\alpha}\binom{2k_2}{\alpha}T_b^{\alpha}=\binom{2k_1+2}{k_1+1}\binom{2k_2}{k_1+b}$$

by looking at the coefficient of $X^{k_1+1}Y^{k_2-b}Z^{k_2+b}$ of the polynomial

$$\begin{aligned}&\sum_{\alpha=0}^{2k_2}(-1)^\alpha\binom{2k_2}{\alpha}F^\alpha(X,Y,Z)\\=&(1+X)^{2k_1-2k_2+2}\sum_{\alpha=1}^{2k_2}(-1)^\alpha\binom{2k_2}{\alpha}(1+XY-Z-ZXY)^\alpha(1+XZ-Y-XYZ)^{2k_2-\alpha}\\=&(1+X)^{2k_1+2}(Y-Z)^{2k_2}.\end{aligned}$$

Summarizing the above calculations, we obtain

$$\begin{aligned}\int_{\mathrm{SU}_2(\mathbf{R})^2}\Psi(u)\mathrm{d}^*u=&(-1)^{k_2}2^{-1}\sum_{b=-k_2}^{k_2}\binom{2k_1}{k_1+b}^{-1}\binom{2k_2}{k_2+b}^{-1}\sum_{\alpha=0}^{2k_2}(-1)^\alpha\binom{2k_2}{\alpha}T_b^\alpha\\=&(-1)^{k_2}2^{-1}\binom{2k_1+2}{k_1+1}\cdot\sum_{b=-k_2}^{k_2}(-1)^{k_2+b}\binom{2k_1}{k_1+b}^{-1}\end{aligned}\tag{6.1}$$

A simple induction argument shows that for any $0\leq k_2\leq k_1$, we have

$$\sum_{b=-k_2}^{k_2}(-1)^{k_2+b}\binom{2k_1}{k_1+b}^{-1}=\frac{2k_1+1}{k_1+1}\cdot\binom{2k_1+1}{k_1-k_2}^{-1}.\tag{6.2}$$

It is clear that (6.1) and (6.2) yield the lemma. □

Proposition 6.2. *We have*

$$\begin{aligned}\mathcal{I}(\varphi_\infty^\star)=&\int_{\mathbf{X}_\infty}\langle\varphi_\infty^\star(x),\varphi_\infty^\star(x)\rangle_{\mathcal{W}\otimes\mathcal{L}}\mathrm{d}x\\=&\frac{(-1)^{k_2}(2k_1+1)}{2^{2k_1+7}}\cdot\frac{\Gamma_{\mathbf{C}}(k_1+k_2+2)\cdot\Gamma_{\mathbf{C}}(k_1-k_2+1)}{\Gamma_{\mathbf{R}}(2)\Gamma_{\mathbf{R}}(4)}\end{aligned}$$

Proof. For $x\in(D_\infty^\times)^2$, we write $x=r\cdot u$ with $r=(r_1,r_2)\in(\mathbf{R}_+)^2$ and $u=(u_1,u_2)\in\mathrm{SU}_2(\mathbf{R})^2$. Then the Haar measue $\mathrm{d}x$ is given by $(r_1r_2)^3\mathrm{d}r\mathrm{d}u$, where $\mathrm{d}r=\mathrm{d}r_1\mathrm{d}r_2$ is the Lebesque measure on $\mathbf{R}_+\times\mathbf{R}_+$ and $\mathrm{d}u=\mathrm{d}u_1\mathrm{d}u_2$ is the Haar measure on $\mathrm{SU}_2(\mathbf{R})^2$ with $\mathrm{vol}(\mathrm{SU}_2(\mathbf{R})^2)=4\pi^4$. We have

$$\begin{aligned}\int_{\mathbf{X}_\infty}\langle\varphi_\infty^\star(x),\varphi_\infty^\star(x)\rangle_{\mathcal{W}\otimes\mathcal{L}}\mathrm{d}x=&\int_{D_\infty^{\times 2}}\langle\varphi_\infty^\star(x),\varphi_\infty^\star(x)\rangle_{\mathcal{W}\otimes\mathcal{L}}\mathrm{d}x\\=&\int_0^\infty\int_0^\infty\int_{\mathrm{SU}_2(\mathbf{R})^2}\langle\varphi_\infty^\star(r\cdot u),\varphi_\infty^\star(r\cdot u)\rangle(r_1r_2)^3\mathrm{d}r\mathrm{d}u.\end{aligned}$$

Note that

$$\begin{aligned}\langle\varphi_\infty^\star(r\cdot u),\varphi_\infty^\star(r\cdot u)\rangle_{\mathcal{W}\otimes\mathcal{L}}=&\langle\rho_{(2k_2,0)}(\begin{pmatrix}r_1&0\\0&r_2\end{pmatrix})\varphi_\infty^\star(u),\rho_{(2k_2,0)}(\begin{pmatrix}r_1&0\\0&r_2\end{pmatrix})\varphi_\infty^\star(u)\rangle_{\mathcal{W}\otimes\mathcal{L}}\\=&(r_1r_2)^{2k_1}e^{-4\pi(r_1^2+r_2^2)}\langle P_{\underline{k}}(u),P_{\underline{k}}(u)\rangle_{\mathcal{W}\otimes\mathcal{L}}.\end{aligned}$$

Hence by Lemma 6.1, we see that

$$\begin{aligned}
&\int_{\mathbf{X}_\infty}\langle\varphi^\star_\infty(x),\varphi^\star_\infty(x)\rangle_{\mathcal{W}\otimes\mathcal{L}}\mathrm{d}x\\
=&\left(\int_0^\infty r_1^{2k_1+3}e^{-4\pi r_1^2}\mathrm{d}r_1\right)^2\cdot(4\pi^4)\cdot\int_{\mathrm{SU}_2(\mathbf{R})^2}\langle P_{\underline{k}}(u),P_{\underline{k}}(u)\rangle_{\mathcal{W}\otimes\mathcal{L}}\mathrm{d}^*u\\
=&\left(1/2\cdot(4\pi)^{-k_1-2}\Gamma(k_1+2)\right)^2\cdot(4\pi^4)\cdot(-1)^{k_2}\cdot(2k_1+1)\cdot\frac{\Gamma(k_1+k_2+2)\Gamma(k_1-k_2+1)}{\Gamma(k_1+2)^2}\\
=&(-1)^{k_2}\frac{\pi^3}{2^{2k_1+7}}\cdot(2k_1+1)\cdot 2(2\pi)^{-k_1-k_2-2}\Gamma(k_1+k_2+2)\cdot 2(2\pi)^{-k_1+k_2-1}\Gamma(k_1-k_2+1)\\
=&(-1)^{k_2}\frac{1}{2^{2k_1+7}}\cdot(2k_1+1)\cdot\frac{\Gamma_{\mathbf{C}}(k_1+k_2+2)\cdot\Gamma_{\mathbf{C}}(k_1-k_2+1)}{\Gamma_{\mathbf{R}}(2)\Gamma_{\mathbf{R}}(4)}.
\end{aligned}$$

