

ON THE NON-VANISHING OF HECKE L -VALUES MODULO p

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ABSTRACT. In this article, we follow Hida's approach to establish an analogue of Washington's theorem on the non-vanishing modulo p of Hecke L -values for CM fields with anticyclotomic twists.

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INTRODUCTION

The purpose of this paper is to study the non-vanishing modulo p property of Hecke L -values for CM fields via arithmetic of Eisenstein series. Let \mathcal{F} be a totally real field of degree d over \mathbf{Q} and \mathcal{K} be a totally imaginary quadratic extension of \mathcal{F} . Let Σ be a CM type of \mathcal{K} . Then we can attach the CM period $\Omega_\infty = (\Omega_{\infty, \sigma})_\sigma \in (\mathbf{C}^\times)^\Sigma$ to a Néron differential on an abelian scheme $\mathcal{A}/\overline{\mathbf{Z}}$ of CM type (\mathcal{K}, Σ) . Let $p > 2$ be a rational prime and let $\ell \neq p$ be a rational prime and \mathfrak{l} be a prime of \mathcal{F} above ℓ . Let c be the nontrivial element in $\text{Gal}(\mathcal{K}/\mathcal{F})$. We fix an arithmetic Hecke character χ of \mathcal{K}^\times with infinity type $k\Sigma + \kappa(1 - c)$, where k is a positive integer and $\kappa = \sum_{\sigma \in \Sigma} \kappa_\sigma \sigma$ with integers $\kappa_\sigma \geq 0$. For a multi-index $\kappa = \sum_{\sigma \in \Sigma} \kappa_\sigma \sigma \in \mathbf{Z}[\Sigma]$, we write $\Omega_\infty^\kappa = \Omega_{\infty, \sigma}^{\kappa_\sigma}$ and $a^\kappa = a^{\sum_{\sigma \in \Sigma} \kappa_\sigma}$ for $a \in \mathbf{C}^\times$.

Let $\mathcal{K}_{\mathfrak{l}^n}$ be the ray class field of conductor \mathfrak{l}^n and let $\mathcal{K}_{\mathfrak{l}^\infty} = \cup_n \mathcal{K}_{\mathfrak{l}^n}$. Let $\mathcal{K}_{\mathfrak{l}^\infty}^-$ be the maximal pro- ℓ anticyclotomic extension of \mathcal{K} in $\mathcal{K}_{\mathfrak{l}^\infty}$ and let $\Gamma^- = \text{Gal}(\mathcal{K}_{\mathfrak{l}^\infty}^-/\mathcal{K})$. Let $\mathfrak{X}_{\mathfrak{l}}^-$ be the set of finite order characters of Γ^- . For every $\nu \in \mathfrak{X}_{\mathfrak{l}}^-$, we consider the complex number

$$L^{\text{alg}, \mathfrak{l}}(0, \chi\nu) := \frac{\pi^\kappa \Gamma_\Sigma(k\Sigma + \kappa) L^{(\mathfrak{l})}(0, \chi\nu)}{\Omega_\infty^{k\Sigma + 2\kappa}},$$

where $\Gamma_\Sigma(k\Sigma + \kappa) = \prod_{\sigma \in \Sigma} \Gamma(k + \kappa_\sigma)$. It is known that $L^{\text{alg}, \mathfrak{l}}(0, \chi\nu) \in \overline{\mathbf{Z}}_{(p)}$ if p is unramified in \mathcal{F} and prime to the conductor of χ . We are interested in the non-vanishing property of $L^{\text{alg}, \mathfrak{l}}(0, \chi\nu)$ modulo p when ν varies in $\mathfrak{X}_{\mathfrak{l}}^-$. To be precise, we fix two embeddings $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ once and for all and let \mathfrak{m} be the maximal ideal of $\overline{\mathbf{Z}}_{(p)}$ induced by ι_p . We ask if the following non-vanishing modulo p property holds for (χ, \mathfrak{l}) .

$$(NV) \quad \iota_\infty^{-1}(L^{\text{alg}, \mathfrak{l}}(0, \chi\nu)) \not\equiv 0 \pmod{\mathfrak{m}} \text{ for almost all } \nu \in \mathfrak{X}_{\mathfrak{l}}^-.$$

Here *almost all* means "except for finitely many $\nu \in \mathfrak{X}_{\mathfrak{l}}^-$ " if $\dim_{\mathbf{Q}_\ell} F_{\mathfrak{l}} = 1$ and "Zariski dense subset of $\mathfrak{X}_{\mathfrak{l}}^-$ " if $\dim_{\mathbf{Q}_\ell} F_{\mathfrak{l}} > 1$ (See [Hid04a, p.737]).

This problem has been studied extensively by Hida for general CM fields in [Hid04a] and [Hid07] under the hypothesis that Σ is p -ordinary and by T. Finis in [Fin06] for imaginary quadratic fields under a different

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hypothesis. Let $\tau_{\mathcal{K}/\mathcal{F}}$ be the quadratic character associated to \mathcal{K}/\mathcal{F} and $\mathcal{D}_{\mathcal{K}/\mathcal{F}}$ be the different of \mathcal{K}/\mathcal{F} . Let \mathfrak{C} be the conductor of χ . The following theorem is proved by Hida in [Hid07].

Theorem. *Suppose that Σ is p -ordinary and $p > 2$ is unramified in \mathcal{F} . If $(p\mathfrak{l}, \mathfrak{C}) = 1$ and \mathfrak{C} is a product of split prime factors over \mathcal{F} , then (NV) holds for (χ, \mathfrak{l}) unless the following three conditions are satisfied simultaneously:*

- (M1) \mathcal{K}/\mathcal{F} is unramified everywhere,
- (M2) $\tau_{\mathcal{K}/\mathcal{F}}(\mathfrak{c})$ has value -1 , where \mathfrak{c} is the polarization ideal of $\mathcal{A}_{\overline{\mathcal{F}}}$,
- (M3) For all ideal \mathfrak{a} of \mathcal{F} prime to $p\mathfrak{C}$, $\chi\mathbf{N}_{\mathcal{F}/\mathcal{Q}}(\mathfrak{a}) \equiv \tau_{\mathcal{K}/\mathcal{F}}(\mathfrak{a}) \pmod{\mathfrak{m}}$.

We shall say χ is *residually self-dual* if the condition (M3) holds for χ . By [Hid10, Lemma 5.2], the hypotheses (M1-3) is equivalent to the condition (V): χ is residually self-dual, and the root number associated to χ is congruent to -1 modulo \mathfrak{m} .

We are mainly concerned about the (NV) property of *self-dual* characters. Recall that χ is self-dual if $\chi|_{\mathbf{A}_{\mathcal{F}}^{\times}} = \tau_{\mathcal{K}/\mathcal{F}}|\cdot|_{\mathbf{A}_{\mathcal{F}}}$. Such characters are of its own interest because an important class of them arises from Hecke characters associated to CM abelian varieties over totally real fields (*cf.* [Shi98, Thm.20.15]). Note that as the conductor of self-dual characters by definition is divisible by ramified primes, these characters in general are not covered in Hida's theorem unless \mathcal{K}/\mathcal{F} is unramified. Our main motivation for the (NV) property of self-dual characters is the application to Iwasawa main conjecture for CM fields (*cf.* [Hid07] and [Hsi11]). In our subsequent work [Hsi11], this property is used to show the non-vanishing modulo p of the period integral of certain theta functions which is related to Fourier-Jacobi coefficients of Eisenstein series on unitary groups of degree three. When \mathcal{K} is an imaginary quadratic field and \mathfrak{l} splits in \mathcal{K} , the problem of the non-vanishing modulo p of Hecke L -values associated to self-dual characters has been solved completely by T. Finis in [Fin06] through direct study on the period integral of theta functions modulo p (self-dual characters are called *anticyclotomic* in [Fin06]).

We shall state our main result after preparing some notation. Write $\mathfrak{C} = \mathfrak{C}^+ \mathfrak{I} \mathfrak{R}$, where \mathfrak{C}^+ , \mathfrak{I} and \mathfrak{R} are a product of split, inert and ramified prime factors over \mathcal{F} respectively. Let v_p be the p -adic valuation induced by ι_p . For each $v|\mathfrak{C}^-$, let $\mu_p(\chi_v)$ be the local invariant defined by

$$\mu_p(\chi_v) := \inf_{x \in \mathcal{K}_v^{\times}} v_p(\chi(x) - 1).$$

Note that $\mu_p(\chi_v)$ agrees with the one defined in [Fin06] when χ is self-dual. Following Hida, we make the following hypotheses for (p, \mathcal{K}, Σ) :

- (unr) $p > 2$ is unramified in \mathcal{F} ;
- (ord) Σ is p -ordinary.

Our main result is as follows.

Theorem A. *Let χ be a self-dual Hecke character of \mathcal{K}^{\times} such that*

- (L) $\mu_p(\chi_v) = 0$ for every $v|\mathfrak{C}^-$,
- (R) The global root number $W(\chi^*) = 1$, where $\chi^* := \chi|\cdot|_{\mathbf{A}_{\mathcal{K}}}^{-\frac{1}{2}}$,
- (C) \mathfrak{R} is square-free.

In addition to (unr), (ord), we further assume

- $(p\mathfrak{l}, \mathcal{D}_{\mathcal{K}/\mathcal{F}}\mathfrak{C}) = 1$,
- \mathfrak{l} splits in \mathcal{K} .

Then (NV) holds for (χ, \mathfrak{l}) .

Note that as χ is self-dual, the assumption (R) is equivalent to Hida's condition (V). Indeed, the assumptions (L) and (R) are necessary for the (NV) property. The assumption (R) is due to the functional equation of the complex L -function $L(s, \chi)$, and the failure of (NV) without (L) has been observed by Gillard (*cf.* [Fin06, Theorem 1.1]). We remark that our result in particular can be applied to Hecke characters attached to certain CM elliptic curves over totally real fields. For example, let E be an elliptic curve over \mathcal{F} with CM by an imaginary quadratic field \mathcal{M} . Let $\mathcal{K} = \mathcal{F}\mathcal{M}$ and let χ be the Hecke character of \mathcal{K}^{\times} such that $L(s, \chi^{-1}) = L(E/\mathcal{F}, s)$. Then it is well known that the assumptions (L) and (C) hold if $(\mathcal{D}_{\mathcal{K}/\mathcal{F}}, \#(\mathcal{O}_{\mathcal{M}}^{\times})) = 1$

and $p > 3$. In general, (C) is expected to be unnecessary. The very reason we impose them is due to the difficulty of the computation of certain Gauss sums $A_\beta(\chi) = A_\beta(\chi_s)|_{s=0}$ defined in (4.14). We leave the removal of (C) to our forthcoming paper [Hsi14, §6].

We also consider the case χ is not residually self-dual. In particular, this implies the failure of (V). We prove the following result in Corollary 6.5, which gives a partial generalization of Hida's theorem.

Theorem B. *Suppose that (unr), (ord) and $(p\mathfrak{l}, \mathcal{D}_{\mathcal{K}/\mathcal{F}}\mathfrak{C}) = 1$. Suppose further that the following conditions hold:*

- (L) $\mu_p(\chi_v) = 0$ for every $v|\mathfrak{C}^-$,
- (N) χ is not residually self-dual.

Then (NV) holds for (χ, \mathfrak{l}) .

The proof is based on Hida's ideas in [Hid04a], where Hida provided a general strategy to study the problem of the non-vanishing of Hecke L -values modulo p via a study on the Fourier coefficients of Eisenstein series. The starting point of Hida is Damerell's formula, which relates a sum of suitable Eisenstein series evaluated at CM points to Hecke L -values for CM fields. And then he proves a key result on Zariski density of CM points in Hilbert modular varieties modulo p , by which he is able to reduce the problem to non-vanishing of an Eisenstein series modulo p using a variant of Sinnott's argument. The assumption that \mathfrak{C} is a product of split primes solely results from the difficulty of the calculation of Fourier coefficients of Eisenstein series. Following Hida's strategy, we first construct an Eisenstein measure which interpolates the Hecke L -values by the evaluation at CM points. The construction of our Eisenstein measure is from representation theoretic point of view, and Damerell's formula is actually a period integral of Eisenstein series against a non-split torus. Fourier coefficients of our Eisenstein series are decomposed into a product of local Whittaker integrals. Through an explicit calculation of these local integrals, we find that some Fourier coefficient is non-zero modulo p provided that certain epsilon dichotomy holds (See Proposition 6.7).

Here is the outline of this article. We fix notation and recall some basic facts about Hilbert modular varieties and CM points in the first three sections. We basically follow the exposition in [Hid04a] except that we use an adelic description of CM points. Readers who are familiar with [Hid04a] may begin with §4, which is the bulk of this paper. In §4, we give the construction of Eisenstein series and the calculation of some local Whittaker integrals. The formulas of the key integrals $\tilde{A}_\beta(\chi)$ are summarized in Proposition 4.4 and Proposition 4.5. The explicit calculation of the period integral of our Eisenstein series is carried out in §5. Finally we show some Fourier coefficient of our Eisenstein series is non-zero modulo p in §6.

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1. NOTATION AND DEFINITIONS

1.1. Throughout \mathcal{F} is a totally real field of degree d over \mathbf{Q} and \mathcal{K} is a totally imaginary quadratic extension of \mathcal{F} . Let c be the complex conjugation, the unique non-trivial element in $\text{Gal}(\mathcal{K}/\mathcal{F})$. Let O (resp. R) be the ring of integer of \mathcal{F} (resp. \mathcal{K}). Let $\mathcal{D}_{\mathcal{F}}$ (resp. $D_{\mathcal{F}}$) be the different (resp. discriminant) of \mathcal{F}/\mathbf{Q} . Let $\mathcal{D}_{\mathcal{K}/\mathcal{F}}$ be the different of \mathcal{K}/\mathcal{F} . For every fractional ideal \mathfrak{b} of O , set $\mathfrak{b}^* = \mathfrak{b}^{-1}\mathcal{D}_{\mathcal{F}}^{-1}$. Denote by $\mathfrak{a} = \text{Hom}(\mathcal{F}, \mathbf{C})$ the set of archimedean places of \mathcal{F} . Denote by \mathfrak{h} (resp. $\mathfrak{h}_{\mathcal{K}}$) the set of finite places of \mathcal{F} (resp. \mathcal{K}). We often write v for a place of \mathcal{F} and w for the place of \mathcal{K} above v . Denote by \mathcal{F}_v the completion of \mathcal{F} at v and by ϖ_v a uniformizer of \mathcal{F}_v . Let $\mathcal{K}_v = \mathcal{F}_v \otimes_{\mathcal{F}} \mathcal{K}$.

Fix two rational primes $p \neq \ell$. Let \mathfrak{l} be a prime of \mathcal{F} above ℓ . Let Σ be a fixed CM type of \mathcal{K} as in the introduction. We shall identify Σ with \mathfrak{a} by the restriction to \mathcal{F} . We assume (unr) and (ord) for (p, \mathcal{K}, Σ) throughout this article. Let

$$\Sigma_p = \{w \in \mathfrak{h}_{\mathcal{K}} \mid w|p \text{ and } w \text{ is induced by } \iota_p \circ \sigma \text{ for } \sigma \in \Sigma\}.$$

We recall that Σ is p -ordinary if $\Sigma_p \cap \Sigma_p c = \emptyset$ and $\Sigma_p \cup \Sigma_p c = \{w \in \mathfrak{h}_{\mathcal{K}} \mid w|p\}$. Note that (ord) implies that every prime of \mathcal{F} above p splits in \mathcal{K} .

1.2. If L is a number field, \mathbf{A}_L is the adèle of L and $\mathbf{A}_{L,f}$ is the finite part of \mathbf{A}_L . The ring of integers of L is denoted by \mathcal{O}_L . For $a \in \mathbf{A}_L$, we put

$$\mathrm{it}_L(a) := a(\mathcal{O}_L \otimes \widehat{\mathbf{Z}}) \cap L.$$

Let $\psi_{\mathbf{Q}}$ be the standard additive character of $\mathbf{A}_{\mathbf{Q}}/\mathbf{Q}$ such that $\psi_{\mathbf{Q}}(x_{\infty}) = \exp(2\pi i x_{\infty})$, $x_{\infty} \in \mathbf{R}$. We define $\psi_L : \mathbf{A}_L/L \rightarrow \mathbf{C}^{\times}$ by $\psi_L(x) = \psi_{\mathbf{Q}} \circ \mathrm{T}_{L/\mathbf{Q}}(x)$. For $\beta \in L$, $\psi_{L,\beta}(x) = \psi_L(\beta x)$. If $L = \mathcal{F}$, we write ψ for $\psi_{\mathcal{F}}$.

We choose once and for all an embedding $\iota_{\infty} : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and an isomorphism $\iota : \mathbf{C} \simeq \mathbf{C}_p$, where \mathbf{C}_p is the completion of an algebraic closure of \mathbf{Q}_p . Let $\iota_p = \iota_{\infty} \circ \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ be their composition. We regard L as a subfield in \mathbf{C} (resp. \mathbf{C}_p) via ι_{∞} (resp. ι_p) and $\mathrm{Hom}(L, \overline{\mathbf{Q}}) = \mathrm{Hom}(L, \mathbf{C}_p)$.

Let $\overline{\mathbf{Z}}$ be the ring of algebraic integers of $\overline{\mathbf{Q}}$ and let $\overline{\mathbf{Z}}_p$ be the p -adic completion of $\overline{\mathbf{Z}}$ in \mathbf{C}_p with the maximal ideal \mathfrak{m}_p . Let $\mathfrak{m} = \iota_p^{-1}(\mathfrak{m}_p)$.

1.3. Let F be a local field. Denote by $|\cdot|_F$ the absolute value of F . We often drop the subscript F if it is clear from the context. We fix the choice of our Haar measure dx on F . If $F = \mathbf{R}$, dx is the Lebesgue measure on \mathbf{R} . If $F = \mathbf{C}$, dx is the twice the Lebesgue measure. If F is a non-archimedean local field, dx (resp. $d^{\times}x$) is the Haar measure on F (resp. F^{\times}) normalized so that $\mathrm{vol}(\mathcal{O}_F, dx) = 1$ (resp. $\mathrm{vol}(\mathcal{O}_F^{\times}, d^{\times}x) = 1$). If $\mu : F^{\times} \rightarrow \mathbf{C}^{\times}$ is a character of F^{\times} , define

$$a(\mu) = \inf \{n \in \mathbf{Z}_{\geq 0} \mid \mu|_{1+\varpi_v^n \mathcal{O}_v} = 1\}.$$

2. HILBERT MODULAR VARIETIES AND HILBERT MODULAR FORMS

2.1. We follow the exposition in [Hid04b, §4.2]. Let $V = \mathcal{F}e_1 \oplus \mathcal{F}e_2$ be a two dimensional \mathcal{F} -vector space and $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathcal{F}$ be the \mathcal{F} -bilinear alternating pairing defined by $\langle e_1, e_2 \rangle = 1$. Let $\mathcal{L} = \mathcal{O}e_1 \oplus \mathcal{O}^*e_2$ be the standard \mathcal{O} -lattice in V . Let $G = \mathrm{GL}_2/\mathcal{F}$. We identify vectors in V with row vectors according to the basis e_1, e_2 , so G has a natural right action on V .

