§ Jordan form I

$\dim_F V = n$

V : finite dim'l vector space

$T : V \to V$ linear transformation

Def. V is T-cyclic if

V is spanned by $\{ v, Tv, \dots, T^{n-1}v \}$

for some v

Theorem ☆☆

Suppose $V = \ker T^m$ for some m

(hence $V = \ker T^n$)

Then V is a direct sum of T-cyclic subspaces.

proof We shall prove by induction on dim'l of V.

If $\dim V = 1$, then V is T-cyclic by definition.

Consider the invariant subspace $\operatorname{Im} T = TV \subset V$

Since $T^n = 0$ ($\Rightarrow \det T = 0 \Rightarrow \ker T \neq 0$

$$\Rightarrow \dim_F \operatorname{Im} T < \dim_F V = n$$

$\operatorname{Im} T \subset V \Rightarrow T^n$ kills $\operatorname{Im} T$

By induction hypothesis, $\operatorname{Im} T$ is a direct sum of T-cyclic subspace. Namely, there exist $w_1, w_2, \dots, w_s \in \operatorname{Im} T$ such that

$$\begin{aligned} A' := \{ & w_1, Tw_1, \dots, T^{m_1-1} w_1, \ (T^{m_1} w_1 = 0) \\ & w_2, Tw_2, \dots, T^{m_2-1} w_2 \ (T^{m_2} w_2 = 0) \\ & \vdots \\ & w_s, Tw_s, \dots, T^{m_s-1} w_s, \ (T^{m_s} w_s = 0) \} \end{aligned}$$

is a basis of $\operatorname{Im} T = TV$.

- $\dim \operatorname{Im} T = m_1 + m_2 + \cdots + m_s$.

Consider $\{ T^{m_1-1} w_1, \ldots, T^{m_s-1} w_s \} \subset \ker T$

By definition, $\{ T^{m_1-1} w_1, \ldots, T^{m_s-1} w_s \}$ is linearly indep.

$\Rightarrow \quad r := \dim \ker T \geq s$.

Let $\{ T^{m_1-1} w_1, \ldots, T^{m_s-1} w_s, w_{s+1}, \ldots, w_r \}$ be

a basis of $\ker T$.

Choose $v_1, \ldots, v_s$ such that $T v_i = w_i$

$i = 1, \ldots, s$.

Set

$$\mathcal{A} = \left\{ \begin{array}{l} v_1, T v_1, \ldots, T^{m_1} v_1, \\ \vdots \\ v_s, T v_s, \ldots, T^{m_s} v_s, \\ v_{s+1} \\ \vdots \\ v_r \end{array} \right\}$$

The number of elements of $\mathcal{A} = m_1 + \cdots + m_s + r$

$= \dim \operatorname{Im} T$

$+ \dim \ker T$

$= n = \dim_F V$

<u>Claim</u> $\mathcal{A}$ is linearly indep.

If the claim is true, then

$$V = \bigoplus_{i=1}^{s} \operatorname{span}_F \{ v_i, Tv_i, \cdots, T^{m_i} v_i \}$$

$$\bigoplus_{k=s+1}^{r} F \cdot v_k \quad \text{is a direct}$$

sum of cyclic subspaces.

<u>Proof of the claim:</u>

If we have a linear relation

$$(*) \quad \sum_{i=1}^{s} \sum_{j=0}^{m_i} \alpha_{ij} T^j v_i + \sum_{k=s+1}^{r} \beta_k v_k = 0,$$

applying T on both sides of $(*)$, we obtain

$$\sum_{i=1}^{s} \sum_{j=0}^{m_i-1} \alpha_{ij} T^j w_i = 0 \quad (Tv_i = w_i)$$

$$\Rightarrow \alpha_{ij} = 0 \quad \forall \quad i = 1, \dots, s$$

$$j = 0, \dots, m_i - 1.$$

$\because \{w_1, \dots, T^{m_i-1}w_1, \dots, w_s, \dots, T^{m_s-1}w_s\}$ is a basis of $\operatorname{Im} T$)

$$(*) \Rightarrow \sum_{i=1}^{s} \alpha_{im_i} T^{m_i} v_i + \sum_{k=s+1}^{r} \beta_k v_k = 0$$

$$\|$$

$$\sum_{i=1}^{s} \alpha_{im_i} \cdot T^{m_i-1} w_i + \sum_{k=s+1}^{r} \beta_k v_k = 0$$

$$\Rightarrow \alpha_{1m_1} = \alpha_{1m_2} = \dots = \beta_{s+1} = \dots = \beta_r = 0$$

$\therefore \{T^{m_1-1}w_1, \dots, T^{m_s-1}w_s, v_{s+1}, \dots, v_r\}$

is a basis of $\operatorname{Ker} T$

$\square$.