This finishes the proof of the proposition. □

6.2. **Local integrals at finite places: preliminary.** In the following two subsections 6.3 and 6.4, we let p be a rational prime and calculate the local zeta integral

$$\mathcal{Z}_p(\varphi^\star_p,f^\dagger_p)=\int_{H^0_1(\mathbf{Q}_p)}\mathcal{B}_{\omega_p}(\omega_p(h_p)\varphi^\star_p,\varphi^\star_p)\mathcal{B}_{\sigma_p}(\sigma_p(h_p)f^\dagger_p,f^\dagger_p)\mathrm{d}h_p.$$

To simply the notation, we often omit the subscript p. For example, we write D_0, F, ω, $\varphi^\star$, $f^\dagger$, h for $D_0\otimes\mathbf{Q}_p$, $F\otimes\mathbf{Q}_p$, ω_p, $\varphi^\star_p$, $f^\dagger_p$ and h_p. Let $\mathcal{U}_p=H^0_1(\mathbf{Q}_p)\cap((R\otimes\mathbf{Z}_p)^\times\times\mathbf{Z}_p^\times)/\mathcal{O}_{F_p}^\times)$ be the local component of the open-compact subgroup $\mathcal{U}$ defined in (4.10). One verifies that $\varphi^\star$ and $f^\dagger$ are $\mathcal{U}_p$-invariant, and hence we have

$$\mathcal{Z}_p(\varphi^\star,f^\dagger)=\sum_h\mathcal{B}_\omega(\omega(h)\varphi^\star,\varphi^\star)\mathcal{B}_\sigma(\sigma(h)f^\dagger,f^\dagger)\cdot\mathrm{vol}(\mathcal{U}_ph\mathcal{U}_p),\tag{6.3}$$

where h runs over a complete set of representatives of the double coset space $\mathcal{U}_p\backslash H^0_1(\mathbf{Q}_p)/\mathcal{U}_p$.

6.3. **Local integrals at finite places: the split case.** In this subsection, we suppose that $p=\mathfrak{p}\mathfrak{p}^c$ is split in F. We shall identity $H^0(\mathbf{Q}_p)$ with $(D_0^\times\times D_0^\times)/\mathbf{Q}_p^\times$ with respect to $\mathfrak{p}$ as in Remark 4.1. First we treat the case $p\nmid N^-$. Then $D_0=\mathrm{M}_2(\mathbf{Q}_p)$ and $H^0(\mathbf{Q}_p)=(\mathrm{GL}_2(\mathbf{Q}_p)\times\mathrm{GL}_2(\mathbf{Q}_p))/\mathbf{Q}_p^\times$. We have $\sigma=\pi_\mathfrak{p}\boxtimes\pi_{\mathfrak{p}^c}$, where $\pi_\mathfrak{p}$ and $\pi_{\mathfrak{p}^c}$ are admissible and irreducible representations of $\mathrm{PGL}_2(\mathbf{Q}_p)$. Then $f^0=f^0_\mathfrak{p}\otimes f^0_{\mathfrak{p}^c}$, where $f^0_\mathfrak{p}$ and $f^0_{\mathfrak{p}^c}$ are new vectors of $\pi_\mathfrak{p}$ and $\pi_{\mathfrak{p}^c}$. For $\pi=\pi_\mathfrak{p}$ or $\pi_{\mathfrak{p}^c}$ and $f^0_?=f^0_\mathfrak{p}$ or $f^0_{\mathfrak{p}^c}$, let $\mathcal{B}_\pi:\pi\otimes\overline{\pi}\to\mathbf{C}$ be the $\mathrm{GL}_2(\mathbf{Q}_p)$-invariant pairing such that $\mathcal{B}_\pi(f^0_?,f^0_?)=1$. We thus have

$$\mathcal{B}_\sigma(a_1\otimes a_2,b_1\otimes b_2)=\mathcal{B}_{\pi_\mathfrak{p}}(a_1,b_1)\mathcal{B}_{\pi_{\mathfrak{p}^c}}(a_2,b_2).$$

In the case $p\mid N^+$, we shall assume $\mathfrak{p}^c\mid\mathfrak{N}^+$. Thus $\pi_{\mathfrak{p}^c}\simeq\mathrm{St}\otimes(\chi_2\circ\det)$ is a special representation associated with a unramified quadratic character $\chi_2:\mathbf{Q}_p^\times\to\{\pm1\}$ and let $\varepsilon\in\{\pm1\}$ be the sign given by

$$\varepsilon:=\begin{cases}1 & \text{if }\pi_\mathfrak{p}\text{ is spherical,}\\ \varepsilon_\mathfrak{p}\varepsilon_{\mathfrak{p}^c} & \text{if }\pi_\mathfrak{p}\text{ is special.}\end{cases}$$

Here we recall that $\varepsilon_\mathfrak{p}$ and $\varepsilon_{\mathfrak{p}^c}$ are the Atkin-Lehner eigenvalues of $\mathbf{f}^\circ$ at $\mathfrak{p}$ and $\mathfrak{p}^c$ in (4.4). Note that $\chi_2(p)=-\varepsilon_{\mathfrak{p}^c}$ (*cf.* [Sch02, Proposition 3.1.2]).

Proposition 6.3. *Suppose that $p\nmid N^-$ is split in F. If $p\nmid N^+$, then*

$$\mathcal{Z}_p(\varphi^\star,f^\dagger)=\mathrm{vol}(\mathcal{U}_p)\cdot\frac{L(1,\mathrm{As}^+(\pi))}{\zeta_p(2)\zeta_p(4)},$$

and if $p\mid N^+$

$$\mathcal{Z}_p(\varphi^\star,f^\dagger)=\mathrm{vol}(\mathcal{U}_p)\cdot p^{-2}(1+\varepsilon)\cdot\frac{L(1,\mathrm{As}^+(\pi_p))}{\zeta_p(1)\zeta_p(2)}\cdot\mathcal{B}_\sigma(f^\dagger,f^\dagger).$$

Proof. For $n, a \in \mathbf{Z}$, put

$$h_{n,a} = (\begin{pmatrix} p^{n+a} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} p^n & 0 \\ 0 & p^a \end{pmatrix}) \in H_1^0(\mathbf{Q}_p).$$

Let

$$t = p^{-1}.$$

For $m \in \mathbf{Z}$, put

$$\mathbf{c}_1(m) = \mathcal{B}_{\pi_\mathfrak{p}}(\pi_\mathfrak{p}(\begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix} f^0_\mathfrak{p}, f^0_\mathfrak{p}); \quad \mathbf{c}_2(m) = \mathcal{B}_{\pi_{\mathfrak{p}^c}}(\pi_{\mathfrak{p}^c}(\begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix} f^0_{\mathfrak{p}^c}, f^0_{\mathfrak{p}^c}).$$

It is well-known that for any $\epsilon > 0$, there exists a constant C_ϵ such that

$$|\mathbf{c}_i(m)| \leq C_\epsilon \cdot t^{|m|(1/2-\epsilon)} \text{ for } i = 1, 2 \tag{6.4}$$

by the Ramanujan conjecture.