For each finite place v of \mathcal{F} , we put

$$K_v^0 = \{g \in G(\mathcal{F}_v) \mid (\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_v)g = \mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_v\}.$$

Let $K^0 = \prod_{v \in \mathfrak{h}} K_v^0$ and $K_p^0 = \prod_{v|p} K_v^0$. For a prime-to- $p\ell$ positive integer N , we define an open-compact subgroup $U(N)$ of $G(\mathbf{A}_{\mathcal{F},f})$ by

$$(2.1) \quad U(N) := \{g \in G(\mathbf{A}_{\mathcal{F},f}) \mid g \equiv 1 \pmod{N\mathcal{L}}\}.$$

Let K be an open-compact subgroup of $G(\mathbf{A}_{\mathcal{F},f})$ such that $K_p = K_p^0$. We assume that $K \supset U(N)$ for some N as above and that K is sufficiently small so that the following condition holds:

$$(neat) \quad K \text{ is neat and } \det(K) \cap \mathcal{O}_+^{\times} \subset (K \cap \mathcal{O}^{\times})^2.$$

2.2. **Kottwitz models.** We first review Kottwitz models of Hilbert modular varieties.

Definition 2.1 (*S*-quadruples). Let \square be a finite set of rational primes and let $\mathcal{W}_{(\square)} = \mathbf{Z}_{(\square)}[\zeta_N]$, $\zeta = \exp(\frac{2\pi i}{N})$. Define the fibered category $\mathcal{A}_K^{(\square)}$ over $SC\mathcal{H}/\mathcal{W}_{(\square)}$ as follows. Let S be a locally noetherian connected $\mathcal{W}_{(\square)}$ -scheme and let \bar{s} be a geometric point of S . Objects are abelian varieties with real multiplication (AVRM) over S of level K , i.e. a *S*-quadruple $\underline{A} = (A, \bar{\lambda}, \iota, \bar{\eta}^{(\square)})_S$ consisting of the following data:

- (1) A is an abelian scheme of dimension d over S .
- (2) $\iota : \mathcal{O} \hookrightarrow \mathrm{End}_S A \otimes_{\mathbf{Z}} \mathbf{Z}_{(\square)}$.
- (3) λ is a prime-to- \square polarization of A over S and $\bar{\lambda}$ is the $\mathcal{O}_{(\square),+}$ -orbit of λ . Namely

$$\bar{\lambda} = \mathcal{O}_{(\square),+}\lambda := \{\lambda' \in \mathrm{Hom}(A, A^t) \otimes_{\mathbf{Z}} \mathbf{Z}_{(\square)} \mid \lambda' = \lambda \circ a, a \in \mathcal{O}_{(\square),+}\}.$$

- (4) $\bar{\eta}^{(\square)} = \eta^{(\square)}K^{(\square)}$ is a $\pi_1(S, \bar{s})$ -invariant $K^{(p)}$ -orbit of isomorphisms of $\mathcal{O}_{\mathcal{K}}$ -modules $\eta^{(\square)} : \mathcal{L} \otimes_{\mathbf{Z}} \mathbf{A}_f^{(\square)} \xrightarrow{\sim} V^{(\square)}(A_{\bar{s}}) := H_1(A_{\bar{s}}, \mathbf{A}_f^{(\square)})$. Here we define $\eta^{(\square)}g$ for $g \in G(\mathbf{A}_{\mathcal{F},f}^{(\square)})$ by $\eta^{(\square)}g(x) = \eta^{(\square)}(g * x)$.

Furthermore, $(A, \bar{\lambda}, \iota, \bar{\eta}^{(\square)})_S$ satisfies the following conditions:

- Let t denote the Rosati involution induced by λ on $\mathrm{End}_S A \otimes_{\mathbf{Z}} \mathbf{Z}_{(\square)}$. Then $\iota(b)^t = \iota(b)$, $\forall b \in \mathcal{O}$.

- Let e^λ be the Weil pairing induced by λ . Lifting the isomorphism $\mathbf{Z}/N\mathbf{Z} \simeq \mathbf{Z}/N\mathbf{Z}(1)$ induced by ζ_N to an isomorphism $\zeta : \widehat{\mathbf{Z}} \simeq \widehat{\mathbf{Z}}(1)$, we can regard e^λ as an \mathcal{F} -alternating form $e^\lambda : V^{(\square)}(A_{\bar{s}}) \times V^{(\square)}(A_{\bar{s}}) \rightarrow \mathcal{D}_{\mathcal{F}}^{-1} \otimes_{\mathbf{Z}} \mathbf{A}_{\mathcal{F},f}^{(\square)}$. Let e^η denote the \mathcal{F} -alternating form on $V^{(\square)}(A_{\bar{s}})$ induced by $e^\eta(x, x') = \langle x\eta, x'\eta \rangle$. Then

$$e^\lambda = u \cdot e^\eta \text{ for some } u \in \mathbf{A}_{\mathcal{F},f}^{(\square)}.$$

- As $O \otimes_{\mathbf{Z}} \mathcal{O}_S$ -modules, we have an isomorphism $\text{Lie } A \simeq O \otimes_{\mathbf{Z}} \mathcal{O}_S$ locally under Zariski topology of S . For two S -quadruples $\underline{A} = (A, \bar{\lambda}, \iota, \bar{\eta}^{(\square)})_S$ and $\underline{A}' = (A', \bar{\lambda}', \iota', (\bar{\eta}')^{(\square)})_S$, we define the morphisms by

$$\text{Hom}_{\mathcal{A}_K^{(\square)}}(\underline{A}, \underline{A}') = \left\{ \phi \in \text{Hom}_O(A, A') \mid \phi^* \bar{\lambda}' = \bar{\lambda}, \phi \circ (\bar{\eta}')^{(\square)} = \bar{\eta}^{(\square)} \right\}.$$

We say $\underline{A} \sim \underline{A}'$ (resp. $\underline{A} \simeq \underline{A}'$) if there exists a prime-to- \square isogeny (resp. isomorphism) in $\text{Hom}_{\mathcal{A}_K^{(\square)}}(\underline{A}, \underline{A}')$.

We consider the cases when $\square = \emptyset$ and $\{p\}$. When $\square = \emptyset$ is the empty set and $\mathcal{W}_{(\square)} = \mathbf{Q}(\zeta_N)$, we define the functor $\mathcal{E}_K : SCH_{/\mathbf{Q}(\zeta_N)} \rightarrow SETS$ by

$$\mathcal{E}_K(S) = \{ \underline{A} = (A, \bar{\lambda}, \iota, \bar{\eta})_S \in \mathcal{A}_K(S) \} / \sim.$$

By the theory of Shimura-Deligne, \mathcal{E}_K is represented by a quasi-projective scheme Sh_K over $\mathbf{Q}(\zeta_N)$. We define the functor $\mathfrak{E}_K : SCH_{/\mathbf{Q}} \rightarrow SETS$ by

$$\mathfrak{E}_K(S) = \left\{ (A, \bar{\lambda}, \iota, \bar{\eta}) \in \mathcal{A}_K^{(\square)}(S) \mid \eta^{(\square)}(\mathcal{L} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}) = H_1(A_{\bar{s}}, \widehat{\mathbf{Z}}) \right\} / \simeq.$$

By the discussion in [Hid04b, p.136], we have $\mathfrak{E}_K \xrightarrow{\sim} \mathcal{E}_K$ under the hypothesis (neat).

When $\square = \{p\}$, we write \mathcal{W} for $\mathcal{W}_{(p)}$ and define functor $\mathcal{E}_K^{(p)} : SCH_{/\mathcal{W}} \rightarrow SETS$ by

$$\mathcal{E}_K^{(p)}(S) = \left\{ \underline{A} = (A, \bar{\lambda}, \iota, \bar{\eta}^{(p)})_S \in \mathcal{A}_{K^{(p)}}^{(p)}(S) \right\} / \sim.$$

In [Kot92], Kottwitz shows $\mathcal{E}_K^{(p)}$ is representable by a quasi-projective scheme $Sh_K^{(p)}$ over \mathcal{W} if K is neat. Similarly we define the functor $\mathfrak{E}_K^{(p)} : SCH_{/\mathcal{W}} \rightarrow SETS$ by

$$\mathfrak{E}_K^{(p)}(S) = \left\{ (A, \bar{\lambda}, \iota, \bar{\eta}^{(p)}) \in \mathcal{A}_K^{(p)}(S) \mid \eta^{(p)}(\mathcal{L} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}^{(p)}) = H_1(A_{\bar{s}}, \widehat{\mathbf{Z}}^{(p)}) \right\} / \simeq.$$

It is shown in [Hid04b, §4.2.1] that $\mathfrak{E}_K^{(p)} \xrightarrow{\sim} \mathcal{E}_K^{(p)}$.

2.3. Igusa schemes. For each positive integer n , put

$$K^n := \left\{ g = (g_v) \in K \mid g_p \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \pmod{p^n} \right\}$$

Definition 2.2 (S -quintuples). Let n be a positive integer. We define the fibered category $\mathcal{A}_{K,n}^{(p)}$ whose objects are AVRMs over an \mathcal{W} -scheme of level K^n , i.e. a S -quintuple $(\underline{A}, j)_S$ consisting of a S -quadruple $\underline{A} = (A, \bar{\lambda}, \iota, \bar{\eta}^{(p)}) \in \mathcal{A}_{K^{(p)}}^{(p)}(S)$ and a monomorphism

$$j : O^* \otimes \mu_{p^n} \hookrightarrow A[p^n]$$

as O -group schemes over S . We call j a level- p^n structure of A . Morphisms are

$$\text{Hom}_{\mathcal{A}_{K,n}^{(p)}}((\underline{A}, j), (\underline{A}', j')) = \left\{ \phi \in \text{Hom}_{\mathcal{A}_{K^{(p)}}^{(p)}}(\underline{A}, \underline{A}') \mid \phi j = j' \right\}.$$

Define the functor $\mathfrak{J}_{K,n}^{(p)} : SCH_{/\mathcal{W}} \rightarrow SETS$ by

$$\mathfrak{J}_{K,n}^{(p)}(S) = \left\{ (\underline{A}, j) = (A, \bar{\lambda}, \iota, \bar{\eta}^{(p)}, j)_S \in \mathcal{A}_{K,n}^{(p)}(S) \mid \eta^{(p)}(\mathcal{L} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}^{(p)}) = T^{(p)}(A) \right\} / \simeq.$$

It is known that $\mathfrak{J}_{K,n}^{(p)}$ are relatively representable over $\mathfrak{E}_K^{(p)}$ (cf. [SGA64, Prop. 3.12]), so it is represented by a scheme over \mathcal{W} , which we denote by $I_{K,n}$.

For $n \geq n' > 0$, the natural morphism $\pi_{n,n'} : I_{K,n} \rightarrow I_{K,n'}$ induced by the inclusion $O^* \otimes \mu_{p^{n'}} \hookrightarrow O^* \otimes \mu_{p^n}$ is finite étale. The forgetful morphism $\pi : I_{K,n} \rightarrow Sh_K^{(p)}$ defined by $\pi : (\underline{A}, j) \mapsto \underline{A}$ are étale for all $n > 0$. Hence $I_{K,n}$ is smooth over $\text{Spec } \mathcal{W}$. The image of π is the pre-image of ordinary abelian schemes in $I_{K,n} \otimes \bar{\mathbb{F}}_p$.

2.4. Complex uniformization. We describe the complex points $Sh_K(\mathbf{C})$. Put

$$X^+ = \{\tau = (\tau_\sigma)_{\sigma \in \mathbf{a}} \in \mathbf{C}^{\mathbf{a}} \mid \text{Im } \tau_\sigma > 0 \text{ for all } \sigma \in \mathbf{a}\}.$$

Let \mathcal{F}_+ be the set of totally positive elements in \mathcal{F} and let $G(\mathcal{F})^+ = \{g \in G(\mathcal{F}) \mid \det g \in \mathcal{F}_+\}$. Define the complex Hilbert modular variety by

$$M(X^+, K) := G(\mathcal{F})^+ \backslash X^+ \times G(\mathbf{A}_{\mathcal{F}, f}) / K.$$

It is well known that $M(X^+, K) \xrightarrow{\sim} Sh_K(\mathbf{C})$ by the theory of abelian varieties over \mathbf{C} .

For $\tau = (\tau_\sigma)_{\sigma \in \mathbf{a}} \in X^+$, we let p_τ be the period map $V \otimes_{\mathbf{Q}} \mathbf{R} \xrightarrow{\sim} \mathbf{C}^{\mathbf{a}}$ defined by $p_\tau(ae_1 + be_2) = a\tau + b$, $a, b \in \mathcal{F} \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{R}^{\mathbf{a}}$. We can associate a AVRMS to $(\tau, g) \in X^+ \times G(\mathbf{A}_{\mathcal{F}, f})$ as follows.

- The complex abelian variety $\mathcal{A}_g(\tau) = \mathbf{C}^{\mathbf{a}} / p_\tau(\mathcal{L}_g)$, where $\mathcal{L}_g := (\mathcal{L} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}})g^{-1} \cap V$.
- The \mathcal{F}_+ -orbit of polarization $\langle \cdot, \cdot \rangle_{can}$ on $\mathcal{A}_g(\tau)$ is given by the Riemann form $\langle \cdot, \cdot \rangle \circ p_\tau^{-1}$.
- The $\iota_{\mathbf{C}} : O \hookrightarrow \text{End } \mathcal{A}_g(\tau) \otimes_{\mathbf{Z}} \mathbf{Q}$ is induced from the pull back of the natural \mathcal{F} -action on V via p_τ .
- The level structure $\eta_g : \mathcal{L} \otimes_{\mathbf{Z}} \mathbf{A}_f \xrightarrow{\sim} (g * \mathcal{L}) \otimes_{\mathbf{Z}} \mathbf{A}_f = H_1(\mathcal{A}_g(\tau), \mathbf{A}_f)$ is defined by $\eta_g(v) = vg^{-1}$.

Let $\underline{\mathcal{A}}_g(\tau)$ denote the \mathbf{C} -quadruple $(\mathcal{A}_g(\tau), \langle \cdot, \cdot \rangle_{can}, \iota, K\eta_g)$. Then $[(\tau, g)] \mapsto [\underline{\mathcal{A}}_g(\tau)]$ gives rise to an isomorphism $M(X^+, K) \xrightarrow{\sim} Sh_K(\mathbf{C})$.

Let $\underline{z} = \{z_\sigma\}_{\sigma \in \mathbf{a}}$ be the standard complex coordinates of $\mathbf{C}^{\mathbf{a}}$ and $d\underline{z} = \{dz_\sigma\}_{\sigma \in \mathbf{a}}$. Then O -action on $d\underline{z}$ is given by $\iota_{\mathbf{C}}(\alpha)^* dz_\sigma = \sigma(\alpha) dz_\sigma$, $\sigma \in \mathbf{a} = \text{Hom}(\mathcal{F}, \mathbf{C})$. Let $z = z_{id}$ be the coordinate corresponding to $\iota_\infty : \mathcal{F} \hookrightarrow \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Then

$$(2.2) \quad (O \otimes_{\mathbf{Z}} \mathbf{C})dz = H^0(\mathcal{A}_g(\tau), \Omega_{\mathcal{A}_g(\tau)/\mathbf{C}}).$$

2.5. Hilbert modular forms.

2.5.1. For $\tau \in \mathbf{C}$ and $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2(\mathbf{R})$, we put

$$(2.3) \quad J(g, \tau) = c\tau + d.$$

For $\tau = (\tau_\sigma)_{\sigma \in \mathbf{a}} \in X^+$ and $g_\infty = (g_\sigma)_{\sigma \in \mathbf{a}} \in G(\mathcal{F} \otimes_{\mathbf{Q}} \mathbf{R})$, we put

$$\underline{J}(g_\infty, \tau) = \prod_{\sigma \in \mathbf{a}} J(g_\sigma, \tau_\sigma).$$

Definition 2.3. Denote by $\mathbf{M}_k(K, \mathbf{C})$ the space of holomorphic Hilbert modular form of parallel weight k and level K . Each $\mathbf{f} \in \mathbf{M}_k(K, \mathbf{C})$ is a \mathbf{C} -valued function $\mathbf{f} : X^+ \times G(\mathbf{A}_{\mathcal{F}, f}) \rightarrow \mathbf{C}$ such that the function $\mathbf{f}(-, g_f) : X^+ \rightarrow \mathbf{C}$ is holomorphic for each $g_f \in G(\mathbf{A}_{\mathcal{F}, f})$ and

$$\mathbf{f}(\alpha(\tau, g_f)u) = \underline{J}(\alpha, \tau)^{k\Sigma} \mathbf{f}(\tau, g_f) \text{ for all } u \in K \text{ and } \alpha \in G(\mathcal{F})^+.$$

2.5.2. *Fourier expansion.* For every $\mathbf{f} \in \mathbf{M}_k(K, \mathbf{C})$, we have the Fourier expansion

$$\mathbf{f}(\tau, g_f) = \sum_{\beta \in \mathcal{F}_+ \cup \{0\}} W_\beta(\mathbf{f}, g_f) e^{2\pi i \Gamma_{\mathcal{F}/\mathbf{Q}}(\beta\tau)}.$$

We call $W_\beta(\mathbf{f}, g_f)$ the β -th Fourier coefficient of \mathbf{f} at g_f .

For a semi-group L in \mathcal{F} , let $L_+ = \mathcal{F}_+ \cap L$ and $L_{\geq 0} = L_+ \cup \{0\}$. If B is a ring, we denote by $B[[L]]$ the set of all formal series

$$\sum_{\beta \in L} a_\beta q^\beta, \quad a_\beta \in B.$$

Let $a, b \in (\mathbf{A}_{\mathcal{F}, f}^{(pN)})^\times$ and let $\mathbf{a} = \text{il}_{\mathcal{F}}(a)$ and $\mathbf{b} = \text{il}_{\mathcal{F}}(b)$. The q -expansion of \mathbf{f} at the cusp (\mathbf{a}, \mathbf{b}) is given by

$$(2.4) \quad \mathbf{f}|_{(\mathbf{a}, \mathbf{b})}(q) = \sum_{\beta \in (N^{-1}\mathbf{a}\mathbf{b})_{\geq 0}} W_\beta(\mathbf{f}, \begin{bmatrix} b^{-1} & 0 \\ 0 & a \end{bmatrix}) q^\beta \in \mathbf{C}[[(N^{-1}\mathbf{a}\mathbf{b})_{\geq 0}]].$$

If B is a \mathcal{W} -algebra in \mathbf{C} , we put

$$\mathbf{M}_k(K, B) = \{\mathbf{f} \in \mathbf{M}_k(K, \mathbf{C}) \mid \mathbf{f}|_{(\mathbf{a}, \mathbf{b})}(q) \in B[[(N^{-1}\mathbf{a}\mathbf{b})_{\geq 0}]] \text{ at all cusps } (\mathbf{a}, \mathbf{b})\}.$$

2.5.3. *Tate objects.* Let \mathcal{S} be a set of d -linear \mathbf{Q} -independent elements in $\text{Hom}(\mathcal{F}, \mathbf{Q})$ such that $l(\mathcal{F}_+) > 0$ for $l \in \mathcal{S}$. If L is a lattice in \mathcal{F} and n a positive integer, let $L_{\mathcal{S}, n} = \{x \in L \mid l(x) > -n \text{ for all } l \in \mathcal{S}\}$ and put $B((L; \mathcal{S})) = \lim_{n \rightarrow \infty} B[L_{\mathcal{S}, n}]$. To a pair (\mathbf{a}, \mathbf{b}) of two prime-to- pN fractional ideals, we can attach the Tate AVRMS $\text{Tate}_{\mathbf{a}, \mathbf{b}}(q) = \mathbf{a}^* \otimes_{\mathbf{Z}} \mathbb{G}_m / q^{\mathbf{b}}$ over $\mathbf{Z}((\mathbf{a}\mathbf{b}; \mathcal{S}))$ with O -action ι_{can} . As described in [Kat78], $\text{Tate}_{\mathbf{a}, \mathbf{b}}(q)$ has a canonical $\mathbf{a}\mathbf{b}^{-1}$ -polarization λ_{can} and also carries ω_{can} a canonical $O \otimes \mathbf{Z}((\mathbf{a}\mathbf{b}; \mathcal{S}))$ -generator of $\Omega_{\text{Tate}_{\mathbf{a}, \mathbf{b}}}$ induced by the isomorphism $\text{Lie}(\text{Tate}_{\mathbf{a}, \mathbf{b}}(q) / \mathbf{Z}((\mathbf{a}\mathbf{b}; \mathcal{S}))) = \mathbf{a}^* \otimes_{\mathbf{Z}} \text{Lie}(\mathbb{G}_m) \simeq \mathbf{a}^* \otimes \mathbf{Z}((\mathbf{a}\mathbf{b}; \mathcal{S}))$. Let $\mathcal{L}_{\mathbf{a}, \mathbf{b}} = \mathcal{L} \cdot \begin{bmatrix} \mathbf{b} \\ \mathbf{a}^{-1} \end{bmatrix} = \mathbf{b}e_1 \oplus \mathbf{a}^*e_2$. Then we have a level N -structure $\eta_{can} : N^{-1}\mathcal{L}_{\mathbf{a}, \mathbf{b}} / \mathcal{L}_{\mathbf{a}, \mathbf{b}} \xrightarrow{\sim} \text{Tate}_{\mathbf{a}, \mathbf{b}}(q)[N]$ over $\mathbf{Z}[\zeta_N]((N^{-1}\mathbf{a}\mathbf{b}; \mathcal{S}))$ induced by the fixed primitive N -th root of unity ζ_N . We write $\underline{\text{Tate}}_{\mathbf{a}, \mathbf{b}}$ for the Tate $\mathbf{Z}((\mathbf{a}\mathbf{b}; \mathcal{S}))$ -quadruple $(\text{Tate}_{\mathbf{a}, \mathbf{b}}(q), \overline{\lambda_{can}}, \iota_{can}, \overline{\eta_{can}}^{(p)})$ at (\mathbf{a}, \mathbf{b}) . In addition, since \mathbf{a} is prime to p , we let $\eta_{p, can}^0 : O^* \otimes_{\mathbf{Z}} \mu_{p^n} = \mathbf{a}^* \otimes_{\mathbf{Z}} \mu_{p^n} \hookrightarrow \text{Tate}_{\mathbf{a}, \mathbf{b}}(q)$ be the canonical level p^n -structure induced by the natural inclusion $\mathbf{a}^* \otimes_{\mathbf{Z}} \mu_{p^n} \hookrightarrow \mathbf{a}^* \otimes_{\mathbf{Z}} \mathbb{G}_m$.

