Cayley Hamilton Theorem

Let $A \in M_n(F)$.

The characteristic polynomial of A

$$ch_A(x) := \det(x \cdot I_n - A)$$

Example $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$ch_A(x) = \det\begin{pmatrix} x-1 & -2 \\ -3 & x-4 \end{pmatrix} = (x-1)(x-4) - 6$$

$$= x^2 - 5x - 2.$$

In general, $ch_A(x)$ is a monic polynomial with coefficient in F.

We can write

$$ch_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

$$a_i \in F.$$

For any matrix $B \in M_n(F)$, we put

$$ch_A(B) := \underbrace{B^n + a_{n-1}B^{n-1} + \cdots + a_1 \cdot B + a_0 \cdot I_n}_{\in M_n(F)}$$

<u>Theorem</u> (Cayley - Hamilton)

$$ch_A(A) = O_n = 0 \cdot I_n.$$

<u>proof</u> We begin with

$$(*) \quad (x \cdot I_n - A)\, \mathrm{adj}(xI_n - A) = \det(xI_n - A) \cdot I_n$$

Write $g(x) := \mathrm{adj}(xI_n - A)$. Then

<u>Note that</u>

→ $g(x)$ is a polynomial with coefficient in $M_n(F)$!

In other words,

$$g(x) = C_{n-1} \cdot x^{n-1} + C_{n-2} \cdot x^{n-2} + \cdots + C_0$$

$$C_i \in M_n(F)$$

Therefore, $\circledast$ tells us that

$$(x \cdot I_n - A) \cdot (C_{n-1} \cdot x^{n-1} + C_{n-2} \cdot x^{n-2} + \cdots + C_0)$$

$$= ch_A(x) \cdot I_n = x^n \cdot I_n + a_{n-1} x^{n-1} + \cdots + a_0 \cdot I_n$$

Comparing the coefficients (in $M_n(F)$) on both sides, we find that

$$C_{n-1} = I_n$$

$$\begin{cases} C_{n-2} - A C_{n-1} = a_{n-1} \cdot I_n, \\ \vdots \\ C_{k-1} - A \cdot C_k = a_k \cdot I_n \\ \vdots \end{cases}$$

$$L \quad O - A \cdot C_0 = a_n \cdot I_n$$

Therefore, if we set $C_n = C_{-1} := O_n$, then

$$ch_A(A) := \sum_{k=0}^{n} a_k \cdot A^k \quad (a_n = 1)$$

$$= \sum_{k=0}^{n} A^k \cdot (C_{k-1} - A C_k)$$

$$= \sum_{k=0}^{n} A^k \cdot C_{k-1} - \sum_{k=0}^{n} A^{k+1} \cdot C_k$$

$$= C_{-1} \cdot A^0 - A^{n+1} \cdot C_n = O_n$$

$\square$.