Case (i) $p \nmid N_F$: In this case, $\pi_\mathfrak{p}$ and $\pi_{\mathfrak{p}^c}$ are both spherical. Let $R_0 = \mathrm{M}_2(\mathbf{Z}_p)$. Then $\mathcal{U}_p = (R_0^\times \times R_0^\times)/\mathbf{Z}_p^\times$, $\varphi^\star$ is the characteristic function of $R_0 \oplus R_0$ and $f^\dagger = f^0 = f^0_\mathfrak{p} \otimes f^0_{\mathfrak{p}^c}$ is the fixed new vector. By Cartan decomposition, the set

$$\{h_{n,a} \mid n + a \geq 0,\, n - a \geq 0\}$$

is a complete set of representatives of $\mathcal{U}_p \backslash H_1^0(\mathbf{Q}_p)/\mathcal{U}_p$. One can verify that

$$\begin{aligned} \mathcal{B}_\omega(\omega(h_{n,a})\varphi^\star, \varphi^\star) =& t^{2(|n|+|a|)}, \\ \mathcal{B}_\sigma(\sigma(h_{n,a})f^0, f^0) =& \mathbf{c}_1(n+a)\mathbf{c}_2(n-a), \\ \#(R_0^\times \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix} R_0^\times / R_0^\times) =& \begin{cases} 1 & \text{if } m = 0, \\ t^{-|m|}(1+t) & \text{if } m \neq 0. \end{cases} \end{aligned}$$

From (6.3) together with the above equations, we see that $\mathrm{vol}(\mathcal{U}_p)^{-1}\mathcal{Z}^0(\varphi^\star, f^\dagger)$ equals

$$\begin{aligned} &\sum_{n+a\geq 0, n-a \geq 0} t^{2(|n|+|a|)} \mathcal{B}_\sigma(\sigma(h_{n,a})f^0), f^0)\#(\mathcal{U}_p h_{n,a} \mathcal{U}_p/\mathcal{U}_p) \\ =& 1 + \sum_{n\geq 1} t^{4n}(\mathbf{c}_1(2n) + \mathbf{c}_2(2n))t^{-2n}(1+t) \\ &+ \sum_{n \geq a+1, a\geq 0} t^{2n+2a}\mathbf{c}_1(n+a)\mathbf{c}_2(n-a)t^{-2n}(1+t)^2 + \sum_{n\geq a+1, a\geq 1} t^{2n+2a}\mathbf{c}_1(n-a)\mathbf{c}_2(n+a)t^{-2n}(1+t)^2. \end{aligned}$$

Here the above series converges absolutely by (6.4). Suppose that $\pi_\mathfrak{p} = \pi(\mu_1, \mu_1^{-1})$ and $\pi_{\mathfrak{p}^c} = \pi(\mu_2, \mu_2^{-1})$. By Macdonald's formula (*cf.* [Bum97, Theorem 4.6.6]), for $i = 1, 2$ letting $\alpha_i = \mu_i(p)$ and $\beta_i = \mu_i(p)^{-1} = \alpha_i^{-1}$, we have

$$\mathbf{c}_i(m) = \frac{t^{\frac{|m|}{2}}}{1+t}(\alpha_i^{|m|}A_i - \beta_i^{|m|}B_i), \tag{6.5}$$

where

$$A_i = \frac{\alpha_i - \beta_i t}{\alpha_i - \beta_i}; \quad B_1 = \frac{\beta_i - \alpha_i t}{\alpha_i - \beta_i}.$$

Note that $|\alpha_i| = |\beta_i| = 1$. Therefore, we obtain

$$\begin{aligned}
&\mathrm{vol}(\mathcal{U}_p)^{-1}\mathcal{Z}_p(\varphi^\star, f^\dagger)\\
=& 1+\sum_{n\geq 1} t^{3n}(\alpha_1^{2n}A_1 - \beta_1^{2n}B_1 + \alpha_2^{2n}A_2 - \beta_2^{2n}B_2)+\\
&+\sum_{n\geq 1, a\geq 0} t^{n+3a}(\alpha_1^{n+2a}\alpha_2^n A_1A_2 - \alpha_2^n\beta_1^{n+2a}A_2B_1 - \alpha_1^{n+2a}\beta_2^n A_1B_2 + \beta_1^{n+2a}\beta_2^n B_1B_2)\\
&+\sum_{n\geq 1, a\geq 1} t^{n+3a}(\alpha_1^{n}\alpha_2^{n+2a} A_1A_2 - \alpha_2^{n+2a}\beta_1^{n}A_2B_1 - \alpha_1^{n}\beta_2^{n+2a} A_1B_2 + \beta_1^{n}\beta_2^{n+2a} B_1B_2)\\
=& 1+\frac{A_1\alpha_1^2t^3}{1-t^3\alpha_1^2} - \frac{B_1\beta_1^2t^3}{1-t^3\beta_1^2} + \frac{A_2\alpha_2^2t^3}{1-t^3\alpha_2^2} - \frac{B_2\beta_2^2t^3}{1-t^3\beta_2^2}\\
&+\frac{A_1A_2\alpha_1\alpha_2t}{(1-\alpha_1^2t^3)(1-\alpha_2\alpha_1t)} - \frac{A_2B_1\beta_1\alpha_2t}{(1-\beta_1^2t^3)(1-\beta_1\alpha_2t)} - \frac{A_1B_2\alpha_1\beta_2t}{(1-\alpha_1^2t^3)(1-\alpha_1\beta_2t)} + \frac{B_1B_2\beta_1\beta_2t}{(1-\beta_1^2t^3)(1-\beta_1\beta_2t)}\\
&+\frac{A_1A_2\alpha_1\alpha_2^3t^4}{(1-\alpha_2^2t^3)(1-\alpha_1\alpha_2t)} - \frac{A_2B_1\beta_1\alpha_2^3t^4}{(1-\alpha_2^2t^3)(1-\beta_1\alpha_2t)} - \frac{A_1B_2\alpha_1\beta_2^3t^4}{(1-\beta_2^2t^3)(1-\alpha_1\beta_2t)} + \frac{B_1B_2\beta_1\beta_2^3t^4}{(1-\beta_2^2t^3)(1-\beta_1\beta_2t)}.
\end{aligned}$$

We use *Mathematica* to factor the above rational expression and find that

$$\mathrm{vol}(\mathcal{U}_p)^{-1}\cdot\mathcal{Z}_p(\varphi^\star, f^\dagger) = \frac{(1-t^4)(1-t^2)}{(1-\alpha_1\alpha_2t)(1-\beta_1\alpha_2t)(1-\alpha_1\beta_2t)(1-\beta_1\beta_2t)} = \frac{L(1,\pi_{\mathfrak{p}}\otimes\pi_{\mathfrak{p}^c})}{\zeta_p(2)\zeta_p(4)}.$$

This completes the proof of the case (i).

