Review

V and W two finite dimensional vector spaces $/F$

$\mathcal{A} = \{ v_1 \dots v_n \}$ basis of V $\quad n = \dim_F V$

$\mathcal{B} = \{ w_1 \dots w_m \}$ " W $\quad m = \dim_F W$.

① $\mathrm{Hom}_F(V, W) \simeq M_{m\times n}(F)$

$$T \longmapsto [T]_{\mathcal{A},\mathcal{B}} = (a_{ij})$$

$$T(v_j) = \sum_{i=1}^{m} a_{ij} \cdot w_i \qquad j = 1, \dots, n.$$

If U is a vector space with a basis $\mathcal{C}$,

and $V \xrightarrow{T} W \xrightarrow{S} U$, $\ell = \dim_F U$, then we have

$$\boxed{\underbrace{[S \circ T]_{\mathcal{A},\mathcal{C}}}_{M_{\ell\times n}(F)} = \underbrace{[S]_{\mathcal{B},\mathcal{C}}}_{M_{\ell\times m}(F)} \cdot \underbrace{[T]_{\mathcal{A},\mathcal{B}}}_{M_{m\times n}(F)}}$$

② $M_{m\times n}(F) \xrightarrow{\sim} \mathrm{Hom}_F(F^n, F^m)$

$v = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} \in F^n$ column vector

$A \longmapsto A : v \mapsto A \cdot v$

matrix multiplication

$A = [A]_{\text{standard bases}}$

$$\begin{tikzcd} M_{l\times m}(F) \times M_{m\times n}(F) \arrow[r] & M_{l\times n}(F) \\ (A, B) \arrow[r, mapsto] \arrow[d, "\wr"] & A \cdot B \arrow[d, "\wr"] \\ \mathrm{Hom}_F(F^m, F^l) \times \mathrm{Hom}_F(F^n, F^m) \arrow[r] & \mathrm{Hom}_F(F^n, F^l) \\ (A, B) \arrow[r, mapsto] & A \circ B \end{tikzcd}$$

③ To each basis $\alpha = \{v_1 \cdots v_n\}$, we can associate an isomorphism: $e_\alpha : F^n \xrightarrow{\sim} V$.

defined by $e_\alpha(e_i) = v_i \quad i = 1, \cdots, n$

e_i is the standard basis $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \to$ i-th.

We have the diagram.

$$\begin{tikzcd} F^n \arrow[r, "e_{\mathcal{A}}", "\sim"'] \arrow[d, "{[T]_{\mathcal{A},\mathcal{B}}}"'] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "T"] \\ F^m \arrow[r, "e_{\mathcal{B}}", "\sim"'] & W \end{tikzcd}$$

Namely, $\forall$ column vector $v \in F^n$,

$$\begin{aligned} [T]_{\mathcal{A},\mathcal{B}} \cdot v &= e_{\mathcal{B}}^{-1}\left(T\left(e_{\mathcal{A}}(v)\right)\right) \\ &= e_{\mathcal{B}}^{-1} \circ T \circ e_{\mathcal{A}} (v) \end{aligned}$$

④ If $\mathcal{A}_1, \mathcal{B}_1$ are another bases of V and W,

then we have

$$[T]_{\mathcal{A},\mathcal{B}} = B^{-1} \cdot [T]_{\mathcal{A}_1,\mathcal{B}_1} \cdot A,$$

where
$$A = e_{\mathcal{A}_1}^{-1} \circ e_{\mathcal{A}} \in M_{n\times n}(F) = \mathrm{Hom}_F(F^n, F^n)$$
$$B = e_{\mathcal{B}_1}^{-1} \circ e_{\mathcal{B}} \in M_{m\times m}(F) = \mathrm{Hom}_F(F^m, F^m)$$

In particular, if $T: F^n \to F^m$ is given by a matrix denoted by T, then

$$[T]_{\mathcal{A},\mathcal{B}} = B^{-1} \cdot T \cdot A$$

$A = e_{\mathcal{A}} = [v_1, \cdots, v_n] \in M_{n\times n}(F)$ $v_i \in F^n$, $w_i \in F^m$ column vectors.

$B = e_{\mathcal{B}} = [w_1 \cdots w_m] \in M_{m\times m}(F)$

(This is because $T = [T]_{\text{Standard bases}}$

$\mathcal{A}_1, \mathcal{B}_1$: standard bases & $e_{\text{stand basis}}$ = the identity matrix.