2.5.4. *Geometric modular forms.* We collect here definitions and basic facts of geometric modular forms. For the precise theory, we refer to [Kat78] or [Hid04b]. Let $T = \text{Res}_{O/\mathbf{Z}} \mathbb{G}_m$ and $\kappa \in \text{Hom}(T, \mathbb{G}_m)$. Let B be a $\mathbf{Z}_{(p)}$ -algebra. Consider $[\underline{A}] = [(A, \bar{\lambda}, \iota, \bar{\eta}^{(p)})] \in \mathfrak{E}_K(C)$ for a B -algebra C with a differential form ω generating $H^0(A, \Omega_{A/C})$ over $O \otimes_{\mathbf{Z}} C$. A geometric modular form f over B of weight κ and level K is a functorial rule of assigning a value $f(\underline{A}, \omega) \in C$ satisfying the following axioms.

- (G1) $f(\underline{A}, \omega) = f(\underline{A}', \omega') \in C$ if $(\underline{A}, \omega) \simeq (\underline{A}', \omega')$ over C ,
- (G2) For a B -algebra homomorphism $\varphi : C \rightarrow C'$, we have

$$f((\underline{A}, \omega) \otimes_C C') = \varphi(f(\underline{A}, \omega)),$$

- (G3) $f(\underline{A}, a\omega) = \kappa(a^{-1})f(\underline{A}, \omega)$ for all $a \in T(C) = (O \otimes_{\mathbf{Z}} C)^\times$,

- (G4) $f(\underline{\text{Tate}}_{\mathbf{a}, \mathbf{b}}, \omega_{can}) \in B[\zeta_N][[(N^{-1}\mathbf{a}\mathbf{b})_{\geq 0}]]$ at all cusps (\mathbf{a}, \mathbf{b}) .

For a positive integer k , we regard $k \in \text{Hom}(T, \mathbb{G}_m)$ as the character $t \mapsto \mathbf{N}_{\mathcal{F}/\mathbf{Q}}(t)^k$. We denote by $\mathcal{M}_k(K, B)$ the space of geometric modular forms over B of weight k and level K .

For each $f \in \mathcal{M}_k(K, \mathbf{C})$, we regard f as a holomorphic Hilbert modular form of weight k and level K by

$$f(\tau, g_f) = f(\mathcal{A}_g(\tau), \langle \cdot, \cdot \rangle_{can}, \iota_{\mathbf{C}}, \bar{\eta}_g, 2\pi idz),$$

where dz is the differential form in (2.2). By GAGA principle, this gives rise to an isomorphism $\mathcal{M}_k(K, \mathbf{C}) \xrightarrow{\sim} \mathbf{M}_k(K, \mathbf{C})$. As discussed in [Kat78, §1.7], the evaluation $\mathbf{f}(\underline{\text{Tate}}_{\mathbf{a}, \mathbf{b}}, \omega_{can})$ is independent of the auxiliary choice of \mathcal{S} in the construction of the Tate object. Moreover, we have the following important identity which bridges holomorphic modular forms and geometric modular forms.

$$\mathbf{f}|_{(\mathbf{a}, \mathbf{b})}(q) = \mathbf{f}(\underline{\text{Tate}}_{\mathbf{a}, \mathbf{b}}, \omega_{can}) \in \mathbf{C}[[(N^{-1}\mathbf{a}\mathbf{b})_{\geq 0}]].$$

By q -expansion principle, if B is \mathcal{W} -algebra in \mathbf{C} , then $\mathcal{M}_k(K, B) = \mathbf{M}_k(K, B)$.

2.5.5. *p -adic modular forms.* Let B be a p -adic ring in \mathbf{C}_p . Let $V(K, B)$ be the space of Katz p -adic modular forms over B defined by

$$V(K, B) := \varprojlim_m \varinjlim_n H^0(I_{K, n/B/p^m B}, \mathcal{O}_{I_{K, n}}).$$

In other words, Katz p -adic modular forms are formal functions on Igusa towers.

Let C be a $B/p^m B$ -algebra. For each C -point $[(\underline{A}, j)] \in \varprojlim_m \varinjlim_n I_{K, n}(C)$, the level p^∞ -structure j induces an isomorphism $j_* : O^* \otimes_{\mathbf{Z}} \text{Lie}(\widehat{\mathbb{G}}_m/C) = O^* \otimes_{\mathbf{Z}} C \xrightarrow{\sim} \text{Lie}(A)$. Let dt/t be the canonical invariant differential form of $\widehat{\mathbb{G}}_m$. Then $j^* dt/t := dt/t \circ j_*$ is a generator of $H^0(A, \Omega_A)$ as a $O \otimes_{\mathbf{Z}} C$ -module. We thus have a natural injection

$$(2.5) \quad \begin{aligned} \mathcal{M}_k(K, B) &\hookrightarrow V(K, B) \\ f &\mapsto \widehat{f}(\underline{A}, j) := f(\underline{A}, j_* dt/t) \end{aligned}$$

which preserves the q -expansions in the sense that $\widehat{f}|_{(\mathbf{a}, \mathbf{b})}(q) := \widehat{f}(\underline{\text{Tate}}_{\mathbf{a}, \mathbf{b}}, \eta_{p, can}^0) = f|_{(\mathbf{a}, \mathbf{b})}(q)$. We will call \widehat{f} the p -adic avatar of f .

2.6. Hecke action. Let $h \in G(\mathbf{A}_{\mathcal{F},f}^{(p)})$ and let ${}_hK := hKh^{-1}$. We define a morphism $|h : \mathcal{E}_{{}_hK}^{(p)} \xrightarrow{\sim} \mathcal{E}_K^{(p)}$ by

$$\underline{A} = (A, \bar{\lambda}, \iota, \bar{\eta}^{(p)}) \mapsto \underline{A}|h = (A, \bar{\lambda}, \iota, h\bar{\eta}^{(p)}).$$

Then $|h$ induces an \mathcal{W} -isomorphism $Sh_K^{(p)} \xrightarrow{\sim} Sh_{{}_hK}^{(p)}$, and $|h$ thus acts on spaces of modular forms. In particular, for $F \in V(K, \mathcal{W})$, we define $F|h \in V({}_hK, \mathcal{W})$ by

$$F|h(\underline{A}) = F(\underline{A}|h).$$

Let $K_0(\mathfrak{l}) := \{g \in K \mid e_2g \in O^*e_2 \pmod{\mathfrak{l}\mathcal{L}}\}$. Define the $U_{\mathfrak{l}}$ -operator on $V(K_0(\mathfrak{l}), \mathcal{W})$ by

$$F|U_{\mathfrak{l}} = \sum_{u \in O^*/\mathfrak{l}O^*} F| \begin{bmatrix} \varpi_{\mathfrak{l}} & u \\ 0 & 1 \end{bmatrix}.$$

Using the description of complex points of $Sh_K^{(p)}(\mathbf{C})$ in §2.4, it is not difficult to verify by definition that for $(\tau, g) \in X^+ \times G(\mathbf{A}_{\mathcal{F},f})$ two pairs $(\underline{A}_g(\tau)|h, \omega)$ and $(\underline{A}_{gh}(\tau), \omega)$ of \mathbf{C} -quadruples and invariant differential forms are $\mathbf{Z}_{(p)}$ -isogenous, so we have the isomorphism:

$$(2.6) \quad \begin{aligned} \mathbf{M}_k(K, \mathbf{C}) &\xrightarrow{\sim} \mathbf{M}_k({}_hK, \mathbf{C}) \\ \mathbf{f} &\mapsto \mathbf{f}|h(\tau, g) = \mathbf{f}(\tau, gh). \end{aligned}$$

3. CM POINTS

3.1. In this section, we give an adelic description of CM points in Hilbert modular varieties. Fix a prime-to- p integral ideal \mathfrak{C} of R such that $(p\mathfrak{l}, \mathfrak{C}\mathcal{D}_{\mathcal{K}/\mathcal{F}}) = 1$. Write $\mathfrak{C} = \mathfrak{C}^+\mathfrak{C}^-$, where $\mathfrak{C}^- = \mathfrak{I}\mathfrak{R}$, \mathfrak{I} (resp. \mathfrak{R}) is a product of inert (resp. ramified) primes in \mathcal{K}/\mathcal{F} and $\mathfrak{C}^+ = \mathfrak{F}\mathfrak{F}_c$ is a product of split primes in \mathcal{K}/\mathcal{F} such that $(\mathfrak{F}, \mathfrak{F}_c) = 1$ and $\mathfrak{F} \subset \mathfrak{F}_c^c$. Recall that we have assumed (unr) and (ord) in the introduction. Let Σ be a p -ordinary CM type of \mathcal{K} and identify Σ with \mathbf{a} by the restriction to \mathcal{F} . We choose $\vartheta \in \mathcal{K}$ such that

- (d1) $\vartheta^c = -\vartheta$ and $\text{Im} \sigma(\vartheta) > 0$ for all $\sigma \in \Sigma$,
- (d2) $\mathfrak{c}(R) := \mathcal{D}_{\mathcal{F}}^{-1}(2\vartheta\mathcal{D}_{\mathcal{K}/\mathcal{F}}^{-1})$ is prime to $p\mathcal{D}_{\mathcal{K}/\mathcal{F}}\mathfrak{l}\mathfrak{C}^c$.

Let $\vartheta^{\Sigma} := (\sigma(\vartheta))_{\sigma \in \Sigma} \in X^+$. Let $D = -\vartheta^2 \in \mathcal{F}_+$ and define $\rho : \mathcal{K} \hookrightarrow M_2(\mathcal{F})$ by

$$\rho(a\vartheta + b) = \begin{bmatrix} b & -Da \\ a & b \end{bmatrix}.$$

Consider the isomorphism $q_{\vartheta} : \mathcal{K} \xrightarrow{\sim} \mathcal{F}^2 = V$ defined by $q_{\vartheta}(a\vartheta + b) = ae_1 + be_2$. It is clear that $(0, 1)\rho(\alpha) = q_{\vartheta}(\alpha)$ and $q_{\vartheta}(x\alpha) = q_{\vartheta}(x)\rho(\alpha)$ for $\alpha, x \in \mathcal{K}$. Let $\mathbf{C}(\Sigma)$ be the \mathcal{K} -module whose underlying space is \mathbf{C}^{Σ} with the \mathcal{K} -action given $\alpha(x_{\sigma}) = (\sigma(\alpha)x_{\sigma})$. Then we have a canonical isomorphism $\mathcal{K} \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{C}(\Sigma)$, and $p_{\vartheta} := q_{\vartheta}^{-1} : V \otimes_{\mathbf{Q}} \mathbf{R} \xrightarrow{\sim} \mathcal{K} \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{C}(\Sigma)$ is the period map associated to ϑ^{Σ} .

3.2. A good level structure.

3.2.1. For each $v|p\mathfrak{F}\mathfrak{F}_c^c$, we decompose $v = w\bar{w}$ into two places w and \bar{w} of \mathcal{K} with $w|\mathfrak{F}\Sigma_p$. Here $w|\mathfrak{F}\Sigma_p$ means $w|\mathfrak{F}$ or $w \in \Sigma_p$. Let e_w (resp. $e_{\bar{w}}$) be the idempotent associated to w (resp. \bar{w}). Then $\{e_w, e_{\bar{w}}\}$ gives an O_v -basis of R_v . Let $\vartheta_w \in \mathcal{F}_v$ such that $\vartheta = -\vartheta_w e_{\bar{w}} + \vartheta_w e_w$.

For inert or ramified place v and w the place of \mathcal{K} above v , we fix a O_v -basis $\{1, \theta_v\}$ such that θ_v is a uniformizer if v is ramified and $\bar{\theta} = -\theta$ if $v \nmid 2$. Let $\delta_v := \theta_v - \bar{\theta}_v$ be a fixed generator of the relative different $\mathcal{D}_{\mathcal{K}_w/\mathcal{F}_v}$.

Fix a finite idele $d_{\mathcal{F}} = (d_{\mathcal{F}_v}) \in \mathbf{A}_{\mathcal{F},f}$ such that $\text{il}_{\mathcal{F}}(d_{\mathcal{F}}) = \mathcal{D}_{\mathcal{F}}$. By (d2), we may choose $d_{\mathcal{F}_v} = 2\vartheta\delta_v^{-1}$ if $v|D_{\mathcal{K}/\mathcal{F}}\mathfrak{I}$ (resp. $d_{\mathcal{F}_v} = -2\vartheta_w$ if $w|\mathfrak{F}\Sigma_p$).

3.2.2. We shall choose a basis $\{e_{1,v}, e_{2,v}\}$ of $R \otimes_O O_v$ for each finite place $v \neq \mathfrak{l}$ of \mathcal{F} . If $v \nmid p\mathfrak{l}\mathfrak{C}^c$, we choose $\{e_{1,v}, e_{2,v}\}$ in $R \otimes O_v$ such that $R \otimes_O O_v = O_v e_{1,v} \oplus O_v^* e_{2,v}$. It is clear that $\{e_{1,v}, e_{2,v}\}$ can be taken to be $\{\vartheta, 1\}$ except for finitely many v . If $v|p\mathfrak{F}\mathfrak{F}_c^c$, let $\{e_{1,v}, e_{2,v}\} = \{e_{\bar{w}}, d_{\mathcal{F}_v} \cdot e_w\}$ with $w|\mathfrak{F}\Sigma_p$. If v is inert or ramified, let $\{e_{1,v}, e_{2,v}\} = \{\theta_v, d_{\mathcal{F}_v} \cdot 1\}$. For every integer $n \geq 0$, we let $R_n = O + \mathfrak{l}^n R$, and let $\{e_{1,\mathfrak{l}}^{(n)}, e_{2,\mathfrak{l}}^{(n)}\} := \{-1, -d_{\mathcal{F}_1} \varpi_{\mathfrak{l}}^n \theta_{\mathfrak{l}}\}$ be a basis of $R_n \otimes_O O_{\mathfrak{l}}$.

For $v \in \mathbf{h}$, let ς_v (resp. $\varsigma_v^{(n)}$) be the element in $\mathrm{GL}_2(\mathcal{F}_v)$ such that $e_i \varsigma_v^{-1} = q_\vartheta(e_{i,v})$ (resp. $e_i(\varsigma_v^{(n)})^{-1} = q_\vartheta(e_{i,v}^{(n)})$). For $v = \sigma \in \mathbf{a}$, let $\varsigma_v = \begin{bmatrix} \mathrm{Im} \sigma(\vartheta) & 0 \\ 0 & 1 \end{bmatrix}$. Define $\varsigma^{(l)} = \prod_{v \neq l} \varsigma_v \in \mathrm{GL}_2(\mathbf{A}_{\mathcal{F}}^{(l)})$ and $\varsigma^{(n)} = \varsigma^{(l)} \times \varsigma_l^{(n)} \in \mathrm{GL}_2(\mathbf{A}_{\mathcal{F}})$. Let ς_f and $\varsigma_f^{(n)}$ be the finite components of ς and $\varsigma^{(n)}$ respectively. By the definition of $\varsigma^{(n)}$, we have

$$(\mathcal{L} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}) \cdot (\varsigma_f^{(n)})^{-1} = q_\vartheta(\Gamma^{-n} R_n \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}).$$

The matrix representation of ς_v according to the basis $\{e_1, e_2\}$ for $v|p|D_{\mathcal{K}/\mathcal{F}}\mathfrak{C}\mathfrak{C}^c$ is given as follows:

$$(3.1) \quad \begin{aligned} \varsigma_v &= \begin{bmatrix} d_{\mathcal{F}_v} & -2^{-1}t_v \\ 0 & d_{\mathcal{F}_v}^{-1} \end{bmatrix}, \quad t_v = \theta_v + \overline{\theta}_v \text{ if } v|D_{\mathcal{K}/\mathcal{F}}\mathfrak{I}, \\ \varsigma_v &= \begin{bmatrix} \frac{d_{\mathcal{F}_v}}{2} & -\frac{1}{2} \\ \frac{d_{\mathcal{F}_v}}{-2\vartheta_w} & \frac{-1}{2\vartheta_w} \end{bmatrix} = \begin{bmatrix} -\vartheta_w & -\frac{1}{2} \\ 1 & \frac{-1}{2\vartheta_w} \end{bmatrix} \text{ if } v|p\mathfrak{I}\mathfrak{C}^c \text{ and } w|\mathfrak{I}\Sigma_p, \\ \varsigma_l^{(n)} &= \begin{bmatrix} -b_l & 1 \\ a_l & 0 \end{bmatrix} \begin{bmatrix} d_{\mathcal{F}_l} \varpi_l^n & 0 \\ 0 & 1 \end{bmatrix} \quad (\theta_l = a_l \vartheta + b_l, a_l \in O_l^\times, b_l \in O_l). \end{aligned}$$

3.3. For every $a \in \mathbf{A}_{\mathcal{K},f}^\times$, we let

$$\underline{A}_n(a)/_{\mathbf{C}} := \underline{\mathcal{A}}_{\rho(a)\varsigma^{(n)}}(\vartheta^\Sigma) = (\mathcal{A}_{\rho(g)\varsigma^{(n)}}(\vartheta^\Sigma), \overline{\langle \cdot, \cdot \rangle}_{can}, \iota_{can}, \overline{\eta(a)}) \in \mathrm{Sh}_{\mathbf{K}}(\mathbf{C})$$

be the \mathbf{C} -quadruple associated to $(\vartheta^\Sigma, \rho(a)\varsigma_f^{(n)})$ as in §2.4. Then $\underline{A}_n(a)/_{\mathbf{C}}$ is an abelian variety with CM by \mathcal{K} . Let W be the p -adic completion of the maximal unramified extension of \mathbf{Z}_p in \mathbf{C}_p . By the general theory of CM abelian varieties, the \mathbf{C} -quadruple $\underline{A}_n(a)/_{\mathbf{C}}$ descends to a W -quadruple $\underline{A}_n(a)$. Moreover, since \mathcal{K} is p -ordinary, $\underline{A}_n(a) \otimes_W \overline{\mathbb{F}}_p$ is an ordinary abelian variety, hence the level p^∞ -structure $\eta(a)_p$ over \mathbf{C} descends to a level p^∞ -structure over W . Thus we obtain a map $x_n : \mathbf{A}_{\mathcal{K},f}^\times \rightarrow \varprojlim_m I_{K,m}(W) \subset I_{K,\infty}(W)$, which factors through $C_{\mathcal{K}} := \mathbf{A}_{\mathcal{K},f}^\times / \mathcal{K}^\times$ the idele class group of \mathcal{K} . The collection of points $Cl^\infty := \sqcup_{n=1}^\infty x_n(C_{\mathcal{K}})$ in $I_{K,\infty}(W)$ is called *CM points* in Hilbert modular varieties.