Case (ii) $p \mid N^+$: In this case, $\pi_{\mathfrak{p}^c} = \mathrm{St}\otimes(\chi_2\circ\det)$ is special. Let R_0 be the standard Eichler order of level p in $\mathrm{M}_2(\mathbf{Z}_p)$ given by

$$R_0 = \left\{ g\in\mathrm{M}_2(\mathbf{Z}_p) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p\mathbf{Z}_p}\right\}.$$

Then $\mathcal{U}_p = H_1^0(\mathbf{Q}_p)\cap(R_0^\times\times R_0^\times)/\mathbf{Z}_p^\times$, $\varphi^\star$ is the characteristic function of $R_0\oplus R_0$ and $f^\dagger = f_{\mathfrak{p}}^\dagger\otimes f_{\mathfrak{p}^c}^0$, where

$$f_{\mathfrak{p}}^\dagger = \begin{cases} f_{\mathfrak{p}}^0 - \chi_2(p)\cdot\pi_{\mathfrak{p}}(\begin{pmatrix} 0 & 1 \\ p & 0\end{pmatrix})f_{\mathfrak{p}}^0 & \text{if } \mathfrak{p}\nmid\mathfrak{N}^+ \iff p\in\mathcal{P},\\ f_{\mathfrak{p}}^0 & \text{if } \mathfrak{p}\mid\mathfrak{N}^+ \iff p\notin\mathcal{P}.\end{cases} \tag{6.6}$$

Here $\mathcal{P}$ is the set defined in (4.5). Let $w\in\mathrm{GL}_2(\mathbf{Q}_p)$, w_1, w_2 in $H_1^0(\mathbf{Q}_p)$ be given by

$$w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix},\quad w_1 = (w, \mathbf{1}_2);\quad w_2 = (\mathbf{1}_2, w).$$

Then one can verify directly that the set

$$\Xi := \{\, h_{n,a},\ w_1h_{n,a},\ w_2h_{n,a},\ w_1w_2h_{n,a}\,\}_{n,a\in\mathbf{Z}}$$

is a complete set of representatives of $\mathcal{U}_p\backslash H_1^0(\mathbf{Q}_p)/\mathcal{U}_p$. For integers a,b,c,d, put

$$\mathbf{S}_{a,b,c,d} := \left\{ \begin{pmatrix} x & y \\ z & w\end{pmatrix}\in\mathrm{M}_2(\mathbf{Z}_p) \mid x\in p^a\mathbf{Z}_p, y\in p^b\mathbf{Z}_p, z\in p\mathbf{Z}_p\cap p^c\mathbf{Z}_p, w\in p^d\mathbf{Z}_p\right\}.$$

A direct computation shows that

$$\begin{aligned}
\mathcal{B}_\omega(\omega(h_{n,a})\varphi^\star,\varphi^\star) &= \mathrm{vol}(\mathbf{S}_{a,n,1-n,-a})^2 = t^{2|n|+2|a|+2},\\
\mathcal{B}_\omega(\omega(w_1h_{n,a})\varphi^\star,\varphi^\star) &= \mathrm{vol}(\mathbf{S}_{1-n,-a,a,n})^2 = t^{2|n-\frac12|+2|a-\frac12|+2},\\
\mathcal{B}_\omega(\omega(w_2h_{n,a})\varphi^\star,\varphi^\star) &= \mathrm{vol}(\mathbf{S}_{n,a,-a,1-n})^2 = t^{2|n-\frac12|+2|a+\frac12|+2},\\
\mathcal{B}_\omega(\omega(w_1w_2h_{n,a})\varphi^\star,\varphi^\star) &= \mathrm{vol}(\mathbf{S}_{-a,1-n,n,a})^2 = t^{2|n-1|+2|a|+2}.
\end{aligned}\tag{6.7}$$

Next we consider $\mathcal{B}_\sigma(\sigma(h_{n,a})f^\dagger, f^\dagger)$. For $\pi = \pi_{\mathfrak{p}}$ or $\pi_{\mathfrak{p}^c}$ and any $f \in \pi$, we put

$$\Phi_f(g) = \mathcal{B}_\pi(\pi(g)f, f) \cdot \#(\mathcal{U}_p g \mathcal{U}_p/\mathcal{U}_p).$$

For $h = (g_1, g_2) \in H_1^0(\mathbf{Q}_p)$, we have

$$\mathcal{B}_\sigma(\sigma(h)f^\dagger, f^\dagger) \cdot \mathrm{vol}(\mathcal{U}_p h \mathcal{U}_p) = \mathrm{vol}(\mathcal{U}_p) \cdot \Phi_{f_{\mathfrak{p}}^\dagger}(g_1) \cdot \Phi_{f_{\mathfrak{p}^c}^0}(g_2).$$

If $\pi = \mathrm{St} \otimes (\chi \circ \det)$ is special, it is well known that

$$\Phi_{f^0}(\begin{pmatrix} p^m & \\ & 1 \end{pmatrix}) = \chi(p)^m; \quad \Phi_{f^0}(w \begin{pmatrix} p^m & \\ & 1 \end{pmatrix}) = -\chi(p)^m \quad (w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}). \tag{6.8}$$

By the definition of $f_{\mathfrak{p}}^\dagger$ in (6.6), it is straightforward to verify that

$$\begin{aligned} \Phi_{f_{\mathfrak{p}}^\dagger}(\begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}) &= \Phi_{f_{\mathfrak{p}}^\dagger}(\begin{pmatrix} p^{|m|} & 0 \\ 0 & 1 \end{pmatrix}), \\ \Phi_{f_{\mathfrak{p}}^\dagger}(w \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}) &= \varepsilon \cdot (-\chi_2(p)) \cdot \Phi_{f_{\mathfrak{p}}^\dagger}(\begin{pmatrix} p^{m-1} & 0 \\ 0 & 1 \end{pmatrix}). \end{aligned} \tag{6.9}$$

Let $\alpha_2 := \chi_2(p)$. By (6.3) and (6.7), we have

$$\begin{aligned} &\mathrm{vol}(\mathcal{U}_p)^{-1} \cdot \mathcal{Z}_v^0(\varphi^\star, f^\dagger) \\ =& \sum_{\varepsilon_1, \varepsilon_2 \in \{0,1\}} \sum_{n,a \in \mathbf{Z}} \mathcal{B}_\omega(\omega(w^{\varepsilon_1}, w^{\varepsilon_2})h_{n,a})\varphi^\star, \varphi^\star) \Phi_{f_1^\dagger}(w^{\varepsilon_1} \begin{pmatrix} p^{n+a} & \\ & 1 \end{pmatrix}) \Phi_{f_2^0}(w^{\varepsilon_2} \begin{pmatrix} p^n & \\ & p^a \end{pmatrix}) \\ =& \sum_{\varepsilon_1 \in \{0,1\}} \sum_{n,a \in \mathbf{Z}} (\mathcal{B}_\omega(\omega((w^{\varepsilon_1}, \mathbf{1}_2)h_{n,a})\varphi^\star, \varphi^\star) - \mathcal{B}_\omega((\omega(w^{\varepsilon_1}, w)h_{n,a})\varphi, \varphi)) \Phi_{f_1^\dagger}(w^{\varepsilon_1} \begin{pmatrix} p^{n+a} & 0 \\ 0 & 1 \end{pmatrix}) \alpha_2^{n-a} \\ =& \sum_{n,a \in \mathbf{Z}} (t^{2|n|+2|a|+2} - t^{2|n-\frac{1}{2}|+2|a+\frac{1}{2}|+2}) \Phi_{f_1^\dagger}(\begin{pmatrix} p^{n+a} & 0 \\ 0 & 1 \end{pmatrix}) \alpha_2^{n-a} \\ &+ \sum_{n,a \in \mathbf{Z}} (t^{2|n-\frac{1}{2}|+2|a-\frac{1}{2}|+2} - t^{2|n-1|+2|a|+2}) \Phi_{f_1^\dagger}(w \begin{pmatrix} p^{n+a} & 0 \\ 0 & 1 \end{pmatrix}) \alpha_2^{n-a}. \end{aligned}$$

