Review :

Trace & Determinant

Theorem : There exists a unique function

$\det : M_n(F) \to F$ such that

① $\det([v_1 \cdots \alpha v_i + \beta v_i' \cdots v_n])$

$= \alpha \det([v_1 \cdots v_i, \cdots v_n]) + \beta \det([v_1 \cdots v_i' \cdots v_n])$

$\forall \quad \alpha, \beta \in F$

② $\det([v_1 \cdots v_n]) = 0$ if $v_i = v_j$ for some $i \neq j$

③ $\det(I_n) = 1$, $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ is the identity matrix.

Note that

② $\Rightarrow \det([v_1 \cdots v_i \quad v_j \cdots v_n]) = (-1) \det([v_1 \cdots v_j \cdots v_i \cdots v_n])$

proof: (Uniqueness) For brevity, we shall write D for the det function

Given $A = (a_{ij}) \in M_n(F)$,

we write $A = [v_1 \cdots v_n]$; $v_j = \sum_{i=1}^{n} a_{ij} e_i$

$$D(A) = D\left(\left[\sum_{i_1=1}^{n} a_{i_1 1} e_{i_1}, \sum_{i_2=2}^{n} a_{i_2 2} e_{i_2}, \cdots, \sum_{i_n=1}^{n} a_{i_n n} e_{i_n}\right]\right)$$

$$\overset{①}{=} \sum_{i_1=1}^{n} a_{i_1 1} D\left(\left[e_{i_1}, \sum_{i_2=1}^{n} a_{i_2} e_{i_2 2}, \sum_{i_3=1}^{n} a_{i_3 3} e_{i_3}, \cdots\right]\right)$$

$$\overset{①}{=} \sum_{\substack{(i_1 \cdots i_n) \\ \text{all permutations of} \\ (1,2,\cdots n)}} a_{i_1 1} a_{i_2 2} a_{i_3 3} \cdots a_{i_n n} D\left([e_{i_1}, e_{i_2}, \cdots, e_{i_n}]\right)$$

$$= \sum a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} D\left([e_1, \cdots, e_n]\right) \cdot (-1)^{\operatorname{sgn}(i_1 \cdots i_n)}$$

$$= \sum_{i} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} \cdot \left(\underbrace{\operatorname{sgn}(i_1 \cdots i_n)}_{\in \{\pm 1\}} \right)$$

This shows the uniqueness of D.

* <u>Existence</u> We construct D by induction on n.

Suppose we have constructed
$$D : M_{n-1}(F) \to F.$$

Define $D : M_n(F) \longrightarrow F$

$$D\left(\begin{bmatrix} a_1 & \cdots & a_n \\ u_1 & \cdots & u_n \end{bmatrix}\right) := a_1 \cdot D([u_2 \cdots u_n])$$

$a_i \in F$

$u_i \in F^{n-1}$

$$- a_2 D([u_1 \ \hat{u}_2 \cdots u_n])$$

$$+ a_3 D([u_1 \ u_2 \ \hat{u}_3 \cdots u_n])$$

$$- \vdots$$

$$+ (-1)^{n+1} a_n D([u_1 \cdots u_{n-1}])$$

$$= \sum_{k=1}^{n} (-1)^{k+1} a_k \cdot D([u_1 \cdots \hat{u}_k \cdots u_n])$$

Then one verifies that this function enjoys

①, ② & ③ easily.

For example, we check ② as follows:

$$D([v_1 \cdots v_i \cdots v_i \cdots v_n]) \overset{?}{=} 0 \qquad v_i := \begin{pmatrix} b \\ y \end{pmatrix} \quad \begin{matrix} b \in F \\ y \in F^{n-1} \end{matrix}$$

$$\|$$

$$\sum_{\substack{k=1 \\ k \neq i,j}}^{n} (-1)^{k+1} \cdot a_k \cdot \underbrace{D([w_1 \cdots \hat{w}_k, \cdots y, \cdots y \; w_n])}_{\overset{/\!/}{0} \text{ by ②}}$$

$$+ \quad (-1)^{i+1} \cdot b \cdot D([w_1, \cdots \hat{w}_i, \cdots y \cdots w_n])$$

$$+ \quad (-1)^{j+1} \cdot b \cdot D([w_1, \cdots y, \cdots \hat{w}_j \cdots w_n])$$

$$D([w_1, \cdots, \hat{w}_i, \cdots y \cdots w_n])$$

$$\|$$

$$0$$

$$= (-1)^{i+j+1} D([w_1 \cdots, y, \cdots \hat{w}_j \cdots w_n])$$

□