3.4. **Polarization ideal.** The alternating pairing $\langle \cdot, \cdot \rangle : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{F}$ defined by $\langle x, y \rangle = (c(x)y - xc(y))/2\vartheta$ induces an isomorphism $R \wedge_O R = \mathfrak{c}(R)^{-1} \mathcal{D}_{\mathcal{F}}^{-1}$ for the fractional ideal $\mathfrak{c}(R) = \mathcal{D}_{\mathcal{F}}^{-1} (2\vartheta \mathcal{D}_{\mathcal{K}/\mathcal{F}}^{-1})$. Then $\mathfrak{c}(R)$ is the polarization of CM points $x_0(1)$. From the equation

$$\mathcal{D}_{\mathcal{F}}^{-1} \det(\varsigma_f) = \wedge^2 \mathcal{L} \varsigma_f^{-1} = \wedge^2 R = \mathfrak{c}(R)^{-1} \mathcal{D}_{\mathcal{F}}^{-1},$$

we find that $\mathfrak{c}(R) = (\det(\varsigma_f))$. Moreover, for $a \in \mathbf{A}_{\mathcal{K}}^\times$, the polarization ideal of $x_0(a)$ is $\mathfrak{c}(a) := \mathfrak{c}(R) N_{\mathcal{K}/\mathcal{F}}(a)$, $\mathfrak{a} = \mathrm{il}_{\mathcal{K}}(a)$.

3.5. Measures associated to U_1 -eigenforms.

3.5.1. We briefly recall Hida's construction of the measure associated to an U_1 -eigenform in [Hid04a, §3]. Define the compact subgroup $U_n = (\mathbf{C}_1)^\Sigma \times (R_n \otimes \widehat{\mathbf{Z}})^\times$ in $\mathbf{A}_{\mathcal{K}}^\times = (\mathbf{C}^\times)^\Sigma \times \mathbf{A}_{\mathcal{K},f}^\times$, where \mathbf{C}_1 is the unit circle in \mathbf{C}^\times . Let $Cl_n = \mathcal{K}^\times \mathbf{A}_{\mathcal{F}}^\times \backslash \mathbf{A}_{\mathcal{K}}^\times / U_n$ and let $[\cdot]_n : \mathbf{A}_{\mathcal{K}}^\times \rightarrow Cl_n$ be the quotient map. Let $Cl_\infty = \varprojlim_n Cl_n$. For $a \in \mathbf{A}_{\mathcal{K}}^\times$, we let $[a] := \varprojlim_n [a]_n \in Cl_\infty$ be the holomorphic image in Cl_∞ . Henceforth, every $\nu \in \mathfrak{X}_1^-$ will be regarded implicitly as a p -adic character of Cl_∞ by *geometrically* normalized reciprocity law.

Let $\mathcal{E} \in V(K_0(\mathfrak{l}), \Theta)$ for some finite extension Θ of \mathbf{Z}_p and let $\widehat{\chi}$ be the p -adic avatar of χ . Assuming the following:

- (i) \mathcal{E} is a U_1 -eigenform with the eigenvalue $a_1(\mathcal{E}) \in \overline{\mathbf{Z}}_p^\times$;
- (ii) $\mathcal{E}(x_n(ta)) = \widehat{\chi}^{-1}(a) \mathcal{E}(x_n(t))$, $a \in U_n \cdot \mathbf{A}_{\mathcal{F}}^\times$,

Hida in [Hid04a, (3.9)] associates a $\overline{\mathbf{Z}}_p$ -valued measure $\varphi_{\mathcal{E}}$ on Cl_∞ to the U_1 -eigenform \mathcal{E} such that for a function $\phi : Cl_n \rightarrow \overline{\mathbf{Z}}_p$, we have

$$(3.2) \quad \int_{Cl_\infty} \phi d\varphi_{\mathcal{E}} := a_1(\mathcal{E})^{-n} \cdot \sum_{[t]_n \in Cl_n} \mathcal{E}(x_n(t)) \widehat{\chi}(t) \phi([t]_n).$$

3.5.2. Let Δ be the torsion subgroup of Cl_∞ . Let Cl^{alg} be the subgroup of Cl_∞ generated by $[a]$ for $a \in (\mathbf{A}_{\mathcal{K}}^{(l)})^\times$ and $\Delta^{\text{alg}} = Cl^{\text{alg}} \cap \Delta$. We choose a set of representatives $\mathcal{B} = \{b\}$ of $\Delta/\Delta^{\text{alg}}$ in Δ and a set of representatives $\mathcal{R} = \{r\}$ of Δ^{alg} in $(\mathbf{A}_{\mathcal{K},f}^{(p_l)})^\times$. Thus $\Delta = \mathcal{B}[\mathcal{R}] = \{b[r]\}_{b \in \mathcal{B}, r \in \mathcal{R}}$. For $a \in (\mathbf{A}_{\mathcal{K},f}^{(p_l)})^\times$, we define

$$\mathcal{E}[a] := \mathcal{E}|\rho_\varsigma(a), \rho_\varsigma(a) := \varsigma^{-1}\rho(a)\varsigma \in G(\mathbf{A}_{\mathcal{F},f}^{(p_l)}).$$

By definition, $\mathcal{E}[a](x_n(t)) = \mathcal{E}(x_n(ta))$. Following Hida (cf. [Hid07, (4.4) p.25]), we put

$$(3.3) \quad \mathcal{E}^{\mathcal{R}} = \sum_{r \in \mathcal{R}} \widehat{\chi}(r) \mathcal{E}[r].$$

In [Hid04a], Hida reduces the non-vanishing of L -values to the non-vanishing of Eisenstein series by proving the following theorem.

Theorem 3.1 (Theorem 3.2 and Theorem 3.3 [Hid04a]). *Suppose the following conditions in addition to (unr) and (ord):*

(H) *Write the order of the Sylow l -subgroup of $\mathbb{F}[\chi]^\times$ as $l^{r(\chi)}$. Then there exists a strict ideal class $\mathfrak{c} \in Cl_{\mathcal{F}}$ such that $\mathfrak{c} = \mathfrak{c}(\mathfrak{a})$ for some R -ideal \mathfrak{a} and for every $u \in O$ prime to \mathfrak{l} , we can find $\beta \equiv u \pmod{l^{r(\chi)}}$ with $\mathfrak{a}_\beta(\mathcal{E}^{\mathcal{R}}, \mathfrak{c}) \not\equiv 0 \pmod{\mathfrak{m}_p}$,*

where $\mathfrak{a}_\beta(\mathcal{E}^{\mathcal{R}}, \mathfrak{c})$ is the β -th Fourier coefficient of $\mathcal{E}^{\mathcal{R}}$ at the cusp (O, \mathfrak{c}^{-1}) . Then

$$\int_{Cl_\infty} \nu d\mathcal{E} \not\equiv 0 \pmod{\mathfrak{m}_p} \text{ for almost all } \nu \in \mathfrak{X}_1^-.$$

Remark. As pointed by the referee, if \mathfrak{l} has degree one over \mathbf{Q} , the above theorem is Theorem 3.2 [Hid04a]. In general, the theorem holds under the assumption (h) in Theorem 3.3 *loc.cit.*, which is slightly weaker than (H) (See the discussion [Hid04a, p.778]).

4. CONSTRUCTION OF THE EISENSTEIN SERIES

4.1. Let χ be a Hecke character of \mathcal{K}^\times with infinity type $k\Sigma + \kappa(1 - c)$, where $k \geq 1$ is an integer and $\kappa = \sum \kappa_\sigma \sigma \in \mathbf{Z}[\Sigma]$, $\kappa_\sigma \geq 0$. Let $\mathfrak{c}(\chi)$ be the conductor of χ . We assume that $\mathfrak{C} = \mathfrak{c}(\chi)\mathfrak{S}$, where \mathfrak{S} is only divisible by primes split in \mathcal{K}/\mathcal{F} and $(\mathfrak{c}(\chi)\mathfrak{l}, \mathfrak{S}) = 1$. Put

$$\chi^* = \chi| \cdot |_{\mathbf{A}_{\mathcal{K}}}^{-\frac{1}{2}} \text{ and } \chi_+ = \chi|_{\mathbf{A}_{\mathcal{F}}}^\times.$$

Let $K_\infty^0 := \prod_{v \in \mathfrak{a}} \text{SO}(2, \mathbf{R})$ be a maximal compact subgroup of $G(\mathcal{F} \otimes_{\mathbf{Q}} \mathbf{R})$. For $s \in \mathbf{C}$, we let $I(s, \chi_+)$ denote the space consisting of smooth and K_∞^0 -finite functions $\phi : G(\mathbf{A}_{\mathcal{F}}) \rightarrow \mathbf{C}$ such that

$$\phi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} g\right) = \chi_+^{-1}(d) \left| \frac{a}{d} \right|_{\mathbf{A}_{\mathcal{F}}}^s \phi(g).$$

Conventionally, functions in $I(s, \chi_+)$ are called *sections*. Let B be the upper triangular subgroup of G . The adelic Eisenstein series associated to a section $\phi \in I(s, \chi_+)$ is defined by

$$E_{\mathbf{A}}(g, \phi) = \sum_{\gamma \in B(\mathcal{F}) \backslash G(\mathcal{F})} \phi(\gamma g).$$

The series $E_{\mathbf{A}}(g, \phi)$ is absolutely convergent for $\text{Re } s \gg 0$.

4.2. **Fourier coefficients of Eisenstein series.** Put $\mathbf{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Let v be a place of \mathcal{F} and let $I_v(s, \chi_+)$ be the local constitute of $I(s, \chi_+)$ at v . For $\phi_v \in I_v(s, \chi_+)$ and $\beta \in \mathcal{F}_v$, we recall that the β -th local Whittaker integral $W_\beta(\phi_v, g_v)$ is defined by

$$W_\beta(\phi_v, g_v) = \int_{\mathcal{F}_v} \phi_v(\mathbf{w} \begin{bmatrix} 1 & x_v \\ 0 & 1 \end{bmatrix} g_v) \psi(-\beta x_v) dx_v,$$

and the intertwining operator $M_{\mathbf{w}}$ is defined by

$$M_{\mathbf{w}}\phi_v(g_v) = \int_{\mathcal{F}_v} \phi_v(\mathbf{w} \begin{bmatrix} 1 & x_v \\ 0 & 1 \end{bmatrix} g_v) dx_v.$$

By definition, $M_{\mathbf{w}}\phi_v(g_v)$ is the 0-th local Whittaker integral. It is well known that local Whittaker integrals converge absolutely for $\operatorname{Re} s \gg 0$, and have meromorphic continuation to all $s \in \mathbf{C}$.

If $\phi = \otimes_v \phi_v$ is a decomposable section, then it is well known that $E_{\mathbf{A}}(g, \phi)$ has the following Fourier expansion:

$$(4.1) \quad E_{\mathbf{A}}(g, \phi) = \phi(g) + M_{\mathbf{w}}\phi(g) + \sum_{\beta \in \mathcal{F}} W_{\beta}(E_{\mathbf{A}}, g), \text{ where}$$

$$M_{\mathbf{w}}\phi(g) = \frac{1}{\sqrt{|D_{\mathcal{F}}|_{\mathbf{R}}}} \cdot \prod_v M_{\mathbf{w}}\phi_v(g_v); \quad W_{\beta}(E_{\mathbf{A}}, g) = \frac{1}{\sqrt{|D_{\mathcal{F}}|_{\mathbf{R}}}} \cdot \prod_v W_{\beta}(\phi_v, g_v).$$

The sum $\phi(g) + M_{\mathbf{w}}\phi(g)$ is called the *constant term* of $E_{\mathbf{A}}(g, \phi)$.

4.3. The choice of local sections and Fourier coefficients. In this subsection, we will choose for each place v a good local section $\phi_{\chi, s, v}$ in $I_v(s, \chi_+)$ and calculate its local β -th Fourier coefficient for $\beta \in \mathcal{F}_v^{\times}$.

4.3.1. We first introduce some notation and definitions. Let $S^{\circ} = \{v \in \mathbf{h} \mid v \nmid |\mathcal{C}\mathcal{C}^c D_{\mathcal{K}/\mathcal{F}}|\}$. Let v be a place of \mathcal{F} . Let L/\mathcal{F}_v be a finite extension and let d_L be a generator of the absolute different \mathcal{D}_L of L . Let $\psi_L := \psi \circ \Gamma_{L/\mathcal{F}_v}$. Given a character $\mu : L^{\times} \rightarrow \mathbf{C}$, we recall that the epsilon factor $\epsilon(s, \mu, \psi_L)$ in [Tat79] is defined by

$$\epsilon(s, \mu, \psi_L) = |c|_L^s \int_{c^{-1}\mathcal{O}_L^{\times}} \mu^{-1}(x) \psi_L(x) d_L x \quad (c = d_L \varpi_L^{a(\mu)}).$$

Here $d_L x$ is the Haar measure on L self-dual with respect to ψ_L . If φ is a Bruhat-Schwartz function on L , the zeta integral $Z(s, \mu, \varphi)$ is given by

$$Z(s, \mu, \varphi) = \int_L \varphi(x) \mu(x) |x|_L^s d^{\times} x \quad (s \in \mathbf{C}).$$

The local root number $W(\mu)$ is defined by

$$W(\mu) := \epsilon\left(\frac{1}{2}, \mu, \psi_L\right)$$

(cf. [MS00, p.281 (3.8)]). It is well known that $|W(\mu)|_{\mathbf{C}} = 1$ if μ is unitary.

To simplify the notation, we let $F = \mathcal{F}_v$ (resp. $E = \mathcal{K} \otimes_{\mathcal{F}} \mathcal{F}_v$) and let $d_F = d_{\mathcal{F}_v}$ be the fixed generator of the absolute different \mathcal{D}_F in §3.2.1. Write χ (resp. χ_+ , χ^*) for χ_v (resp. $\chi_{+,v}$, χ_v^*). If $v \in \mathbf{h}$, we let $O_v = \mathcal{O}_F$ (resp. $R_v = R \otimes_{\mathcal{O}} O_v$) and let $\varpi = \varpi_v$ be a uniformizer of F . For a set Y , denote by \mathbb{I}_Y the characteristic function of Y .

4.3.2. v is archimedean. Let $v = \sigma \in \Sigma$ and $F = \mathbf{R}$. For $g \in G(F) = \operatorname{GL}_2(\mathbf{R})$, we put

$$\delta(g) = |\det(g)| \cdot \left| J(g, i) \overline{J(g, i)} \right|^{-1}.$$

Define the section $\phi_{k, s, \sigma}^h \in I_v(s, \chi_+)$ of weight k by

$$(4.2) \quad \phi_{k, s, \sigma}^h(g) := J(g, i)^{-k} \delta(g)^s.$$

The intertwining operator $M_{\mathbf{w}}\phi_{k, s, \sigma}$ is given by

$$(4.3) \quad M_{\mathbf{w}}\phi_{k, s, \sigma}^h(g) = i^k (2\pi) \frac{\Gamma(k+2s-1)}{\Gamma(k+s)\Gamma(s)} \cdot \overline{J(g, i)}^k \det(g)^{-k} \delta(g)^{1-s}.$$

For $(x, y) \in \mathbf{R} \times \mathbf{R}_+$ and $\beta \in \mathbf{R}^{\times}$, it is well known that

$$(4.4) \quad W_{\beta}(\phi_{k, s, \sigma}^h, \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix})|_{s=0} = \frac{(2\pi i)^k}{\Gamma(k)} \sigma(\beta)^{k-1} \exp(2\pi i \sigma(\beta)(x + iy)) \cdot \mathbb{I}_{\mathbf{R}_+}(\sigma(\beta)).$$

Define the section $\phi_{k, \kappa_{\sigma}, s, \sigma}^{n.h.} \in I(s, \chi_+)$ of weight $k + 2\kappa_{\sigma}$ by

$$(4.5) \quad \phi_{k, \kappa_{\sigma}, s, \sigma}^{n.h.}(g) := J(g, i)^{-k - \kappa_{\sigma}} \overline{J(g, i)}^{\kappa_{\sigma}} \delta(g)^s.$$

Let V_+ be the *weight raising* differential operator in [JL70, p.165] given by

$$V_+ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes 1 + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes i \in \operatorname{Lie}(\operatorname{GL}_2(\mathbf{R})) \otimes_{\mathbf{R}} \mathbf{C}.$$

Denote by $V_+^{\kappa\sigma}$ the operator $(V^+)^{\kappa\sigma}$ acting on $I_v(s, \chi_+)$. By [JL70, Lemma 5.6 (iii)], we have

$$(4.6) \quad V_+^{\kappa\sigma} \phi_{k,s,\sigma}^h = \frac{2^{\kappa\sigma} \Gamma(k + \kappa_\sigma + 2s)}{\Gamma(k + 2s)} \phi_{k,\kappa_\sigma,s,\sigma}^{n.h.}$$

4.3.3. $v \in S^\circ$. In this case, χ is unramified. Define $\phi_{\chi,s,v}(g)$ to be the spherical Godement section in $I_v(s, \chi_+)$. To be precise, put

$$\begin{aligned} \phi_{\chi,s,v}(g) &= f_{\Phi_v}(g) := |\det g|^s \int_{F^\times} \Phi_v((0, t)g) \chi_+(t) |t|^{2s} d^\times t, \text{ where} \\ \Phi_v &= \mathbb{I}_{O_v \oplus O_v^*}. \end{aligned}$$

It is well known that the local Whittaker integral is

$$(4.7) \quad W_\beta(\phi_{\chi,s,v}, \begin{bmatrix} \mathbf{c}_v & \\ & 1 \end{bmatrix})|_{s=0} = \frac{1 - \chi^*(\varpi)^{v(\beta\mathbf{c}_v)+1}}{1 - \chi^*(\varpi)} \cdot |\mathcal{D}_{\mathcal{F}}|^{-1} \cdot \mathbb{I}_{O_v}(\beta\mathbf{c}_v),$$

and the intertwining operator is given by

$$(4.8) \quad M_{\mathbf{w}} \phi_{\chi,s,v} \left(\begin{bmatrix} \mathbf{c}_v & \\ & 1 \end{bmatrix} \right) = L_v(2s - 1, \chi_+) |\mathbf{c}_v|^{1-s} \chi_+^{-1}(\mathbf{c}_v).$$