Note that the absolute convergence of the above series follows from the Ramanujan conjecture. By (6.9), the first summation of the above equation is given by

$$\begin{aligned} &\sum_{n,a \in \mathbf{Z}} (t^{2|n|+2|a|+2} - t^{2|n-\frac{1}{2}|+2|a+\frac{1}{2}|+2}) \Phi_{f_1^\dagger}(\begin{pmatrix} p^{n+a} & 0 \\ 0 & 1 \end{pmatrix}) \alpha_2^{n-a} \\ =& \sum_{n,a \geq 0} (t^{2n+2a+2} - t^{2n+2a+4}) \Phi_{f_1^\dagger}(\begin{pmatrix} p^{-n+a} & 0 \\ 0 & 1 \end{pmatrix}) \alpha_2^{-n-a} \\ &+ \sum_{n,a \geq 1} (t^{2n+2a+2} - t^{2n+2a}) \Phi_{f_1^\dagger}(\begin{pmatrix} p^{n-a} & 0 \\ 0 & 1 \end{pmatrix}) \alpha_2^{n+a} \\ =& t^2(1-t^2) \sum_{0 \leq n,a} \alpha_2^{n+a} t^{2n+2a} \Phi_{f_1^\dagger}(\begin{pmatrix} p^{-n+a} & 0 \\ 0 & 1 \end{pmatrix}) \\ &+ t^4(t^2-1) \sum_{0 \leq n,a} \alpha_2^{n+a} t^{2n+2a} \Phi_{f_1^\dagger}(\begin{pmatrix} p^{n-a} & \\ & 1 \end{pmatrix}) \\ =& t^2(1-t^2)^2 \sum_{0 \leq n,a} \alpha_2^{n+a} t^{2n+2a} \Phi_{f_1^\dagger}(\begin{pmatrix} p^{n-a} & \\ & 1 \end{pmatrix}), \end{aligned}$$

and the second summation is given by

$$\begin{aligned}
&\sum_{n,a\in\mathbf{Z}}(t^{2|n-\frac{1}{2}|+2|a-\frac{1}{2}|+2}-t^{2|n-1|+2|a|+2})\Phi_{f_1^\dagger}(w\begin{pmatrix}p^{n+a}&\\&1\end{pmatrix})\alpha_2^{n-a}\\
=&\sum_{n\geq 0,a\geq 1}(t^{2n+2a+2}-t^{2n+2a+4})\Phi_{f_1^\dagger}(w\begin{pmatrix}p^{-n+a}&\\&1\end{pmatrix})\alpha_2^{-n-a}\\
&+\sum_{n\geq 1,a\geq 0}(t^{2n+2a+2}-t^{2n+2a})\Phi_{f_1^\dagger}(w\begin{pmatrix}p^{n-a}&\\&1\end{pmatrix})\alpha_2^{n+a}\\
=&\alpha_2t^4(1-t^2)\sum_{n,a\geq 0}\alpha_2^{n+a}t^{2n+2a}\Phi_{f_1^\dagger}(w\begin{pmatrix}p^{-n+a+1}&\\&1\end{pmatrix})\\
&+\alpha_2t^2(t^2-1)\sum_{n,a\geq 0}\alpha_2^{n+a}t^{2n+2a}\Phi_{f_1^\dagger}(w\begin{pmatrix}p^{n-a+1}&\\&1\end{pmatrix})\\
=&\varepsilon\cdot t^2(1-t^2)^2\sum_{n,a\geq 0}\alpha_2^{n+a}t^{2n+2a}\Phi_{f_1^\dagger}(\begin{pmatrix}p^{n-a}&\\&1\end{pmatrix}).
\end{aligned}$$

We thus conclude that

$$\mathrm{vol}(\mathcal{U}_p)^{-1}\cdot\mathcal{Z}_p(\varphi^\star,f^\dagger)=(1+\varepsilon)\cdot t^2(1-t^2)^2\sum_{n,a\geq 0}(\alpha_2t^2)^{n+a}\cdot\Phi_{f_1^\dagger}(\begin{pmatrix}p^{n-a}&0\\0&1\end{pmatrix}).\tag{6.10}$$

To evaluate the last infinite series in the above equation, we will use the following elementary identity:

Lemma 6.4. *For indeterminates T,X,Y, we have*

$$\sum_{n,a\geq 0}X^{n+a}Y^{|n-a|}T^{|n-a-1|}=\frac{XY+T}{(1-X^2)(1-XYT)}.$$

Suppose that $\pi_\mathfrak{p}=\pi(\mu_1,\mu_1^{-1})$ is spherical with Satake parameters $\alpha_1=\mu(p)$ and $\beta_1=\mu_1(p)^{-1}$. We have $\#(\mathcal{U}_p\begin{pmatrix}p^m&0\\0&1\end{pmatrix}\mathcal{U}_p/\mathcal{U}_p)=t^{-|m|}$ and

$$\begin{aligned}
\Phi_{f_\mathfrak{p}^\dagger}(\begin{pmatrix}p^m&0\\0&1\end{pmatrix})=&t^{-|m|}\cdot\mathcal{B}_{\pi_\mathfrak{p}}(\pi_\mathfrak{p}(\begin{pmatrix}p^m&0\\0&1\end{pmatrix})f_\mathfrak{p}^0-\alpha_2\pi_\mathfrak{p}(\begin{pmatrix}p^{m-1}&0\\0&1\end{pmatrix})f_\mathfrak{p}^0,f_\mathfrak{p}^0-\alpha_2\pi_\mathfrak{p}(\begin{pmatrix}1&0\\0&p\end{pmatrix})f_\mathfrak{p}^0)\\
=&t^{-|m|}\cdot(2\mathbf{c}_1(m)-\alpha_2\mathbf{c}_1(m+1)-\alpha_2\mathbf{c}_1(m-1)).
\end{aligned}$$

By Macdonald's formula (6.5), we thus find that the last inifnite series in (6.10) equals

$$\begin{aligned}
&\sum_{n,a\geq 0}(\alpha_2t^2)^{n+a}\Phi_{f_\mathfrak{p}^\dagger}(\begin{pmatrix}p^{n-a}&0\\0&1\end{pmatrix})\\
=&\sum_{n,a\geq 0}(\alpha_2t^2)^{n+a}\cdot t^{-|n-a|}\cdot(2\mathbf{c}(n-a)-\alpha_2\mathbf{c}(n-a+1)-\alpha_2\mathbf{c}(n-a-1))\\
=&2\sum_{n,a\geq 0}(\alpha_2t^2)^{n+a}\cdot t^{-|n-a|}\cdot(\mathbf{c}(n-a)-\alpha_2\mathbf{c}(n-a-1))\\
=&\frac{2}{1+t}\sum_{n,a\geq 0}(\alpha_2t^2)^{n+a}\left\{(\alpha_1t^{-\frac{1}{2}})^{|n-a|}A_1+(\beta_1t^{-\frac{1}{2}})^{|n-a|}B_1-\alpha_2t^{-|n-a|}(\alpha_1t^{\frac{1}{2}})^{|n-a-1|}A_1+(\beta_1t^{\frac{1}{2}})^{|n-a-1|}B)\right\}.
\end{aligned}$$