<u>Example</u> $T: F^2 \to F^2$ $T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\mathcal{A} = \left\{ v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$ $\Rightarrow [T]_{\mathcal{A},\mathcal{B}} = \underbrace{\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}}_{B=[w_1\ w_2]}^{-1} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}}_{A=[v_1, v_2]}$

$\mathcal{B} = \left\{ w_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \right\}$

1) Gauss elimination. (高斯消去法)

Given $T: F^n \to F^m$, we can use "Gauss elimination" to find $\mathcal{A}$, $\mathcal{B}$ such that

$$[T]_{\mathcal{A},\mathcal{B}} = \begin{pmatrix} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \end{pmatrix}$$

<u>Example:</u>

$T: F^4 \to F^3$, $T = \begin{pmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & -2 & 2 \\ 3 & 11 & 6 & 34 \end{pmatrix}$.

$$\begin{array}{r} \\ \times(-1) \\ \times(-3) \end{array} \begin{pmatrix} 1 & 3 & 1 & 9 \\ 1 & 1 & -2 & 2 \\ 3 & 11 & 6 & 34 \end{pmatrix} \longrightarrow \begin{array}{r} \\ \\ \times(-1) \end{array} \begin{pmatrix} 1 & 3 & 1 & 9 \\ 0 & -2 & -3 & -7 \\ 0 & 2 & 3 & 7 \end{pmatrix} \begin{array}{l} \\ \times(-\frac{1}{2}) \\ \\ \end{array}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \longleftarrow \begin{pmatrix} 1 & 3 & 1 & 9 \\ 0 & 1 & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{cccc} & & \times(-\frac{3}{2}) & \times(-\frac{7}{2}) \end{array} \qquad\qquad \begin{array}{cccc} & \times(-3) & \times(-1) & \times(-9) \end{array}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This suggests

$$B^{-1} = \begin{pmatrix} 1 & & \\ & -\frac{1}{2} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -3 & -1 & -9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & -\frac{3}{2} & -\frac{7}{2} \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and

$$B^{-1} \cdot T \cdot A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By direct computation,

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & 0 \\ & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & -2 & \\ & & 1 \end{pmatrix} = \begin{array}{ccc} \left(\begin{matrix} 1 \\ 1 \\ 3 \end{matrix}\right. & \begin{matrix} 0 \\ -2 \\ -2 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right) \\ \| & \| & \| \\ w_1 & w_2 & w_3 \end{array}$$

$$A = \begin{pmatrix} 1 & -3 & \frac{7}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{7}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{cccc} \| & \| & \| & \| \\ v_1 & v_2 & v_3 & v_4 \end{array}$$

$\mathcal{A} = \{ v_1, v_2, v_3, v_4 \}$ basis of F^4

$\mathcal{B} = \{ w_1, w_2, w_3 \}$ basis of F^3.

Then $$[T]_{\mathcal{A}, \mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This shows $T(v_1) = w_1$; $T(v_2) = w_2$

$$T(v_3) = T(v_4) = 0$$

Thus we obtain

$$\ker T = \operatorname{span}_F \{ v_3, v_4 \}$$

$$\operatorname{Im} T = \operatorname{span}_F \{ w_1, w_2 \}$$

$\Rightarrow$ nullity of $T = 2$; $\operatorname{rank} T = 2$

Elementary matrices

(I) $\begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array}\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \vdots & \ddots & & \\ & & c & \cdots & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$

(II) $\begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array}\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & \cdots & 1 & \\ & & & \ddots & & \\ & & 1 & \cdots & 0 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$

(III) $\begin{array}{c} \\ \\ i \\ \\ \\ \end{array}\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & a & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$

Given a matrix $T \in M_{m\times n}(F)$

$$\begin{matrix}\text{column} \\ \text{row}\end{matrix} \text{ operations on } T = \begin{matrix}\text{right} \\ \text{left}\end{matrix} \text{ multiplication by elementary matrices.}$$

Theorem Let $T \in M_{m\times n}(F)$.

There exist $B \in M_{m\times m}(F)$, $A \in M_{n\times n}(F)$ products of elementary matrices such that

$$B^{-1} \cdot T \cdot A = \left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & 0 \\ & & 1 & \\ \hline & 0 & & 0 \end{array}\right)$$