4.3.4. $v|\mathfrak{F}\mathfrak{F}^c$. If $v|\mathfrak{F}\mathfrak{F}^c$ is split in \mathcal{K} , write $v = w\bar{w}$ with $w|\mathfrak{F}$ and $\chi_v = (\chi_w, \chi_{\bar{w}})$. Then $a(\chi_w) \geq a(\chi_{\bar{w}})$. We shall define our local section at v to be the Godement section associated to certain Bruhat-Schwartz functions. We first introduce some Bruhat-Schwartz functions. For a character $\mu : F^\times \rightarrow \mathbf{C}^\times$, we define

$$\varphi_\mu(x) = \mathbb{I}_{O_v^\times}(x) \mu(x) \quad (x \in F).$$

Define $\varphi_w = \varphi_{\chi_w}$ and

$$\varphi_{\bar{w}} = \begin{cases} \varphi_{\chi_{\bar{w}}^{-1}} & \text{if } \chi_{\bar{w}} \text{ is ramified,} \\ \mathbb{I}_{O_v} & \text{if } \chi_{\bar{w}} \text{ is unramified.} \end{cases}$$

Let $\Phi_v(x, y) = \varphi_{\bar{w}}(x) \widehat{\varphi}_w(y)$, where $\widehat{\varphi}_w$ is the Fourier transform of φ_w defined by

$$\widehat{\varphi}_w(y) = \int_F \varphi_w(x) \psi(yx) dx.$$

Define $\phi_{\chi,s,v} \in I_v(s, \chi_+)$ by

$$(4.9) \quad \phi_{\chi,s,v}(g) = f_{\Phi_v}(g) := |\det g|^s \int_{F^\times} \Phi_v((0, t)g) \chi_+(t) |t|^{2s} d^\times t.$$

A straightforward calculation shows that the local Whittaker integral is

$$(4.10) \quad \begin{aligned} W_\beta(\phi_{\chi,s,v}, 1) &= \int_{F^\times} \varphi_{\bar{w}}(x) \widehat{\varphi}_w(-\beta x^{-1}) \cdot \chi_+(x) |x|^{2s-1} d^\times x \\ &= \int_{F^\times} \varphi_{\bar{w}}(x) \varphi_w(\beta x^{-1}) \cdot |\mathcal{D}_{\mathcal{F}}|^{-1} \cdot \chi_+(x) |x|^{2s-1} d^\times x \\ &= \chi_+(\beta) \varphi_{\bar{w}}(\beta) |\beta|^{2s-1} \cdot |\mathcal{D}_{\mathcal{F}}|^{-1}, \end{aligned}$$

and the intertwining operator is given by

$$(4.11) \quad M_{\mathbf{w}} \phi_{\chi,s,v}(1) = 0.$$

4.3.5. $v = \mathfrak{l}$. Let $\phi_{\chi,v,s} \in I_v(s, \chi_+)$ be the unique $N(O_v)$ -invariant section supported in the big cell $B(F)\mathbf{w}N(O_v^*)$ and $\phi_{\chi,v,s}(\mathbf{w}) = 1$. One checks easily that $\phi_{\chi,s,v}|U_{\mathfrak{l}}$ given by

$$\phi_{\chi,s,v}|U_{\mathfrak{l}}(g) = \sum_{u \in O_v^*/\mathfrak{l}O_v^*} \phi_{\chi,s,v} \left(g \begin{bmatrix} \varpi & u \\ 0 & 1 \end{bmatrix} \right)$$

is also supported in the big cell and is invariant by $N(O_v^*)$. In particular, $\phi_{\chi,s,v}$ is an $U_{\mathfrak{l}}$ -eigenform, and the eigenvalue is $\chi_+^{-1}(\varpi)$.

The local β -th Whittaker integral is given by

$$(4.12) \quad W_\beta(\phi_{\chi,v,s}, \begin{bmatrix} \mathbf{c} & \\ & 1 \end{bmatrix})|_{s=0} = \int_F \phi_{\chi,v,s}(\mathbf{w} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} \mathbf{c} & \\ & 1 \end{bmatrix}) \psi(-\beta x) dx|_{s=0} \\ = \mathbb{I}_{O_v}(\beta \mathbf{c}),$$

and the intertwining operator is given by $M_{\mathbf{w}} \phi_{\chi,s,v}(\begin{bmatrix} \mathbf{c} & 0 \\ 0 & 1 \end{bmatrix})|_{s=0} = 1$.

4.3.6. $v|D_{\mathcal{K}/\mathcal{F}}\mathfrak{C}^-$. In this case, E is a field and $G(F) = B(F)\rho(E^\times)$. Let w be the place of E above v and let ϖ_E be a uniformizer of E . Let \mathfrak{R}' be the product of ramified primes where χ is unramified, i.e. $\mathfrak{R}' = \prod_{q|D_{\mathcal{K}/\mathcal{F}}, q \nmid \mathfrak{C}^-} q$. Let $\phi_{\chi,s,v}$ be the unique function on $G(F)$ such that

$$(4.13) \quad \phi_{\chi,s,v}(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}) \rho(z)_{\mathfrak{C}_v} = L(s, \chi_v) \cdot \chi_+^{-1}(d) \left| \frac{a}{d} \right|^s \cdot \chi^{-1}(k), \quad b \in B(F), z \in E^\times,$$

where $L(s, \chi_v)$ is the local Euler factor of χ_v defined by

$$L(s, \chi_v) = \begin{cases} 1 & \text{if } v|\mathfrak{R}\mathfrak{I}, \\ \frac{1}{1 - \chi_v(\varpi_E)|\varpi_E|_E^s} & \text{if } v|\mathfrak{R}'. \end{cases}$$

Then $\phi_{\chi,s,v}$ defines a smooth section in $I_v(s, \chi_+)$.

To calculate the local β -th Whittaker integral of $\phi_{\chi,s,v}$, we recall that in §3.2.1, we have fixed $\delta = \delta_v = 2\vartheta d_F^{-1}$ a generator of $\mathcal{D}_{\mathcal{K}/\mathcal{F}}$ and an O_v -basis $\{1, \boldsymbol{\theta}\} = \{1, \boldsymbol{\theta}_v\}$ of R_v so that $\delta = 2\boldsymbol{\theta}$ if $v \nmid 2$ and $\delta = \boldsymbol{\theta} - \bar{\boldsymbol{\theta}}$ if $v|2$. In addition, $\boldsymbol{\theta}$ is a uniformizer of R_v if v is ramified. Let $t = t_v = 2\boldsymbol{\theta} - \delta = \boldsymbol{\theta} + \bar{\boldsymbol{\theta}} \in O_v$. Let $\psi^\circ(x) := \psi(-d_F^{-1}x)$ and $\chi_s = \chi|\cdot|_E^s$ for $s \in \mathbf{C}$. For $\text{Re } s \gg 0$, we have

$$\begin{aligned} & \frac{1}{L(s, \chi_v)} \cdot W_\beta(\phi_{\chi,s,v}, 1) \\ &= \int_F \phi_{\chi,s,v}(\mathbf{w} \begin{bmatrix} 1 & x + 2^{-1}td_F^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_F^{-1} & 0 \\ 0 & d_F \end{bmatrix})_{\mathfrak{C}_v} \psi^\circ(d_F \beta x) dx \\ &= \psi^\circ(-2^{-1}t\beta) \chi_s(d_F) |d_F^{-2}| \cdot \int_F \phi_{\chi,s,v}(\begin{bmatrix} \frac{1}{x^2+D} & \frac{-x}{x^2+D} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & -D \\ 1 & x \end{bmatrix})_{\mathfrak{C}_v} \psi^\circ(-d_F^{-1}\beta x) dx \\ &= \psi^\circ(-2^{-1}t\beta) \chi_s(d_F) |d_F^{-2}| \cdot \int_F \chi_s^{-1}(x + \vartheta) \psi^\circ(d_F^{-1}\beta x) dx \\ &= \psi^\circ(-2^{-1}t\beta) |d_F^{-1}| \cdot \int_F \chi_s^{-1}(x + 2^{-1}\delta) \psi^\circ(\beta x) dx. \end{aligned}$$

We put

$$(4.14) \quad A_\beta(\chi_s) := \int_F \chi_s^{-1}(x + 2^{-1}\delta) \psi^\circ(\beta x) dx.$$

Making change of variable, we find that

$$(4.15) \quad \begin{aligned} A_\beta(\chi_s) &= \psi^\circ(2^{-1}t\beta) \cdot \tilde{A}_\beta(\chi_s), \text{ where} \\ \tilde{A}_\beta(\chi_s) &= \int_F \chi_s^{-1}(x + \boldsymbol{\theta}) \psi^\circ(\beta x) dx. \end{aligned}$$

In particular, the intertwining operator $M_{\mathbf{w}} \phi_{\chi,s,v}(1) = |d_F^{-1}| \cdot L(s, \chi_v) \tilde{A}_0(\chi_s)$. We investigate the analytic behavior of $\tilde{A}_\beta(\chi_s)$. For $\beta \in F$ and $M \geq v(\mathfrak{C}^-)$, we have

$$\tilde{A}_\beta(\chi_s) = \int_{\varpi^{-M}O_v} \chi_s^{-1}(x + \boldsymbol{\theta}) \psi^\circ(\beta x) dx + \sum_{j>M} \chi^*|\cdot|^s(\varpi^j) \int_{O_v^\times} \chi^{-1}(x) \psi^\circ(\beta \varpi^{-j}x) dx$$

for $\text{Re } s$ sufficiently large. This shows that $\tilde{A}_\beta(\chi_s)$ has meromorphic continuation to all $s \in \mathbf{C}$, and $\tilde{A}_\beta(\chi_s)$ is holomorphic at $s = 0$ except when $\beta = 0$, χ_v is unramified and $\chi^*(\varpi) = 1$. In particular, when $k \geq 2$, $\tilde{A}_0(\chi_s)$ and the intertwining operator $M_{\mathbf{w}} \phi_{\chi,s,v}(1)$ are finite at $s = 0$.

Now we have

$$(4.16) \quad W_\beta(\phi_{\chi,s,v}, 1)|_{s=0} = |d_F^{-1}| \cdot L(s, \chi_v) \tilde{A}_\beta(\chi_s)|_{s=0}.$$

In what follows, we let $\beta \in F^\times$ and let $\tilde{A}_\beta(\chi) := \tilde{A}_\beta(\chi_s)|_{s=0}$ (resp. $A_\beta(\chi) := A_\beta(\chi_s)|_{s=0}$). Then $A_\beta(\chi) = \psi^\circ(2^{-1}t\beta) \cdot \tilde{A}_\beta(\chi)$. On the other hand, by our choice of θ it is clear that $|x + \theta|_E \geq |\theta|_E$ for all $x \in \varpi^{-M}O_v$. Let $M_{\mathfrak{C},\beta} = \max\{v(\mathfrak{C}^-), v(\mathfrak{C}^-) + v(\beta)\}$. Then for $M \geq M_{\mathfrak{C},\beta}$, we have

$$(4.17) \quad \begin{aligned} \tilde{A}_\beta(\chi) &= \int_{\varpi^{-M}O_v} \chi^{-1}(x + \theta) \psi^\circ(\beta x) dx \\ &= |\varpi|^{1+M} \sum_{x \in O_v/(\varpi^{1+2M})} \chi^{-1}(\varpi^{-M}x + \theta) \psi^\circ(\beta \varpi^{-M}x). \end{aligned}$$

The following lemma shows the p -integrality of these local Whittaker integrals $W_\beta(\phi_{\chi,s,v}, 1)|_{s=0}$.

Lemma 4.1. *Suppose that $v|D_{\mathcal{K}/\mathcal{F}}\mathfrak{C}^-$. There exists a finite extension \mathcal{O} of $\mathcal{O}_{\mathcal{F},(p)}$ such that for all $\beta \in F^\times$*

$$W_\beta(\phi_{\chi,s,v}, 1)|_{s=0} \in \mathcal{O}.$$

In addition, if $v|\mathfrak{C}^-$, then

$$(4.18) \quad W_\beta(\phi_{\chi,s,v}, 1)|_{s=0} = \psi^\circ(-2^{-1}t\beta) |d_F^{-1}| \cdot A_\beta(\chi) = |d_F^{-1}| \cdot \tilde{A}_\beta(\chi).$$

If $v|\mathfrak{R}'$ and $v(\beta) = -1$, then

$$(4.19) \quad W_\beta(\phi_{\chi,s,v}, 1)|_{s=0} = |d_F^{-1}| \chi^{-1}(\theta) |\varpi|.$$

PROOF. First note that it is not difficult to deduce from (4.17) that $\tilde{A}_\beta(\chi) = 0$ if $v(\beta) < -1 - M_{\mathfrak{C},\beta}$, and that $\tilde{A}_\beta(\chi)$ belongs to the $\mathcal{O}_{\mathcal{F},(p)}$ -algebra generated by the values of χ and $\psi^\circ(\varpi^{-2v(\mathfrak{C}^-)-1})$ for all $\beta \in F^\times$. Then the assertions are clear if $v|\mathfrak{C}^-$. Suppose that $v|\mathfrak{R}'$. Then v is ramified in E and χ is unramified at v . For $s \in \mathbf{C}$, let $\alpha_s := \chi_s(\varpi) |\varpi|^{-1}$. For $\text{Re } s \gg 0$, we have

$$\begin{aligned} \tilde{A}_\beta(\chi_s) &= \int_F \chi_s^{-1}(x + \theta) \psi^\circ(\beta x) \\ &= \int_{\varpi O_v} \chi_s^{-1}(x + \theta) \psi^\circ(\beta x) dx + \sum_{j=0}^{v(\beta)+1} |\varpi^{-j}| \chi_s(\varpi^j) \int_{O_v^\times} \psi^\circ(\beta \varpi^{-j}x) dx \\ &= \chi_s^{-1}(\theta) \int_{\varpi O_v} \psi^\circ(\beta x) + \sum_{j=0}^{v(\beta)+1} \alpha_s^j \cdot \left(\int_{O_v} \psi^\circ(\beta \varpi^{-j}x) dx - \int_{\varpi O_v} \psi^\circ(\beta \varpi^{-j}x) dx \right). \end{aligned}$$

If $v(\beta) \geq 0$, we find that

$$\begin{aligned} \tilde{A}_\beta(\chi_s) &= \chi_s^{-1}(\theta) |\varpi| + \frac{1 - \alpha_s^{v(\beta)+1}}{1 - \alpha_s} \cdot (1 - |\varpi|) - |\varpi| \alpha_s^{v(\beta)+1} \\ &= (1 - \chi_s(\theta))(1 + |\varpi| \chi_s^{-1}(\theta)) \cdot \frac{1 - \alpha_s^{v(\beta)+2}}{1 - \alpha_s}. \end{aligned}$$

In addition, we have $\tilde{A}_\beta(\chi_s) = 0$ if $v(\beta) < -1$ and

$$\tilde{A}_\beta(\chi_s) = (1 - \chi_s(\theta)) \cdot \chi_s^{-1}(\theta) |\varpi| \text{ if } v(\beta) = -1.$$

In any case, the assertions in the case $v|\mathfrak{R}'$ follow immediately from (4.16) and the formulas of $\tilde{A}_\beta(\chi_s)$. \square

4.3.7. Calculation of $\tilde{A}_\beta(\chi)$. We give an explicit calculation of the local integral $\tilde{A}(\chi)$ under the assumption $w(\mathfrak{C}^-) = 1$. Introduce an auxiliary integral $\mathcal{I}(\beta)$ for $\beta \in F$ defined by

$$\mathcal{I}(\beta) := \int_{O_v} \chi^{-1}(x + \theta) \psi^\circ(\beta x) dx.$$

The explicit formulas of $\tilde{A}_\beta(\chi)$ are deduced from the following two lemmas.

Lemma 4.2. *Suppose $w(\mathfrak{C}^-) \leq 1$.*

(1) If $v(\beta) \geq 0$ and $\chi|_{O_v^\times} \neq 1$, then

$$\tilde{A}_\beta(\chi) = \mathcal{I}(0) + \chi^{-1} |(-d_F^{-1}\beta) \cdot \epsilon(1, \chi_+ |\cdot|^{-1}, \psi).$$

(2) If $v(\beta) \geq 0$ and $\chi|_{O_v^\times} = 1$, then

$$\tilde{A}_\beta(\chi) = \mathcal{I}(0) + \sum_{j=1}^{v(\beta)} \chi^*(\varpi^j) \cdot (1 - |\varpi|) - \chi^*(\varpi^{v(\beta)+1}) \cdot |\varpi| \quad (\chi^* = \chi|_{E^{\frac{1}{2}}}).$$

(3) If $v(\beta) < 0$, then $\tilde{A}_\beta(\chi) = \mathcal{I}(\beta)$.

PROOF. It is clear that under our assumption on the conductor of χ , we have

$$\begin{aligned} \tilde{A}_\beta(\chi) &= \int_F \chi^{-1}(x + \theta) \psi^\circ(\beta x) \\ &= \mathcal{I}(\beta) + \sum_{j=1}^{v(\beta)+1} |\varpi|^{-j} \int_{O_v^\times} \chi^{-1}\left(\frac{x}{\varpi^j}\right) \psi^\circ\left(\frac{\beta x}{\varpi^j}\right) dx. \end{aligned}$$

Then the lemma follows immediately. \square

Lemma 4.3.

(1) If $v(\beta) < -1$, then $\mathcal{I}(\beta) = 0$.

(2) If v is ramified, then $\mathcal{I}(0) = \chi^*(\theta^{-1}) |\varpi|^{\frac{1}{2}}$.

(3) If v is inert and $\chi|_{O_v^\times} = 1$, then $\mathcal{I}(0) = -|\varpi|$.

PROOF. Note that if v is ramified, then $\chi|_{O_v^\times} \neq 1$. (1) and (2) follows from the assumption on the conductor of χ and a simple calculation. (3) follows from the following equation:

$$\begin{aligned} 0 &= \int_{R_v^\times} \chi^{-1}(r) dr = \int_{O_v} da \int_{O_v^\times} db \chi^{-1}(a + b\theta) + \int_{O_v^\times} da \int_{\varpi O_v} db \chi^{-1}(a + b\theta) \\ &= (1 - |\varpi|) \int_{O_v} \chi^{-1}(a + \theta) da + |\varpi| (1 - |\varpi|). \end{aligned} \quad \square$$

We summarize the formulas of $\tilde{A}_\beta(\chi)$ in the following two propositions.

Proposition 4.4. *Suppose that $v|\mathfrak{C}^-$ is ramified such that $w(\mathfrak{C}^-) = 1$. Then the formula of $\tilde{A}_\beta(\chi)$ is given as follows.*

(1) If $v(\beta) \geq -1$, then

$$\tilde{A}_\beta(\chi) = \chi^*(\theta^{-1}) |\varpi|^{\frac{1}{2}} + \chi^*(-\beta d_F^{-1}) \epsilon(1, \chi_+ |\cdot|^{-1}, \psi).$$

(2) If $v(\beta) < -1$, then $\tilde{A}_\beta(\chi) = 0$.

(3) If $v \nmid 2$ and $v(\beta) \geq -1$, then

$$\tilde{A}_\beta(\chi) = (\chi^*(-2\delta^{-1}d_F) + \chi^*(2^{-1}\beta)W(\chi^*)) \cdot \chi(-d_F^{-1}) |\varpi|^{\frac{1}{2}}.$$

PROOF. In this case, θ is a uniformizer of R_v . It is straightforward to verify that if $v(\beta) = -1$, then

$$\mathcal{I}(\beta) = \chi^*(\beta d_F^{-1}) \epsilon(1, \chi_+ |\cdot|^{-1}, \psi) + \chi^*(\theta^{-1}) |\varpi|^{\frac{1}{2}}.$$

Thus (1) and (2) follows from Lemma 4.2 and Lemma 4.3.