Applying Lemma 6.4, the above equation equals

$$\begin{aligned}
&\frac{2A_1}{(1+t)(1-t^4)(1-\alpha_1\alpha_2t^{\frac{3}{2}})}\left\{(1+\alpha_1\alpha_2t^{\frac{3}{2}})-\alpha_2(\alpha_2t+\alpha_1t^{\frac{1}{2}})\right\}\\
&+\frac{2B_1}{(1+t)(1-t^4)(1-\beta_1\alpha_2t^{\frac{3}{2}})}\left\{(1+\beta_1\alpha_2t^{\frac{3}{2}})-\alpha_2(\alpha_2t+\beta_1t^{\frac{1}{2}})\right\}
\end{aligned}$$

$$
\begin{aligned}
=&\frac{2A_1(1-t)(1-\alpha_1\alpha_2 t^{\frac12})}{(1+t)(1-t^4)(1-\alpha_1\alpha_2 t^{\frac32})}+\frac{2B_1(1-t)(1-\beta_1\alpha_2 t^{\frac12})}{(1+t)(1-t^4)(1-\beta_1\alpha_2 t^{\frac32})}\\
=&\frac{2(1-t)}{(1+t)(1-t^4)}\left\{\frac{A_1(1-\alpha_1\alpha_2 t^{\frac12})}{1-\alpha_1\alpha_2 t^{\frac32}}+\frac{B_1(1-\beta_1\alpha_2 t^{\frac12})}{1-\beta_1\alpha_2 t^{\frac32}}\right\}\\
=&\frac{2(1-t)}{(1+t)(1-t^4)}\cdot\frac{(\alpha_1-\beta_1 t)(1-\alpha_1\alpha_2 t^{\frac12})(1-\beta_1\alpha_2 t^{\frac32})-(\beta_1-\alpha_1 t)(1-\beta_1\alpha_2 t^{\frac12})(1-\alpha_1\alpha_2 t^{\frac32})}{(\alpha_1-\beta_1)(1-\alpha_1\alpha_2 t^{\frac32})(1-\beta_1\alpha_2 t^{\frac32})}\\
=&\frac{2(1-t)}{(1+t)(1-t^4)}\cdot\frac{1+t-(\alpha_1+\beta_1)\alpha_2 t^{\frac12}(1+t^2)+t^2+t^3}{(1-\alpha_1\alpha_2 t^{\frac32})(1-\beta_1\alpha_2 t^{\frac32})}\\
=&\frac{2}{(1+t)^2}\cdot\frac{(1-\alpha_1\alpha_2 t^{\frac12})(1-\beta_1\alpha_2 t^{\frac12})}{(1-\alpha_1\alpha_2 t^{\frac32})(1-\beta_1\alpha_2 t^{\frac32})}.
\end{aligned}
$$

Therefore, (6.10) yields

$$
\mathcal{Z}_p(\varphi^\star,f^\dagger)=\mathrm{vol}(\mathcal{U}_p)t^2\cdot\frac{L(1,\mathrm{As}^+(\pi))}{\zeta_p(1)\zeta_p(2)}\cdot\frac{4(1-\alpha_1\alpha_2 t^{\frac12})(1-\beta_1\alpha_2 t^{\frac12})}{1+t},
$$

and we complete the proof in this case by noting that

$$
\mathcal{B}_\sigma(f^\dagger,f^\dagger)=\mathcal{B}_{\pi_\mathfrak{p}}(f^\dagger_\mathfrak{p},f^\dagger_\mathfrak{p})=2(1-\alpha_2\cdot\mathcal{B}_{\pi_\mathfrak{p}}(\pi_\mathfrak{p}(\begin{pmatrix}1&0\\0&p\end{pmatrix})f^0_\mathfrak{p},f^0_\mathfrak{p}))=\frac{2(1-t^{\frac12}\alpha_2(\alpha_1+\beta_1))}{1+t}.
$$

Suppose that $\pi_\mathfrak{p}=\mathrm{St}\otimes(\chi_1\circ\det)$ is special. Let $\alpha_1=\chi_1(p)$. Then $\varepsilon=\varepsilon_\mathfrak{p}\varepsilon_{\mathfrak{p}^c}=\alpha_1\alpha_2$. We have

$$
\sum_{n,a\geq 0}(\alpha_2 t^2)^{n+a}\Phi_{f_1^\dagger}(\begin{pmatrix}p^{n-a}&0\\0&1\end{pmatrix}))=\sum_{n,a\geq 0}(\alpha_2 t^2)^{n+a}\alpha_1^{|n-a|}=\frac{1+\alpha_1\alpha_2 t^2}{(1-t^4)(1-\alpha_1\alpha_2 t^2)}.
$$

Recall that

$$
L(s,\mathrm{As}^+(\pi))=L(1,\pi_\mathfrak{p}\otimes\pi_{\mathfrak{p}^c})=(1-\alpha_1\alpha_2 t^s)^{-1}(1-\alpha_1\alpha_2 t^{s+1})^{-1}.
$$

We find that

$$
\begin{aligned}
\mathcal{Z}_p(\varphi^\star,f^\dagger)=&\mathrm{vol}(\mathcal{U}_p)\cdot(1+\varepsilon)\cdot t^2(1-t^2)^2\frac{(1+\varepsilon t^2)(1-\varepsilon t)}{1-t^4}\cdot L(1,\mathrm{As}^+(\pi))\\
=&\mathrm{vol}(\mathcal{U}_p)t^2\cdot(1+\varepsilon)\cdot\frac{L(1,\mathrm{As}^+(\pi))}{\zeta_p(1)\zeta_p(2)}.
\end{aligned}
$$

This finishes the proof of the case (ii). □

Proposition 6.5. *If $p\mid N^-$, then we have*

$$
\mathcal{Z}_p(\varphi^\star,f^\dagger)=\mathrm{vol}(\mathcal{U}_p)\cdot p^{-2}(1+\varepsilon_\mathfrak{p}\varepsilon_{\mathfrak{p}^c})\cdot\frac{L(1,\mathrm{As}^+(\pi))}{\zeta_p(1)\zeta_p(2)}.
$$

Proof. In this case, $D_{0,p}$ is the division quaternion algebra over $\mathbf{Q}_p$ and $\mathcal{U}_p=H^0_1(\mathbf{Q}_p)\cap(R_0^\times\times R_0^\times)/\mathbf{Z}_p^\times$ where $R_0=\{a\in D_0\mid \mathrm{n}(a)\in\mathbf{Z}_p\}$ is the maximal order of D_0. Note that $\varphi^\star$ is the characteristic function on $R_0\oplus R_0$ and $H^0_1(\mathbf{Q}_p)=\{(a,d)\in H^0(\mathbf{Q}_p)\mid \mathrm{n}(a)=\mathrm{n}(d)\}$, it follows that for $h\in H^0_1(\mathbf{Q}_p)$, $\omega(h)\varphi^\star=\varphi^\star$. In addition, let $\varpi_p\in D_0$ with $\mathrm{n}(\varpi_p)=p$. Then $\{(1,1),(\varpi_p,\varpi_p)\}$ is a complete set of representatives of the double coset space $\mathcal{U}_p\backslash H^0_1(\mathbf{Q}_p)/\mathcal{U}_p$. As $f^\dagger=f^0$ is invariant by $\mathcal{U}_p$ and is also an eigenvector of (ϖ_p,ϖ_p) with eigenvalue $\varepsilon_\mathfrak{p}\varepsilon_{\mathfrak{p}^c}$, we find that

$$
\mathcal{Z}_p(\varphi^\star,f^\dagger)=\mathrm{vol}(\mathcal{U}_p)\,\mathrm{vol}(R_0)^2(1+\chi_1\chi_2(p))=\mathrm{vol}(\mathcal{U}_p)p^{-2}\cdot(1+\varepsilon_\mathfrak{p}\varepsilon_{\mathfrak{p}^c}).
$$

Now the proposition follows from the fact that

$$
L(1,\mathrm{As}^+(\pi))=(1-\varepsilon_\mathfrak{p}\varepsilon_{\mathfrak{p}^c}p^{-1})^{-1}(1-\varepsilon_\mathfrak{p}\varepsilon_{\mathfrak{p}^c}p^{-2})^{-1}.
$$

□

6.4. **Local integrals at finite places: the non-split case.** Let p be a prime inert or ramified in F. Then F is a quadratic extension over $\mathbf{Q}_p$, $D = \mathrm{M}_2(F)$, $H^0(\mathbf{Q}_p) = (\mathrm{GL}_2(F) \times \mathbf{Q}_p^\times)/F^\times$, and $\sigma = \pi \boxtimes \mathbf{1}$, where π is an admissible and irreducible representation of $\mathrm{PGL}_2(F)$. Let $\mathcal{O} = \mathcal{O}_F$. For $m \in \mathbf{Z}$, we define

$$u_m = (\begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}, p^m), \quad w = (\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1) \in H^0_1(\mathbf{Q}_p).$$

Proposition 6.6. *Suppose that p is inert in F. If $p \nmid N^+$, then*

$$\mathcal{Z}_p(\varphi^\star, f^\dagger) = \mathrm{vol}(\mathcal{U}_p) \cdot \frac{L(1, \mathrm{As}^+(\pi))}{\zeta_p(2)\zeta_p(4)},$$

and if $p \mid N^+$, then

$$\mathcal{Z}_p(\varphi^\star, f^\dagger) = \mathrm{vol}(\mathcal{U}_p) \cdot p^{-2}(1+\varepsilon_p) \cdot \frac{L(1, \mathrm{As}^+(\pi))}{\zeta_p(1)\zeta_p(2)}.$$

Proof. We let

$$t = p^{-2}.$$

Case (i) $p \nmid N_F$: Then $\pi = \pi(\mu, \mu^{-1})$ is an unramified principal series. Let $R = \mathrm{M}_2(\mathcal{O})$. Then $\mathcal{U}_p = H^0_1(\mathbf{Q}_p) \cap (R^\times \times \mathbf{Q}_p^\times)/F^\times$ and $f^\dagger = f^0$. The set $\{u_m \mid m \in \mathbf{Z}_{\geq 0}\}$ is a complete set of representatives of $\mathcal{U}_p \backslash H^0_1(\mathbf{Q}_p)/\mathcal{U}_p$ by Cartan decomposition, and we have

$$\mathcal{B}_\omega(\omega(u_m)\varphi^\star, \varphi^\star) = t^{|m|} \text{ for } m \in \mathbf{Z}.$$

Let $\alpha = \mu(p)$ and $\beta = \mu(p)^{-1}$. Then $|\alpha|, |\beta| \leq t^{1/5-1/2}$ by the Ramanujaun bound in [LRS99]. By (6.3) and Macdonald's formula, $\mathrm{vol}(\mathcal{U}_p)^{-1}\mathcal{Z}_p(\varphi^\star, f^\dagger)$ equals

$$\begin{aligned}
&1 + \sum_{m \geq 1} \mathcal{B}_\omega(\omega(u_m)\varphi^\star, \varphi^\star)\mathcal{B}_\pi(\pi(u_m)f^0, f^0)\#(\mathcal{U}_p u_m \mathcal{U}_p/\mathcal{U}_p) \\
=&1 + \sum_{m \geq 1} t^m \cdot t^{m/2}(\alpha^m \frac{\alpha - \beta t}{\alpha - \beta} - \beta^m \frac{\beta - \alpha t}{\alpha - \beta}) \cdot t^{-m}(1+t) \\
=&1 + \frac{(\alpha+\beta)t^{\frac{1}{2}} - t - t^2}{(1-\alpha t^{\frac{1}{2}})(1-\beta t^{\frac{1}{2}})} \\
=&\frac{(1-t^2)(1-t)}{(1-\alpha t^{\frac{1}{2}})(1-\beta t^{\frac{1}{2}})(1-t)} = \frac{L(1, \mathrm{As}^+(\pi))}{\zeta_p(2)\zeta_p(4)}.
\end{aligned}$$

Case (ii) $p \mid \mathfrak{N}^+$: In this case, $\pi = \mathrm{St} \otimes \chi \circ \det$ is the unramified special representation of $\mathrm{GL}_2(F)$ attached to a quadratic character $\chi : F^\times \to \{\pm 1\}$. Let R be the Eichler order of p given by

$$R = \left\{ g \in \mathrm{M}_2(\mathcal{O}) \mid g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p\mathcal{O}} \right\}.$$

Then $\mathcal{U}_p = H^0_1(\mathbf{Q}_p) \cap (R^\times \times \mathbf{Z}_p^\times)/\mathcal{O}^\times$, and the set

$$\{u_m, wu_m\}_{m \in \mathbf{Z}}$$

is a complete set of representatives of the double coset space $\mathcal{U}_p \backslash H^0_1(\mathbf{Q}_p)/\mathcal{U}_p$. Put

$$\mathbf{S}'_{a,b} = \left\{ \begin{pmatrix} x & \delta y \\ \delta z & x^c \end{pmatrix} \mid x \in \mathcal{O}, y \in \mathbf{Z}_p \cap p^a\mathbf{Z}_p,\, z \in p\mathbf{Z}_p \cap p^b\mathbf{Z}_p \right\}.$$

Then we have

$$\begin{aligned}
\mathcal{B}_\omega(\omega(u_m)\varphi^\star, \varphi^\star) &= \mathrm{vol}(\mathbf{S}'_{-m,m+1})^2 = t^{|m|+1}, \\
\mathcal{B}_\omega(\omega(wu_m)\varphi^\star, \varphi^\star) &= \mathrm{vol}(\mathbf{S}'_{1-m,m})^2 = t^{|m-1|+1}.
\end{aligned} \tag{6.11}$$