Suppose $v \nmid 2$. Then $\delta = 2\theta$, and (3) follows from (1) and the identity ([Roh82, Prop.8])

$$W(\chi^*) = \chi^*(2)W(\chi^*|_F) = \chi(2) |\varpi|^{-\frac{1}{2}} \epsilon(1, \chi_+ |\cdot|^{-1}, \psi). \quad \square$$

Proposition 4.5. *Suppose that $v|\mathfrak{C}^-$ is inert such that $w(\mathfrak{C}^-) = 1$. Then the formula of $\tilde{A}_\beta(\chi)$ is given as follows.*

(1) If $v(\beta) = -1$, then

$$\tilde{A}_\beta(\chi) = |\varpi| \cdot \sum_{a \in O_v/(\varpi)} \chi^{-1}(a + \theta) \psi^\circ(\beta a).$$

(2) If $v(\beta) < -1$, then $\tilde{A}_\beta(\chi) = 0$.

(3) If $v(\beta) \geq 0$ and $\chi|_{\mathcal{O}_v^\times} = 1$, then

$$\tilde{A}_\beta(\chi) = -|\varpi| + \sum_{j=1}^{v(\beta)} \chi^*(\varpi^j) \cdot (1 - |\varpi|) - \chi^*(\varpi^{v(\beta)+1}) |\varpi|.$$

(4) If $v(\beta) \geq 0$ and $\chi|_{\mathcal{O}_v^\times} \neq 1$, then

$$\tilde{A}_\beta(\chi) = \mathcal{I}(0) + \chi^*(-\beta d_F^{-1}) \epsilon(1, \chi_+ |\cdot|^{-1}, \psi).$$

PROOF. In this case, both δ and θ are units of R_v^\times . It follows from the definition of $\mathcal{I}(\beta)$ that if $v(\beta) = -1$, then

$$\mathcal{I}(\beta) = |\varpi| \cdot \sum_{a \in \mathcal{O}_v / (\varpi)} \chi^{-1}(a + \theta) \psi^\circ(\beta a).$$

The proposition follows from Lemma 4.2 and Lemma 4.3 immediately. \square

4.4. Normalization of Eisenstein series.

Definition 4.6. For $\bullet = h$ or $n.h.$, we put

$$\phi_{\chi,s}^\bullet = \bigotimes_{\sigma \in \mathbf{a}} \phi_{k,s,\sigma}^\bullet \bigotimes_{v \in \mathbf{h}} \phi_{\chi,s,v}.$$

Define the adelic Eisenstein series E_χ^\bullet by

$$(4.20) \quad E_\chi^\bullet(g) = E_{\mathbf{A}}(g, \phi_{\chi,s}^\bullet)|_{s=0}, \quad \bullet = h, n.h.$$

We define the holomorphic (resp. nearly holomorphic) Eisenstein series \mathbb{E}_χ^h (resp. $\mathbb{E}_\chi^{n.h.}$) by

$$\begin{aligned} \mathbb{E}_\chi^h(\tau, g_f) &:= \frac{\Gamma_\Sigma(k\Sigma)}{\sqrt{|D_{\mathcal{F}}|_{\mathbf{R}}(2\pi i)^{k\Sigma}}} \cdot E_\chi^h(g_\infty, g_f) \cdot \underline{J}(g_\infty, \mathbf{i})^{k\Sigma}, \\ \mathbb{E}_\chi^{n.h.}(\tau, g_f) &:= \frac{\Gamma_\Sigma(k\Sigma)}{\sqrt{|D_{\mathcal{F}}|_{\mathbf{R}}(2\pi i)^{k\Sigma}}} \cdot E_\chi^{n.h.}(g_\infty, g_f) \cdot \underline{J}(g_\infty, \mathbf{i})^{k\Sigma+2\kappa} (\det g_\infty)^{-\kappa}, \\ &((\tau, g_f) \in X^+ \times G(\mathbf{A}_{\mathcal{F},f}), g_\infty \in G(\mathcal{F} \otimes_{\mathbf{Q}} \mathbf{R}), g_\infty \mathbf{i} = \tau, \mathbf{i} = (i)_{\sigma \in \Sigma}). \end{aligned}$$

Choose $N = \mathbf{N}_{\mathcal{K}/\mathbf{Q}}(\mathcal{C}D_{\mathcal{K}/\mathcal{F}})^m$ for a sufficiently large integer m so that $\phi_{\chi,s,v}$ are invariant by $U(N)$ for every $v|N$, and put $K := U(N)$. Then the section $\phi_{\chi,s}$ is invariant by $K_0(\mathfrak{l})$.

Proposition 4.7. Let $\mathbf{c} = (\mathbf{c}_v) \in \mathbf{A}_{\mathcal{F},f}^\times$ be a finite idele such that $\mathbf{c}_v = 1$ at $v|p|\mathcal{C}\mathcal{C}^c D_{\mathcal{K}/\mathcal{F}}$. Let $\mathfrak{c} = \mathfrak{l}_{\mathcal{F}}(\mathbf{c})$. Suppose either of the following conditions holds:

- (1) $k > 2$,
- (2) $\mathfrak{F} \neq \mathcal{O}$,
- (3) $\chi_+ = \tau_{\mathcal{K}/\mathcal{F}}|_{\mathbf{A}_{\mathcal{F}}}$ and \mathfrak{l} is split in \mathcal{K} .

Then $\mathbb{E}_\chi^h \in \mathbf{M}_k(K_0(\mathfrak{l}), \mathbf{C})$. The q -expansion of \mathbb{E}_χ^h at the cusp $(\mathcal{O}, \mathfrak{c}^{-1})$ has no constant term and is given by

$$\mathbb{E}_\chi^h|_{(\mathcal{O}, \mathfrak{c}^{-1})} = \sum_{\beta \in (N^{-1}\mathfrak{c}^{-1})_+} \mathbf{a}_\beta(\mathbb{E}_\chi^h, \mathfrak{c}) \cdot q^\beta \in \mathcal{O}[(N^{-1}\mathfrak{c}^{-1})_+]$$

for some finite extension \mathcal{O} of $\mathcal{O}_{\mathcal{F},(p)}$ in $\overline{\mathbf{Z}}$, where

$$\mathbf{a}_\beta(\mathbb{E}_\chi^h, \mathfrak{c}) = \frac{1}{|D_{\mathcal{F}}|_{\mathbf{R}}} \cdot \mathbf{N}_{\mathcal{F}/\mathbf{Q}}(\beta)^{k-1} \cdot \prod_{v \in \mathbf{h}} W_\beta(\phi_{\chi,s,v}, \begin{bmatrix} \mathbf{c}_v & \\ & 1 \end{bmatrix})|_{s=0}.$$

PROOF. Let $\phi_{\chi,s}^{(\infty)} = \bigotimes_{v \in \mathbf{h}} \phi_{\chi,s,v}$. First we claim that for $g_\infty = (g_\sigma)_{\sigma \in \Sigma} \in G(\mathcal{F} \otimes_{\mathbf{Q}} \mathbf{R})$,

$$(4.21) \quad M_{\mathbf{w}} \phi_{\chi,s}^{(\infty)}((g_\infty, \begin{bmatrix} \mathbf{c} & \\ & 1 \end{bmatrix}))|_{s=0} = \prod_{v \in \Sigma} M_{\mathbf{w}} \phi_{k,s,\sigma}^h(g_\sigma) \cdot M_{\mathbf{w}} \phi_{\chi,s}^{(\infty)}(\begin{bmatrix} \mathbf{c} & \\ & 1 \end{bmatrix})|_{s=0} = 0.$$

Indeed, if $k > 2$, then (4.21) follows from (4.3) together with the finiteness of $L^{(p\mathfrak{e})}(-1, \chi_+)$, while if there exists some place $v|\mathfrak{f}$, then we find (4.21) in view of (4.11) immediately. If $\chi_+ = \tau_{\mathcal{K}/\mathcal{F}}| \cdot |_{\mathbf{A}_{\mathcal{F}}}$ and \mathfrak{l} splits in \mathcal{K} , then $k = 1$ and χ is ramified at $v|D_{\mathcal{K}/\mathcal{F}}$. By (4.8), we find that

$$M_{\mathbf{w}}\phi_{\chi,s}^{(\infty)}\left(\begin{bmatrix} \mathbf{c} \\ 1 \end{bmatrix}\right)|_{s=0} = \prod_{v|\mathfrak{c}\mathfrak{l}} M_{\mathbf{w}}\phi_{\chi,s,v}(1)|_{s=0} \cdot L^{(\mathfrak{e})}(0, \tau_{\mathcal{K}/\mathcal{F}})|_{\mathbf{c}}|_{\mathbf{A}_{\mathcal{F}}},$$

so (4.21) follows from the fact that

$$L^{(\mathfrak{e})}(0, \tau_{\mathcal{K}/\mathcal{F}}) = L^{(\mathfrak{e})}(0, \tau_{\mathcal{K}/\mathcal{F}})(1 - \tau_{\mathcal{K}/\mathcal{F}}(\mathfrak{l})) = 0.$$

This proves the claim. In addition, we have $\phi_{\chi,s}^{(\infty)}\left(\begin{bmatrix} \mathbf{c} \\ 1 \end{bmatrix}\right) = 0$ since $\phi_{\chi,s}$ is supported in the big cell.

Therefore, we derive the q -expansion of \mathbb{E}_{χ}^h from (4.1) and (4.4).

We study the p -integrality of $\mathbf{a}_{\beta}(\mathbb{E}_{\chi}^h, \mathbf{c})$. It is well known that χ takes value in a number field. By the inspection of formulas in (4.7), (4.10), (4.12), Lemma 4.1, (4.17), we find that

$$\prod_{v \nmid p\infty} W_{\beta}(\phi_{\chi,s,v}, \begin{bmatrix} \mathbf{c}_v \\ 1 \end{bmatrix})|_{s=0} \in \mathcal{O}$$

for some finite extension \mathcal{O} of $\mathcal{O}_{\mathcal{F},(p)}$. Since χ is unramified at p , we have $\mathbf{a}_{\beta}(\mathbb{E}_{\chi}^h, \mathbf{c}) = 0$ if $\beta \notin O \otimes \mathbf{Z}_p$ by (4.7). Note that $v(\chi^*(\varpi_v)) = 1 - k$ and $v'(\chi^*(\varpi_v)) = 0$ if $v|p$ and $v' \neq v$. Thus if $\beta \in O \otimes \mathbf{Z}_p$, then

$$\begin{aligned} & |D_{\mathcal{F}}|_{\mathbf{R}}^{-1} \mathbf{N}_{\mathcal{F}/\mathbf{Q}}(\beta)^{k-1} \prod_{v|p} W_{\beta}(\phi_{\chi,s,v}, 1)|_{s=0} \\ &= \mathbf{N}_{\mathcal{F}/\mathbf{Q}}(\beta)^{k-1} \prod_{v|p} (1 + \chi^*(\varpi_v) + \cdots + \chi^*(\varpi_v)^{v(\beta)}) \cdot |D_{\mathcal{F}}|_{\mathbf{R}}^{-1} |D_{\mathcal{F}}|_{\mathbf{Q}_p}^{-1} \in \mathcal{O}. \end{aligned}$$

□

The following proposition is directly deduced from the construction of the section $\phi_{\chi,s}$ and the description of Hecke action (2.6). The details are omitted

Proposition 4.8. *Set $\mathbf{D}_{\mathfrak{c}} := \prod_{v|\mathfrak{c}-D_{\mathcal{K}/\mathcal{F}}} \mathcal{K}_v^{\times}$. Under the assumptions in Proposition 4.7, we have*

- (1) \mathbb{E}_{χ}^h is an $U_{\mathfrak{l}}$ -eigenform in $\mathcal{M}_k(K_0(\mathfrak{l}), \mathcal{O})$ with the eigenvalue $\chi_+^{-1}(\varpi_{\mathfrak{l}})$,
- (2) $\mathbb{E}_{\chi}^h| [r] = \chi^{-1}(r)\mathbb{E}_{\chi}^h$ for $r \in \mathbf{D}_{\mathfrak{c}}$,
- (3) $\mathbb{E}_{\chi}^h(x_n(ta)) = \chi^{-1}(a)\mathbb{E}_{\chi}^h(x_n(t))$ for $a \in \mathbf{D}_{\mathfrak{c}}\mathbf{A}_{\mathcal{F}}^{\times}U_n$.

4.5. For $\sigma \in \mathbf{a}$ and an integer n , let

$$\delta_n^{\sigma} := \frac{1}{2\pi i} \left(\frac{\partial}{\partial \tau_{\sigma}} + \frac{n}{2iy_{\sigma}} \right)$$

be the Maass-Shimura's differential operator. Put

$$\delta_k^{\kappa\sigma\sigma} := \delta_{k+2\kappa_{\sigma}}^{\sigma} \circ \cdots \circ \delta_{k+2}^{\sigma} \circ \delta_k^{\sigma} \text{ and } \delta_k^{\kappa} = \prod_{\sigma \in \Sigma} \delta_k^{\kappa\sigma\sigma}.$$

Then the weight raising differential operator $V_+^{\kappa\sigma}$ in §4.3.2 is the representation theoretic avatar of $\delta_k^{\kappa\sigma\sigma}$ in virtue of the following identity:

$$\delta_k^{\kappa} = \frac{1}{(-8\pi)^{\kappa}} V_+^{\kappa}.$$

We thus have

$$(4.22) \quad \delta_k^{\kappa}\mathbb{E}_{\chi}^h = \frac{1}{(-4\pi)^{\kappa}} \cdot \frac{\Gamma_{\Sigma}(k\Sigma + \kappa)}{\Gamma_{\Sigma}(k\Sigma)} \cdot \mathbb{E}_{\chi}^{n.h.}.$$

5. EVALUATION OF EISENSTEIN SERIES AT CM POINTS

5.1. Period integral. Let $\mathcal{E}_\chi := \widehat{\mathbb{E}}_\chi^h$ be the p -adic avatar of \mathbb{E}_χ^h as in (2.5). Let $\{\theta(\sigma)\}_{\sigma \in \Sigma}$ be the Dwork-Katz p -adic differential operators on p -adic modular forms introduced in [Kat78, Cor.(2.6.25)] and let $\theta^\kappa = \prod_{\sigma \in \Sigma} (\theta(\sigma))^{k_\sigma}$. We consider the Hida's measure $\varphi_\chi^\kappa := \varphi_{\theta^\kappa \mathcal{E}_\chi}$ attached to the U_Γ -eigenform $\theta^\kappa \mathcal{E}_\chi \in V(K_0(\mathfrak{l}), \overline{\mathbf{Z}}_p)$ as in (3.2). Let ν be a character on Cl_n . We have

$$(5.1) \quad \int_{Cl_\infty} \nu d\varphi_\chi^\kappa = \chi_+(\varpi_\Gamma)^n \cdot \sum_{[t]_n \in Cl_n} \theta^\kappa \mathcal{E}_\chi(x_n(t)) \widehat{\chi} \nu(t).$$

Let $(\Omega_\infty, \Omega_p) \in (\mathbf{C}^\times)^\Sigma \times (\overline{\mathbf{Z}}_p^\times)^\Sigma$ be the complex and p -adic CM periods of (\mathcal{K}, Σ) introduced in [HT93, (4.4 a,b) p.211] (cf. (Ω, c) in [Kat78, (5.1.46), (5.1.48)]). From [Kat78, (2.4.6), (2.6.8), (2.6.33)] we can deduce the following important identity:

$$(5.2) \quad \frac{1}{\Omega_p^{k_\Sigma + 2\kappa}} \cdot \theta^\kappa \mathcal{E}_\chi(x_n(t)) = \frac{(2\pi i)^{k_\Sigma + 2\kappa}}{\Omega_\infty^{k_\Sigma + 2\kappa}} \cdot \delta_k^\kappa \mathbb{E}_\chi^h(x_n(t)) \quad (t \in (\mathbf{A}_{\mathcal{K},f}^{(p)})^\times).$$

Therefore, it follows from (4.22) and (5.1) that

$$(5.3) \quad \begin{aligned} \int_{Cl_\infty} \nu d\varphi_\chi^\kappa &= \chi_+(\varpi_\Gamma)^n \cdot \frac{(2\pi i)^{k_\Sigma + 2\kappa}}{\Omega_\infty^{k_\Sigma + 2\kappa}} \cdot \sum_{[t]_n \in Cl_n} \delta_k^\kappa \mathbb{E}_\chi^h(x_n(t)) \chi \nu(t) \\ &= \chi_+(\varpi_\Gamma)^n \cdot \frac{\pi^\kappa \Gamma_\Sigma(k_\Sigma + 2\kappa)}{\sqrt{|D_{\mathcal{F}}|_{\mathbf{R}}} \operatorname{Im}(\vartheta)^\kappa \cdot \Omega_\infty^{k_\Sigma + 2\kappa}} \cdot \sum_{[t]_n \in Cl_n} E_\chi^{n,h.}(\rho(t)\varsigma^{(n)}) \chi \nu(t). \end{aligned}$$

Here we choose $t \in (\mathbf{A}_{\mathcal{K},f}^{(p)})^\times$ for a set of representatives $[t]_n \in Cl_n$ (so $\widehat{\chi}(t) = \chi(t)$).

We shall relate (5.3) to certain period integral of Eisenstein series. First we fix the choices of measures. For each finite place v of \mathcal{F} , let $d^\times z_v$ be the normalized Haar measure on \mathcal{K}_v^\times so that $\operatorname{vol}(R_v^\times, d^\times z_v) = 1$ and let $d^\times t_v = d^\times z_v / d^\times x_v$ be the quotient measure on $\mathcal{K}_v^\times / \mathcal{F}_v^\times$. If v is archimedean, let $d^\times t_v$ be the Haar measure on $\mathcal{K}_v^\times / \mathcal{F}_v^\times = \mathbf{C}^\times / \mathbf{R}^\times$ normalized so that $\operatorname{vol}(\mathbf{C}^\times / \mathbf{R}^\times, d^\times t_v) = 1$. Let $d^\times t = \prod'_v d^\times t_v$ be a Haar measure on $\mathbf{A}_{\mathcal{K}}^\times / \mathbf{A}_{\mathcal{F}}^\times$ and let $d^\times \bar{t}$ be the quotient measure of $d^\times t$ on $\mathcal{K}^\times \mathbf{A}_{\mathcal{F}}^\times \backslash \mathbf{A}_{\mathcal{K}}^\times$ by the counting measure on \mathcal{K}^\times . We define the period integral of $E_\chi^{n,h.}$ by

$$l_{\mathcal{K}}(E_\chi^{n,h.}, \nu) := \int_{\mathcal{K}^\times \mathbf{A}_{\mathcal{F}}^\times \backslash \mathbf{A}_{\mathcal{K}}^\times} E_\chi^{n,h.}(\rho(t)\varsigma^{(n)}) \chi \nu(t) d^\times \bar{t}.$$

Then the last term in (5.3) can be expressed as the following period integral

$$\sum_{[t]_n \in Cl_n} E_\chi^{n,h.}(\rho(t)\varsigma^{(n)}) \chi \nu(t) = \frac{1}{\operatorname{vol}(\overline{U}_n, d^\times \bar{t})} l_{\mathcal{K}}(E_\chi^{n,h.}, \nu).$$

The rest of this section is devoted to the calculation of $l_{\mathcal{K}}(E_\chi^{n,h.}, \nu)$. For brevity, we write ϕ_v for $\phi_{\chi,s,v}$ if $v \in \mathbf{h}$ and for $\phi_{k,\kappa_\sigma,s,\sigma}^{n,h.}$ in (4.5) if $v = \sigma \in \mathbf{a}$. The first step is to decompose $l_{\mathcal{K}}(E_\chi^{n,h.}, \nu)$ into a product of local integrals $l_{\mathcal{K}_v}(\phi_v, \nu)$, where

$$l_{\mathcal{K}_v}(\phi_v, \nu) = \int_{\mathcal{K}_v^\times / \mathcal{F}_v^\times} \phi_v(\rho(t)\varsigma_v^{(n)}) \chi \nu(t_v) d^\times t_v.$$

Since $\rho : \mathcal{F}^\times \backslash \mathcal{K}^\times \rightarrow B(\mathcal{F}) \backslash G(\mathcal{F})$ is a bijection, we find that

$$\begin{aligned} l_{\mathcal{K}}(E_\chi^{n,h.}, \nu) &= \int_{\mathcal{K}^\times \mathbf{A}_{\mathcal{F}}^\times \backslash \mathbf{A}_{\mathcal{K}}^\times} E_{\mathbf{A}}(\rho(t)\varsigma^{(n)}, \phi_{\chi,s}^{n,h.}) \chi \nu(t) d^\times \bar{t}|_{s=0} \\ &= \int_{\mathcal{K}^\times \mathbf{A}_{\mathcal{F}}^\times \backslash \mathbf{A}_{\mathcal{K}}^\times} \sum_{\gamma \in G(\mathcal{F})/B(\mathcal{F})} \phi_{\chi,s}^{n,h.}(\gamma \rho(t)\varsigma^{(n)}) \chi \nu(t) d^\times \bar{t}|_{s=0} \\ &= \int_{\mathbf{A}_{\mathcal{F}}^\times \backslash \mathbf{A}_{\mathcal{K}}^\times} \phi_{\chi,s}^{n,h.}(\rho(t)\varsigma^{(n)}) \chi \nu(t) d^\times t|_{s=0} \\ &= \prod_v \int_{\mathcal{K}_v^\times / \mathcal{F}_v^\times} \phi_v(\rho(t_v)\varsigma_v^{(n)}) \chi \nu(t_v) d^\times t_v|_{s=0} = \prod_v l_{\mathcal{K}_v}(\phi_v, \nu)|_{s=0}. \end{aligned}$$

Write E for \mathcal{K}_v and F for \mathcal{F}_v . In what follows, we suppress v from the notation and proceed to calculate the local integral $l_E(\phi, \nu)$.