For $h = (g, \alpha) \in H^0(\mathbf{Q}_p)$, we put

$$\Phi_{f^0}(h) = \mathcal{B}_\sigma(\sigma(h)f^0, f^0) \cdot \#(\mathcal{U}_p h \mathcal{U}_p/\mathcal{U}_p). \tag{6.12}$$

For $m \in \mathbf{Z}$, we have

$$\Phi_{f^0}(u_m) = \chi(p)^m; \quad \Phi_{f^0}(wu_m) = -\chi(p)^m.$$

The above equations along with (6.3) imply that

$$\mathrm{vol}(\mathcal{U}_p)^{-1}\mathcal{Z}_p(\varphi^\star, f^0) = \sum_{m\in\mathbf{Z}} \mathcal{B}_\omega(\omega(u_m)\varphi^\star, \varphi^\star)\Phi_{f^0}(u_m) + \mathcal{B}(\omega(wu_m)\varphi^\star, \varphi^\star)\Phi_{f^0}(wu_m).$$

By (6.11) and (6.12), we have

$$\begin{aligned}\sum_{m\in\mathbf{Z}} \mathcal{B}_\omega(\omega(u_m)\varphi^\star, \varphi^\star)\Phi_{f^0}(u_m) =& \sum_{m\geq 0} t^{m+1}\cdot\chi(p)^m + \sum_{m\geq 1} t^{m+1}\cdot\chi(p)^m \\ =&\frac{t+\chi(p)t^2}{1-\chi(p)t}, \\ \sum_{m\in\mathbf{Z}} \mathcal{B}_\omega(\omega(wu_m)\varphi^\star, \varphi^\star)\Phi_{f^0}(wu_m) =& -\sum_{m\geq 1} t^{m}\cdot\chi(p)^m - \sum_{m\geq 0} t^{m+2}\cdot\chi(p)^m \\ =& -\frac{\chi(p)t+t^2}{1-\chi(p)t}.\end{aligned}$$

We thus obtain

$$\begin{aligned}\mathrm{vol}(\mathcal{U}_p)^{-1}\mathcal{Z}_p(\varphi^\star, f^0) =& \frac{t(1-\chi(p))(1-t)}{1-\chi(p)t} \\ =& t(1-\chi(p))\cdot\frac{1+p^{-1}\chi(p)}{1-p^{-1}}\cdot\frac{(1-t)(1-p^{-1})}{(1-\chi(p)t)(1+p^{-1}\chi(p))} \\ =& p^{-2}(1-\chi(p))\cdot\frac{L(1,\mathrm{As}^+(\pi))}{\zeta_p(2)\zeta_p(1)}.\end{aligned}$$

This completes the proof. □

Proposition 6.7. *Suppose that p is ramified in F. Then we have*

$$\mathcal{Z}_p(\varphi^\star, f^\dagger) = \mathrm{vol}(\mathcal{U}_p)\cdot\left|2^{-4}\Delta_F^3\right|_p(1+p^{-1})\frac{L(1,\mathrm{As}^+(\pi))}{\zeta_p(1)\zeta_p(2)}.$$

Proof. In this case, $(p, N^+N^-) = 1$, and hence $\sigma = \pi\boxtimes\mathbf{1}$, where $\pi = \pi(\mu,\nu)$ is a spherical representation of $\mathrm{PGL}_2(F)$. Let $R = \mathrm{M}_2(\mathcal{O})$. Then $\mathcal{U}_p = H_1^0(\mathbf{Q}_p)\cap(R^\times\times\mathbf{Z}_p^\times)/\mathcal{O}^\times$ and $\{u_m\}_{m\in\mathbf{Z}_{\geq 0}}$ is a complete set of representatives of $\mathcal{U}_P\backslash H_1^0(\mathbf{Q}_p)/\mathcal{U}_p$. Write $\mathcal{O} = \mathbf{Z}_p\oplus\mathbf{Z}_p\theta$ and put $\delta = \theta-\theta^c$. Note that $\delta^{-1}\mathcal{O}$ is the different of $F/\mathbf{Q}_p$. Let $t = p^{-1}$. Since $\varphi^\star$ is the characteristic function of $L_p\oplus L_p$, where

$$L_p = \left\{\begin{pmatrix} x & \delta y\\ \delta z & x^c\end{pmatrix} \mid x\in\mathcal{O}, y,z\in 2^{-1}\mathbf{Z}_p\right\},$$

we see that $\mathcal{B}_\omega(\omega(u_m)\varphi^\star, \varphi^\star) = t^{2m}\cdot\left|2^{-4}(\delta\bar{\delta})^3\right|_p$. On the other hand, we have

$$\begin{aligned}\Phi_{f^0}(u_m) =& \mathcal{B}_\sigma(\sigma(u_m)f^0, f^0)\cdot\#(\mathcal{U}_pu_m\mathcal{U}_p/\mathcal{U}_p) \\ =& \mathcal{B}_\pi(\pi(\begin{pmatrix} p^m & 0\\ 0 & 1\end{pmatrix})f^0, f^0)\cdot t^{-2m}(1+t).\end{aligned}$$

Let ϖ be a uniformizer of $\mathcal{O}$. Let $\alpha = \mu(\varpi)$ and $\beta = \nu(\varpi)$. Then $|\alpha|, |\beta| \leq t^{1/5-1/2}$ by [LRS99]. We have $\Phi_{f^0}(u_0) = 1$, and for $m > 0$, by Macdonald's formula

$$\Phi_{f^0}(u_m) = \frac{t^{-m}}{\alpha-\beta}(\alpha^{2m}(\alpha-\beta t)-\beta^{2m}(\beta-\alpha t)).$$

By (6.3) and the above equations, we see that

$$\begin{aligned}\left|2^4\Delta_F^{-3}\right|_p \mathrm{vol}(\mathcal{U}_p)^{-1}\mathcal{Z}_p(\varphi^\star, f^\dagger) =& \sum_{m\geq 0}\left|2^4\Delta_F^{-3}\right|_p \mathcal{B}_\omega(\omega(u_m)\varphi^\star,\varphi^\star)\Phi_{f^0}(u_m).\\ =&1+\sum_{m>0}\frac{t^m}{\alpha-\beta}\cdot(\alpha^{2m+1}-\alpha^{2m-1}t-\beta^{2m+1}+\beta^{2m-1}t))\\ =&1+\frac{1}{\alpha-\beta}\cdot(\frac{\alpha^3t-\alpha t^2}{1-\alpha^2 t}-\frac{\beta^3 t-\beta t^2}{1-\beta^2 t})\\ =&1+\frac{t(\alpha^2+1+\beta^2-2t-t^2)}{(1-\alpha^2t)(1-\beta t^2)}\\ =&\frac{1+t-t^2-t^3}{(1-\alpha^2t)(1-\beta^2t)}=\frac{(1-t^2)(1+t)(1-t)}{(1-\alpha^2t)(1-t)(1-\beta^2t)}\\ =&(1+t)\cdot\frac{L(1,\mathrm{As}^+(\pi))}{\zeta_p(1)\zeta_p(2)}.\end{aligned}$$

This proves the proposition. □

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