5.2. $v \in S^\circ$ or $v|\mathfrak{f}\mathfrak{f}^c$. In this case, $\phi = f_{\Phi_v, s}$ is the Godement section associated to the Bruhat-Schwartz function Φ_v defined in §4.3.4. We have

$$\begin{aligned} l_E(\phi, \nu) &= \int_{E^\times/F^\times} f_{\Phi}(\rho(t))\chi\nu(t)d^\times t \\ &= \int_{E^\times/F^\times} \chi\nu(t)|t|_E^s d^\times t \int_{F^\times} \Phi_v((0, x)\rho(t)\varsigma)\chi_+(x)|x|_F^{2s} d^\times x \\ &= \int_{E^\times/F^\times} \int_{F^\times} \chi\nu(tx)|tx|_E^s \Phi_v((0, 1)\rho(tx)\varsigma)d^\times t d^\times x \\ &= \int_{E^\times} \chi\nu(z)|z|_E^s \Phi_v((0, 1)\rho(z)\varsigma)d^\times z \\ &= Z(s, \chi\nu, \Phi_E), \end{aligned}$$

where Φ_E is defined by

$$\Phi_E(z) := \Phi_v((0, 1)\rho(z)\varsigma).$$

Suppose $v \in S^\circ$. By definition, $\Phi_v = \mathbb{1}_{O_v \oplus O_v^*}$, and hence $\Phi_E = \mathbb{1}_{R_v}$ is the characteristic function of R_v . It is clear that

$$Z(s, \chi\nu, \Phi_E) = L(s, \chi\nu).$$

Suppose $v|\mathfrak{f}\mathfrak{f}$ is a split prime. We write $E = Fe_{\bar{w}} \oplus Fe_w$ and $\vartheta = -\vartheta_w e_{\bar{w}} + \vartheta_w e_w$ with $w|\mathfrak{f}$ as in §3.2. Then by definition,

$$\Phi_E(z) = \Phi_v((0, 1)\rho(z) \begin{bmatrix} -\vartheta_w & -\frac{1}{2} \\ 1 & -\frac{1}{2\vartheta_w} \end{bmatrix}) = \varphi_{\bar{w}}(x)\widehat{\varphi}_w(-\frac{y}{2\vartheta_w}) \quad (z = xe_{\bar{w}} + ye_w).$$

As $\mathfrak{f} \nmid \mathfrak{C}$, $\nu = (\nu_{\bar{w}}, \nu_w)$ is unramified at v . Therefore,

$$\begin{aligned} Z(s, \chi\nu, \Phi_E) &= \chi_w(-2\vartheta_w) \int_{E^\times} \chi_{\bar{w}}\nu_{\bar{w}}(x)\chi_w\nu_w(y)\varphi_{\bar{w}}(x)\widehat{\varphi}_w(y)|xy|^s d^\times x d^\times y \\ &= \chi_w(-2\vartheta_w) Z(s, \chi_{\bar{w}}\nu_{\bar{w}}, \varphi_{\bar{w}}) Z(s, \chi_w\nu_w, \widehat{\varphi}_w). \end{aligned}$$

By Tate's local functional equation, we have

$$Z(s, \chi_w\nu_w, \widehat{\varphi}_w) = \frac{L(s, \chi_w\nu_w) \cdot \chi_w\nu_w(-1)}{\epsilon(s, \chi_w\nu_w, \psi)L(1-s, \chi_w^{-1}\nu_w^{-1})} \cdot Z(1-s, \chi_w^{-1}\nu_w^{-1}, \varphi_w).$$

We find that

$$Z(s, \chi\nu, \Phi_E)|_{s=0} = L(0, \chi\nu) \cdot L_{\mathfrak{S}} \cdot C_{\mathfrak{f}},$$

where

$$(5.4) \quad L_{\mathfrak{S}} = \prod_{\mathfrak{q}|\mathfrak{S}} L(1, \chi_{\mathfrak{q}}^{-1}\nu_{\mathfrak{q}}^{-1})^{-1} \text{ and } C_{\mathfrak{f}} = \prod_{w|\mathfrak{f}} \frac{\chi_w(2\vartheta_w) \cdot \nu_w^{-1}(d_F \varpi^{a(\chi_w)})}{\epsilon(0, \chi_w, \psi)}.$$

Note that $C_{\mathfrak{f}} \in \overline{\mathbf{Z}}_p^\times$.

5.3. v is archimedean or $v|D_{\mathcal{K}/\mathcal{F}}\mathfrak{C}^-$. Note that ν is trivial on E^\times if v is inert or ramified because ν is a character of $\text{Gal}(\mathcal{K}_{\infty}^-/\mathcal{K})$. If $v|D_{\mathcal{K}/\mathcal{F}}\mathfrak{J}$, by the very definition of $\phi = \phi_{\chi, s, v}$ in (4.13), we find that

$$\begin{aligned} l_E(\phi, \nu) &= \int_{E^\times/F^\times} \phi(\rho(t)\varsigma_v)\chi\nu(t)d^\times t \\ &= \text{vol}(E^\times/F^\times, d^\times t) \cdot \begin{cases} 1 & \cdots v|\mathfrak{C}^-, \\ L(s, \chi_v) & \cdots v|\mathfrak{A}'. \end{cases} \end{aligned}$$

If $v = \sigma|_\infty$ and $\phi = \phi_{k,\kappa,s,\sigma}^{n.h.}$ is defined in (4.5), then

$$\begin{aligned} l_E(\phi, \nu) &= \int_{E^\times/F^\times} \phi_{k,\kappa,s,\sigma}^{n.h.}(\rho(t) \begin{bmatrix} \text{Im } \sigma(\vartheta) & \\ & 1 \end{bmatrix}) \chi \nu(t) d^\times t \\ &= \text{vol}(\mathbf{C}^\times/\mathbf{R}^\times, d^\times t). \end{aligned}$$

5.4. $v = \mathbf{l}$. A direct computation shows that

$$\rho(t)\varsigma_{\mathbf{l}}^{(n)} = \rho(x + y\boldsymbol{\theta}_{\mathbf{l}}) \begin{bmatrix} -b_{\mathbf{l}}d_{\mathcal{F}_{\mathbf{l}}}\varpi_{\mathbf{l}}^n & 1 \\ a_{\mathbf{l}}d_{\mathcal{F}_{\mathbf{l}}}\varpi_{\mathbf{l}}^n & 0 \end{bmatrix} = \begin{bmatrix} * & * \\ xd_{\mathcal{F}_{\mathbf{l}}}\varpi_{\mathbf{l}}^n a_{\mathbf{l}} & yb_{\mathbf{l}} \end{bmatrix} \quad (t = x + y\boldsymbol{\theta}_{\mathbf{l}}).$$

We thus find that

$$\rho(t)\varsigma_{\mathbf{l}}^{(n)} \in B(F)\mathbf{w}N(O_{\mathbf{l}}^*) \iff t \in \mathcal{F}_{\mathbf{l}}^\times(1 + \varpi_{\mathbf{l}}^n\boldsymbol{\theta}_{\mathbf{l}}O_{\mathbf{l}}).$$

Let $R_{n,\mathbf{l}} = R_n \otimes_O O_{\mathbf{l}}$. Then $R_{n,\mathbf{l}}^\times = O^\times(1 + \varpi_{\mathbf{l}}^n\boldsymbol{\theta}_{\mathbf{l}}O_{\mathbf{l}})$. Let $\pi : E^\times \rightarrow E^\times/F^\times$ be the quotient map. Thus we have

$$\begin{aligned} l_E(\phi, \nu) &= \int_{E^\times/F^\times} \phi(\rho(t)\varsigma_{\mathbf{l}}^{(n)}) \chi \nu(t) d^\times t \\ &= \chi_+(\varpi)^{-n} \int_{E^\times/F^\times} \mathbb{I}_{\pi(R_{n,\mathbf{l}}^\times)}(t) d^\times t = \chi_+(\varpi)^{-n} \text{vol}(\pi(R_{n,\mathbf{l}}^\times), d^\times t). \end{aligned}$$

5.5. **Evaluation formula.** We summarize our local calculations in the following proposition.

Proposition 5.1. *Suppose that either of the conditions (1-3) in Proposition 4.7 holds. Then we have the following evaluation formula:*

$$\frac{1}{\Omega_p^{k\Sigma+2\kappa}} \cdot \int_{Cl_\infty} \nu d\varphi_\chi^\kappa = \frac{\pi^\kappa \Gamma_\Sigma(k\Sigma + \kappa) L^{(1)}(0, \chi\nu)}{\Omega_\infty^{k\Sigma+2\kappa}} \cdot \frac{2^r L_{\mathfrak{S}} C_{\mathfrak{F}}}{\sqrt{|D_{\mathcal{F}}|_{\mathbf{R}}(\text{Im } \vartheta)^\kappa}},$$

where r is the number of prime factors of $D_{\mathcal{K}/\mathcal{F}}$.

PROOF. We note that

$$\begin{aligned} \sum_{[t]_n \in Cl_n} E_\chi^{n.h.}(\rho(t)\varsigma^{(n)}) \chi \nu(t) &= \frac{1}{\text{vol}(\overline{U}_n, d^\times t)} \cdot l_{\mathcal{K}}(E_\chi^{n.h.}, \nu) \\ &= \frac{1}{\text{vol}(\overline{U}_n, d^\times t)} \cdot \prod_v l_{\mathcal{K}_v}(\phi_{\chi,s,v}, \nu_v)|_{s=0} \\ &= L^{(1)}(0, \chi\nu) \cdot 2^r L_{\mathfrak{S}} C_{\mathfrak{F}} \cdot \chi_+(\varpi)^{-n}. \end{aligned}$$

The proposition follows from (5.3) immediately. \square

6. NON-VANISHING OF EISENSTEIN SERIES MODULO p

6.1. Throughout this section, we retain the assumptions (unr), (ord) and $(p\mathbf{l}, D_{\mathcal{K}/\mathcal{F}}\mathfrak{C}) = 1$. Let χ be a Hecke character of \mathcal{K}^\times and take \mathfrak{S} to be a split prime \mathfrak{q} as in §4.1. We remark that an auxiliary split prime \mathfrak{q} is introduced to assure the assumption (2) in Proposition 4.7, so the L -value in the evaluation formula Proposition 5.1 has an extra local factor $L_{\mathfrak{q}} = L(1, \chi_{\mathfrak{q}}^{-1}\nu_{\mathfrak{q}}^{-1})^{-1}$. However, this is harmless to (NV) property since the closed subgroup generated by associated Frobenius $\text{Frob}_{\mathfrak{q}}$ in $\text{Gal}(\mathcal{K}_{\mathbf{l}}^-/\mathcal{K})$ is non-trivial, and hence the set $\{\nu \in \mathfrak{X}_{\mathbf{l}}^- \mid L(1, \nu_{\mathfrak{q}}^{-1}\chi_{\mathfrak{q}}^{-1})^{-1} \equiv 0 \pmod{\mathfrak{m}}\}$ is a proper closed subset of $\mathfrak{X}_{\mathbf{l}}^-$.

To establish (NV) property for (χ, \mathbf{l}) , by Hida's non-vanishing criterion of a p -adic measure associated to eigenforms (Theorem 3.1) and the evaluation formula of our Eisenstein measure $d\varphi_\chi^\kappa$ (Proposition 5.1), it suffices to show the non-vanishing modulo p of some Fourier coefficient of $(\theta^\kappa \mathcal{E}_\chi)^\mathcal{R} = \theta^\kappa \mathcal{E}_\chi^\mathcal{R}$ at some cusp $(O, \mathfrak{c}(\mathfrak{a})^{-1})$.

Lemma 6.1. *Put $(\mathbb{E}_\chi^h)^\mathcal{R} = \sum_{r \in \mathcal{R}} \mathbb{E}_\chi^h|_r$. Then we have*

$$(\mathbb{E}_\chi^h)^\mathcal{R} = \#\Delta^{\text{alg}} \cdot \mathbb{E}_\chi^h.$$

PROOF. It can be shown that Δ_{alg} is generated by ramified primes, so \mathcal{R} can be chosen from elements in $\prod_{v|D_{\mathcal{K}/\mathcal{F}}} \mathcal{K}_v^\times$. The lemma follows from Proposition 4.8 (2). \square

Remark 6.2. Since $\mathcal{E}_\chi^{\mathcal{R}}$ is the p -adic avatar of $(\mathbb{E}_\chi^h)^{\mathcal{R}}$ and $\#\Delta_{\text{alg}}$ is a power of 2, from the above lemma and the following identity (cf. [HT93, (1.23)])

$$\mathbf{a}_\beta(\theta^{\mathcal{R}} \mathcal{E}_\chi, \mathbf{c}) = \beta^{\mathcal{R}} \mathbf{a}_\beta(\mathbb{E}_\chi^h, \mathbf{c}),$$

we conclude that (NV) property for (χ, \mathfrak{l}) holds if the following hypothesis (H') is verified:

(H') For every $u \in O_{\mathfrak{l}}$ and a positive integer r , there exist $\beta \in O_{(p)}^\times$ and $\mathbf{c} = \mathbf{c}(\mathfrak{a})$ such that $\beta \equiv u \pmod{\mathfrak{l}^r}$ and

$$\mathbf{a}_\beta(\mathbb{E}_\chi^h, \mathbf{c}) \not\equiv 0 \pmod{\mathfrak{m}}.$$

6.2. Let $\bar{\chi}$ be the reduction modulo \mathfrak{m} of the p -adic avatar of χ and let $\bar{\chi}_+ = \bar{\chi}|_{\mathbf{A}_{\mathcal{F}}^\times}$. Let $\omega_{\mathcal{F}} : \mathbf{A}_{\mathcal{F}}^\times / \mathcal{F}^\times \rightarrow \mu_{p-1}$ be the Teichmüller character regarded as a Hecke character of \mathcal{F}^\times via geometrically normalized reciprocity law. We first treat the case χ is not *residually self-dual*, namely

$$\bar{\chi}_+ \not\equiv \tau_{\mathcal{K}/\mathcal{F}} \omega_{\mathcal{F}} \pmod{\mathfrak{m}}.$$

The following proposition is due to Hida [Hid04a].

Proposition 6.3 (Hida). *Suppose that χ is not residually self-dual. Then (NV) holds for (χ, \mathfrak{l}) if for every $v|\mathfrak{C}^-$ there exists $\eta_v \in \mathcal{F}_v^\times$ such that*

$$(6.1) \quad A_{\eta_v}(\chi_v) \not\equiv 0 \pmod{\mathfrak{m}}.$$

PROOF. We have to verify the hypothesis (H') in Remark 6.2. Given $u \in O_{\mathfrak{l}}$ and a positive integer r , we extend $\eta_{\mathfrak{C}^-} = (\eta_v)_{v|\mathfrak{C}^-}$ to an idele $\eta = (\eta_v)$ in $\mathbf{A}_{\mathcal{F}}^\times$ by taking $\eta_v = 1$ for $v \nmid \mathfrak{l}\mathcal{D}_{\mathcal{K}/\mathcal{F}}$ or $v|\mathfrak{F}\mathfrak{F}^c$, $\eta_{\mathfrak{l}} = u$ and $\eta_v = \varpi_v^{-1}$ as in (4.19) if $v|\mathfrak{R}'$. Let $\mathfrak{b}^- := \prod_{\mathfrak{q}|\mathfrak{C}^-} \mathfrak{q}^{M_{\mathfrak{q}}}$, $M_{\mathfrak{q}} = \max\{v_{\mathfrak{q}}(\mathfrak{C}^-), v_{\mathfrak{q}}(\mathfrak{C}^-) - v_{\mathfrak{q}}(\eta_v)\}$ and put

$$U = \left\{ (x_\infty, x_{\mathcal{F}}) \in \mathbf{R}_+^{[\mathcal{F}:\mathbf{Q}]} \times (O \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}})^\times \mid x_{\mathcal{F}} \equiv 1 \pmod{D_{\mathcal{K}/\mathcal{F}} \mathfrak{l}^r \mathfrak{b}^-} \right\}.$$

Let $\mathbf{c} = \mathbf{c}(\mathfrak{a})$ and \mathbf{c} be the associated idele as in Proposition 4.7 and consider the idele class $[\mathbf{c}\eta^{-1}] := \mathcal{F}^\times \mathbf{c}\eta^{-1}U$ in $\mathbf{A}_{\mathcal{F}}^\times$. For each idele $a \in O \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}$ in the class $[\mathbf{c}\eta^{-1}]$ such that each local component $a_v = 1$ at $v|pD_{\mathcal{K}/\mathcal{F}}\mathfrak{C}\mathfrak{C}^c$, we can write $a = \beta \mathbf{c}\eta^{-1}u$ for $\beta \in O_{(p)}^\times$ and $u \in U$, and from the explicit formula of $\mathbf{a}_\beta(\mathbb{E}_\chi^h, \mathbf{c})$ (Proposition 4.7 combined with Lemma 4.1, (4.7), (4.10) and (4.12)), we find that

$$(6.2) \quad \begin{aligned} \mathbf{a}_\beta(\mathbb{E}_\chi^h, \mathbf{c}) &= C_\beta \cdot \prod_{v \in S^o} \frac{1 - \chi^*(\varpi_v)^{v(a_v)+1}}{1 - \chi^*(\varpi_v)} \cdot \chi_+(\mathbf{c}_v), \text{ where} \\ C_\beta &= |D_{\mathcal{F}}|_{\mathbf{R}}^{-1} \cdot \mathbf{N}_{\mathcal{F}/\mathbf{Q}}(\beta)^{k-1} \cdot \prod_{w|\mathfrak{F}} \chi_+(\beta) \varphi_{\bar{w}}(\beta) |D_{\mathcal{F}}^{-1}|_{\mathcal{F}_v} \cdot |\mathbf{c}|_{\mathcal{F}_v} \cdot \prod_{w|\mathfrak{R}'} \chi^{-1}(\theta_w) |\varpi_w D_{\mathcal{F}}^{-1}|_{\mathcal{F}_v} \\ &\quad \times \prod_{v|\mathfrak{C}^-} A_{\eta_v}(\chi_v) \cdot |D_{\mathcal{F}}^{-1}|_{\mathcal{F}_v} \psi^o(-2^{-1}t_v \beta). \end{aligned}$$

By our choices of η and a , $C_\beta \not\equiv 0 \pmod{\mathfrak{m}}$.

Suppose that $\mathbf{a}_\beta(\mathbb{E}_\chi^h, \mathbf{c}) \equiv 0 \pmod{\mathfrak{m}}$ for all $\beta \in O_{(p)}^\times$ such that $\beta \equiv u \pmod{\mathfrak{l}^r}$. In particular, for every uniformizer $\varpi_v \in [\mathbf{c}\eta^{-1}]$ at $v \nmid p\mathfrak{C}\mathfrak{C}^c D_{\mathcal{K}/\mathcal{F}}$, we deduce from (6.2) that $\chi^*(\varpi_v) \equiv \bar{\chi}_+ \omega_{\mathcal{F}}^{-1}(\varpi_v) \equiv -1 \pmod{\mathfrak{m}}$. Moreover, the argument in [Hid04a, p.780] shows that $\bar{\chi}_+ \omega_{\mathcal{F}}^{-1}$ is a quadratic character of level U and takes value -1 on $[\mathbf{c}\eta^{-1}]$. Moving $\mathbf{c} = \mathbf{c}(\mathfrak{a})$ among prime-to- $p\mathfrak{C}\mathfrak{D}_{\mathcal{K}/\mathcal{F}}$ ideals \mathfrak{a} of R , we conclude that $\bar{\chi}_+ \omega_{\mathcal{F}}^{-1} \equiv \tau_{\mathcal{K}/\mathcal{F}} \pmod{\mathfrak{m}}$, which is a contradiction. \square

Lemma 6.4. *Let $v|\mathfrak{C}^-$ and w be the place of \mathcal{K} above v . Suppose that $\mu_p(\chi_v) = 0$. Then there exists $\eta_v \in \mathcal{F}_v^\times$ such that*

$$A_{\eta_v}(\chi_v) \not\equiv 0 \pmod{\mathfrak{m}}.$$

Moreover, if v is inert and $\chi_v|_{\mathcal{F}_v^\times} = \tau_{\mathcal{K}_v/\mathcal{F}_v}$, then η_v can be further chosen so that $v(\eta_v) = -w(\mathfrak{C}^-)$.

PROOF. First we make some observations. Notation is as in §4.3.6. We let $F = \mathcal{F}_v$ and $E = \mathcal{K}_v$. Let $\varpi = \varpi_v$ be a uniformizer of F and $\theta = \theta_v$. Recall that $\mu_p(\chi_v) = \inf_{x \in E^\times} v_p(\chi_v(x) - 1)$, so the assumption

$\mu_p(\chi_v) = 0$ is equivalent to $\chi|_{E^\times} \not\equiv 1 \pmod{\mathfrak{m}}$. Since $A_\beta(\chi) = \psi^\circ(t\beta)\tilde{A}_\beta(\chi)$, it is equivalent to showing the lemma for $\tilde{A}_\beta(\chi)$. For an integer m and $a \in F$, we put

$$c_m(a) = \int_{O_v} \chi^{-1}(a + \varpi^m x + \boldsymbol{\theta}) dx.$$

By (4.17), for $\eta \in \varpi^{-m} O_v^\times$ and every sufficiently large positive integer M (depending on m) we have

$$\tilde{A}_\eta(\chi) = \int_{\varpi^{-M} O_v} \chi^{-1}(x + \boldsymbol{\theta}) \psi^\circ(\eta x) dx.$$

Thus for each $a \in F$ we find that

$$\begin{aligned} \int_{\varpi^{-m} O_v^\times} \tilde{A}_\eta(\chi) \psi^\circ(-\eta a) d\eta &= \int_{\varpi^{-M} O_v} \chi^{-1}(x + \boldsymbol{\theta}) dx \int_{\varpi^{-m} O_v^\times} \psi^\circ(\eta(x - a)) d\eta \\ (6.3) \qquad \qquad \qquad &= \int_{O_v} \chi^{-1}(a + \varpi^m x + \boldsymbol{\theta}) dx - \int_{O_v} \chi^{-1}(a + \varpi^{m-1} x + \boldsymbol{\theta}) dx \\ &= c_m(a) - c_{m-1}(a). \end{aligned}$$

Now we prove the first assertion by contradiction. Suppose that $\tilde{A}_\eta(\chi) \equiv 0 \pmod{\mathfrak{m}}$ for all $\eta \in F^\times$. The equation (6.3) implies that for every $a \in F$, $c_m(a) \pmod{\mathfrak{m}}$ is a constant independent of m . Taking a sufficiently large m_0 , we find that $c_m(a) \equiv c_{m_0}(a) = \chi^{-1}(a + \boldsymbol{\theta}) \pmod{\mathfrak{m}}$ for every integer m and $a \in F$. On the other hand, it is clear that $c_m(a) = c_m(a')$ whenever $a, a' \in \varpi^m O_v$, so we conclude that the function $a \mapsto \chi^{-1}(a + \boldsymbol{\theta}) \pmod{\mathfrak{m}}$ is the constant function $\chi^{-1}(\boldsymbol{\theta})$ on F , and hence $\chi(1 + a\boldsymbol{\theta}) \equiv 1 \pmod{\mathfrak{m}}$ for all $a \in F$. This implies that $\chi_v \equiv 1 \pmod{\mathfrak{m}}$, which is a contradiction.

We proceed to prove the second assertion. Suppose that v is inert and $\chi_v|_F = \tau_{E/F}$. Note that in this case $\mu_p(\chi_v) = 0$ is equivalent to $\chi_v|_{\mathcal{O}_E^\times} \not\equiv 1 \pmod{\mathfrak{m}}$. Let $m = w(\mathfrak{C}^-) \geq 1$. If $\tilde{A}_\eta(\chi) \equiv 0 \pmod{\mathfrak{m}}$ for all $\eta \in \varpi^{-m} O_v^\times$, then it follows from (6.3) that

$$\chi^{-1}(a + \boldsymbol{\theta}) = c_m(a) \equiv c_{m-1}(0) \pmod{\mathfrak{m}} \text{ for } a \in \varpi^{m-1} O_v.$$

Therefore, $a \mapsto \chi^{-1}(a + \boldsymbol{\theta}) \pmod{\mathfrak{m}}$ is the constant function $\chi^{-1}(\boldsymbol{\theta})$ on $\varpi^{m-1} O_v$, and hence $\chi(1 + a\boldsymbol{\theta}) \equiv 1 \pmod{\mathfrak{m}}$ for all $a \in \varpi^{m-1} O_v$. If $m = w(\mathfrak{C}^-) > 1$, this is impossible, and if $m = 1$, this contradicts to the assumption that $\chi|_{\mathcal{O}_E^\times} \not\equiv 1 \pmod{\mathfrak{m}}$. \square

The following corollary is an immediate consequence of Proposition 6.3 and Lemma 6.4, which gives a partial generalization of Hida's theorem.

Corollary 6.5. *Suppose that the following conditions hold:*

- (L) $\mu_p(\chi_v) = 0$ for every $v|\mathfrak{C}^-$,
- (N) χ is not residually self-dual.

Then **(NV)** holds for (χ, \mathfrak{l}) .

6.3. We consider the self-dual case. First we recall the following lemma on local root numbers of self-dual characters.

Lemma 6.6 (Prop. 3.7 [MS00]). *Let χ be a self-dual character, i.e. $\chi|_{\mathbf{A}_F} = \tau_{\mathcal{K}/F} \cdot | \cdot |_{\mathbf{A}_F}$. Then*

- (1) $W(\chi_v^*) = \pm \chi_v^*(2\vartheta)$.
- (2) If v is split, $W(\chi_v^*) = \chi_v^*(2\vartheta)$.
- (3) If v is inert, $W(\chi_v^*) = (-1)^{a(\chi_v^*) + v(\mathfrak{c}(R))} \chi_v^*(2\vartheta)$ ($\mathfrak{c}(R) = \mathcal{D}_F^{-1}(2\vartheta \mathcal{D}_{\mathcal{K}/F}^{-1})$).

Proposition 6.7. *Let χ be a self-dual character of the global root number $W(\chi^*) = +1$ ($\chi^* = \chi| \cdot |_{\mathbf{A}_K}^{-\frac{1}{2}}$). Suppose that \mathfrak{l} splits in \mathcal{K} and that there exists $\eta_v \in \mathcal{F}_v^\times$ for each $v|\mathfrak{C}^-$ such that*

- (i) $A_{\eta_v}(\chi_v) \not\equiv 0 \pmod{\mathfrak{m}}$,
- (ii) $W(\chi_v^*) \tau_{\mathcal{K}/F}(\eta_v) = \chi_v^*(2\vartheta)$.

Then **(NV)** holds for (χ, \mathfrak{l}) .

PROOF. We need to verify the hypothesis (H') in Remark 6.2. Given $u \in O_{\mathfrak{l}}$ and a positive integer r , we extend $(\eta_v)_{v|\mathfrak{C}^-}$ to an idele $\eta = (\eta_v)$ in \mathbf{A}_F^\times such that

- $\eta_l \equiv u \pmod{l^r}$ and $\eta_v = 1$ for every split prime $v \neq l$,
- $W(\chi_v^*)\tau_{\mathcal{K}/\mathcal{F}}(\eta_v) = \chi_v^*(2\vartheta)$ for every $v|\mathbf{h}$.

By Lemma 6.6, this is possible since l splits in \mathcal{K} . On the other hand, it is well known that $W(\chi_\sigma^*) = i^{2\kappa_\sigma+1} = \chi_\sigma^*(\vartheta)$ since $\chi_\sigma^*(z) = \frac{z}{|z|} \cdot (\frac{z}{\bar{z}})^{\kappa_\sigma}$ for $\sigma \in \Sigma$ (cf. [Tat79, p.13]). From the assumption on the global root number $W(\chi^*) = \prod_v W(\chi_v^*) = 1$, we deduce that

$$\prod_{v \in \mathbf{h}} W(\chi_v^*) = \prod_{v \in \mathbf{h}} \chi_v^*(2\vartheta).$$

This implies that $\tau_{\mathcal{K}/\mathcal{F}}(\eta) = 1$, so we can write

$$\eta = \beta N_{\mathcal{K}/\mathcal{F}}(a), \beta \in \mathcal{F}_+, a \in \mathbf{A}_{\mathcal{K}}^\times.$$

Moreover by the approximation theorem, the idele a can be further chosen so that $a \equiv 1 \pmod{p^{\ell^r} \mathfrak{C}^N}$ for any sufficiently large N . Note that

$$W(\chi_v^*)\tau_{\mathcal{K}_w/\mathcal{F}_v}(\beta) = W(\chi_v^*)\tau_{\mathcal{K}/\mathcal{F}}(\eta_v) = \chi_v^*(2\vartheta).$$

For every sufficiently small ϵ , we have thus constructed $\beta \in \mathcal{F}_+ \cap O_{(p\mathfrak{F}\mathfrak{C}^c)}^\times$ such that

- $\beta \equiv u \pmod{l^r}$,
- $|\beta - \eta_v|_{\mathcal{F}_v} < \epsilon$ for all $v|p\mathfrak{C}\mathfrak{C}^c$,
- $W(\chi_v^*)\chi_v^*(\beta) = \chi_v^*(2\vartheta)$ for all $v \in \mathbf{h}$.

Here we let ϵ be sufficiently small so that $A_\beta(\chi_v) = A_{\eta_v}(\chi_v)$ for $v|\mathfrak{C}^-$. Recall that $v(\mathfrak{c}(R)) = 0$ for $v|p\mathfrak{C}\mathfrak{C}^c$ by our choice of ϑ . By Lemma 6.6 (3), we find that $v(\beta) \equiv v(\mathfrak{c}(R)) \pmod{2}$ for every inert place $v \nmid \mathfrak{C}^-$. It follows that there exists a fractional \mathfrak{a} of R such that

$$\prod_{\mathfrak{q}|\mathfrak{C}^-} \mathfrak{q}^{v_{\mathfrak{q}}(\beta)} = (\beta)\mathfrak{c}(R)N_{\mathcal{K}/\mathcal{F}}(\mathfrak{a})^{-1} = (\beta)\mathfrak{c}(\mathfrak{a}).$$

Define $\mathfrak{c} \in \mathbf{A}_{\mathcal{F},f}^\times$ as follows: $\mathfrak{c}_v = \beta^{-1}$ if $v \nmid p\mathfrak{C}\mathfrak{C}^c$, $\mathfrak{c}_v = 1$ if $v|p\mathfrak{C}\mathfrak{C}^c$. Then $\mathfrak{a}(\mathfrak{c}) = \mathfrak{c}(\mathfrak{a})$ by the choice of β and $\mathfrak{c}(\mathfrak{a})$. From Proposition 4.7, (4.7), (4.10), (4.12) and (4.18), we find that the β -th Fourier coefficient $\mathfrak{a}_\beta(\mathbb{E}_\chi^h, \mathfrak{c})$ of \mathbb{E}_χ^h at the cusp (O, \mathfrak{c}^{-1}) is given by

$$\begin{aligned} \mathfrak{a}_\beta(\mathbb{E}_\chi^h, \mathfrak{c}) &= \frac{1}{|D_{\mathcal{F}}|_{\mathbf{R}}} \cdot \prod_{v \in \mathbf{h}} W_\beta(\phi_{\chi, s, v}, \begin{bmatrix} \mathfrak{c}_v & \\ & 1 \end{bmatrix})|_{s=0} \\ &= \prod_{w|\mathfrak{F}} \chi_w(\beta) \cdot \prod_{v|\mathfrak{C}^-} A_\beta(\chi_v) \cdot |D_{\mathcal{F}}^{-1}|_{\mathcal{F}_v} \psi^\circ(-2^{-1}t_v\beta). \end{aligned}$$

It is clear that the non-vanishing of $\mathfrak{a}_\beta(\mathbb{E}_\chi^h, \mathfrak{c}) \pmod{\mathfrak{m}}$ is equivalent to

$$A_\beta(\chi_v) = A_{\eta_v}(\chi_v) \not\equiv 0 \pmod{\mathfrak{m}} \text{ for every } v|\mathfrak{C}^-. \quad \square$$

Now we are ready to prove our main result.

Theorem 6.8. *Suppose that l splits in \mathcal{K} . Let χ be a self-dual Hecke character such that*

- (L) $\mu_p(\chi_v) = 0$ for every $v|\mathfrak{C}^-$,
- (R) The global root number $W(\chi^*) = 1$,
- (C) \mathfrak{R} is square-free.

Then (NV) holds for (χ, l) .

PROOF. It suffices to verify that for each $v|\mathfrak{C}^-$ there exists $\eta_v \in \mathcal{F}_v^\times$ which satisfies (i) and (ii) in Proposition 6.7. For $v|\mathfrak{R}$, we take $\eta_v \in \mathcal{F}_v^\times$ such that $W(\chi_v^*) = \tau_{\mathcal{K}/\mathcal{F}}(\eta_v)\chi_v^*(2\vartheta)$. Note that the assumption (C) implies that $v \nmid 2$. By Proposition 4.4 (3) we find that

$$\begin{aligned} A_{\eta_v}(\chi) &= (\chi_v^*(-2\delta_v^{-1}d_{\mathcal{F}_v}) + \chi_v^*(2^{-1}\eta_v)W(\chi_v^*)) \cdot \chi_v(-2^{-1}d_F^{-1})|\varpi|^{\frac{1}{2}} \\ &= (\chi_v^*(\vartheta) + \chi^*(\vartheta))\chi_v(-2^{-1}d_F^{-1})|\varpi|^{\frac{1}{2}} \quad (2\vartheta = d_{\mathcal{F}_v}\delta_v) \\ &= 2\chi_v^*(\vartheta)\chi_v(-2^{-1}d_F^{-1})|\varpi|^{\frac{1}{2}} \not\equiv 0 \pmod{\mathfrak{m}}. \end{aligned}$$

For $v|\mathfrak{J}$, we choose η_v to be as in Lemma 6.4, so $A_{\eta_v}(\chi_v) \not\equiv 0 \pmod{\mathfrak{m}}$ and

$$v(\eta_v) = w(\mathfrak{C}^-) = a(\chi_v^*) + v(\mathfrak{c}(R)).$$

It follows from Lemma 6.6 (3) that $W(\chi_v^*) = \tau_{\mathcal{K}/\mathcal{F}}(\eta_v)\chi_v^*(2\vartheta)$. \square

Remark 6.9. We give a few remarks on Theorem 6.8:

- (1) The assumption (C) has been removed in view of [Hsi14, Prop. 6.3].
- (2) Let χ_1 be a self-dual character and ν be a finite order character such that ν has prime-to- p conductor and $\nu \equiv 1 \pmod{\mathfrak{m}}$. As pointed out by the referee, one can prove Theorem 6.8 for $\chi := \chi_1\nu$, keeping (L) and (C) but replacing (R) by the condition (Rm): $W(\chi^*) \equiv 1 \pmod{\mathfrak{m}}$, which implies the condition (R) for χ_1 . Indeed, as ν must have square-free conductor, (C) holds for χ_1 , and (L) obviously holds for χ_1 as well. Thus χ_1 satisfies the hypothese in Theorem 6.8, and for every $u \in O_{\mathfrak{l}}$ and r , we can choose $\beta \in \mathcal{F}_+$ as in the proof of Proposition 6.7 such that $\mathfrak{a}_{\beta}(\mathbb{E}_{\chi_1}^h, \mathfrak{c}) \not\equiv 0 \pmod{\mathfrak{m}}$. By the condition (L) the supports of the conductors of χ and χ_1 only differ by split primes, we find that $\mathfrak{a}_{\beta}(\mathbb{E}_{\chi}^h, \mathfrak{c}) \not\equiv 0 \pmod{\mathfrak{m}}$.
- (3) From the expression of $A_{\beta}(\chi)$ in (4.17), it is not difficult to deduce that if $\mu_p(\chi_v) > 0$ for some place $v|\mathfrak{C}^-$, then $A_{\beta}(\chi_v) \equiv 0 \pmod{\mathfrak{m}}$ for all $\beta \in \mathcal{F}_v^{\times}$, and hence $\mathbb{E}_{\chi}^h \equiv 0 \pmod{\mathfrak{m}}$ by q -expansion principle (cf. [Hsi14, Prop.6.2] for the self-dual case). In addition, in the self-dual case, we can deduce from [Hsi14, Lemma 6.1] that if $W(\chi^*) = -1$, then $\mathfrak{a}_{\beta}(\mathbb{E}_{\chi}^h, \mathfrak{c}) = 0$ for all β . It follows that $\mathbb{E}_{\chi}^h|_{\mathfrak{c}} = 0$.
- (4) The assumption that \mathfrak{l} splits in \mathcal{K} is only used in Proposition 6.7 to assure the local root number $W(\chi_{\mathfrak{l}}^*)$ for all $u \in O_{\mathfrak{l}}$ and r satisfies certain epsilon dichotomy $W(\chi_{\mathfrak{l}}^*)\tau_{\mathcal{K}_{\mathfrak{l}}/\mathcal{F}_{\mathfrak{l}}}(\eta_v) = \chi_{\mathfrak{l}}^*(2\vartheta)$ whenever $\eta_v \equiv u \pmod{\mathfrak{l}'}$. This is false for nonsplit \mathfrak{l} . For example, when \mathfrak{l} is inert, this dichotomy holds precisely when $v_{\mathfrak{l}}(\eta_v) \equiv v_{\mathfrak{l}}(\mathfrak{c}(R)) \pmod{2}$. To treat nonsplit \mathfrak{l} , it seems that one has to refine Theorem 3.2 in [Hid04a] (at least when \mathfrak{l} has degree one over \mathbf{Q}).

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