Last time

(因隔離無法來習題課交作業同學，請將作業照相後email給助教)

V : vector space $/F$, $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$

- subspace
- $v_1, \cdots, v_n \in V$, $\underbrace{\text{span}_F\{v_1, \cdots, v_n\}}_{\text{subspace}} \subset V$
- V is finite dimensional $/F$

$\Leftrightarrow$ V is generated by finitely many vectors $/F$

($V = \text{span}_F\{v_1, \cdots, v_n\}$ for some non-negative integer n)

if $n = 0$, we set $V = \{0\}$. $\boxed{\dim_F V = 0 \Leftrightarrow V = \{0\}}$

- Basis : A subset $\{v_1, \cdots, v_n\} \subset V$ is called a basis of V if

(i) $V = \text{span}_F\{v_1, \cdots, v_n\}$

(ii) $\{v_1, \cdots, v_n\}$ is linearly independent.

- Replacement lemma :

If $V = \text{span}_F\{v_1, \cdots, v_n\}$, let $\{u_1, \cdots, u_k\}$ be a subset of V which is linearly independent, then (i) $k \le n$

(ii) $\exists$ k vectors in $\{v_1, \dots, v_n\}$, say

$v_1 \cdots v_k$, such that

$$\mathrm{span}_F\{u_1, \dots, u_k, v_{k+1}, \dots, v_n\} = \mathrm{span}_F\{v_1, \dots, v_n\}$$

- If V is finite dimensional, then $\overset{=V}{V}$ must have a basis,

Define $\dim_F V$ = the cardinality of a basis.

(= the number of distinct elements in a basis)

If V is not finite dimensional, then define $\dim_F V = \infty$.

Examples

① $V = F^n$, $e_1 = (1, 0 \cdots 0)$ $e_2 = (0, 1, 0 \cdots 0)$, $\cdots$

$e_n = (0, 0, \cdots, 1)$

Then $\{e_1, e_2, \dots, e_n\}$ is a basis of F^n.

$$\Rightarrow \dim_F F^n = n.$$

② $V = \{ (x, y, z) \in F^3 \mid 3x + 2y + z = 0 \}$

$= \{ (x, y, -3x - 2y) \mid x, y \in F \}$

$= \{ x \cdot (1, 0, -3) + y \cdot (0, 1, -2) \mid x, y \in F \}$

$= \mathrm{span}_F \{ (1, 0, -3), (0, 1, -2) \}.$

One checks that $\{ (1, 0, -3), (0, 1, -2) \}$

is linearly indep. $/F$, so

$\dim_F V = 2.$

③ $V = \left\{ (x_1, x_2, \cdots) \;\middle|\; \begin{array}{l} x_i \in F, \\ x_n + x_{n+1} = x_{n+2} \\ n = 1, 2, 3, \cdots \end{array} \right\}$

$\underline{+}$: $(x_1, x_2, \cdots) + (y_1, y_2, \cdots) := (x_1 + y_1, x_2 + y_2, \cdots)$

Scalar product $\alpha \in F$

$\alpha \cdot (x_1, x_2, \cdots) := (\alpha x_1, \alpha x_2, \alpha x_3, \cdots)$

One checks ($+$, scalar product) equips V with a structure of vector space $/F$.

$\dim_F V = 2$. In fact,

$V = \mathrm{span}_F\{ \underline{(0,1,1,2,3,\cdots)} ; \underline{(1,0,1,1,2,3,\cdots)} \}$.

<u>Theorem</u> ☆ Let V be a finite dimensional vector space $/F$ with $\dim_F V = n$.

Let $\{v_1, \cdots, v_m\}$ be a subset of V with $m > n$.

Then $\{v_1, \cdots, v_m\}$ is <u>linearly dependent</u>.

<u>proof</u>: If $\{v_1, \cdots, v_m\}$ is linearly indep., then $m \leq n$ by replacement lemma.

This is a contradiction. $\square$

Corollary. Consider the following system of linear equations.

$$(*)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots \ a_{nm}x_m = 0 \end{cases}$$

$a_{ij} \in F$.

If $m > n$, then

$(*)$ must have a non-trivial solution.

(非顯然)

or 非自明

(The trivial solution to $(*)$ is $x_1 = x_2 = \cdots = x_m = 0$).

$\Leftrightarrow \exists\ (x_1, \cdots, x_n) \neq (0, 0, \cdots, 0)$ satisfies $(*)$.

~~proof~~: Consider

$$v_1 = (a_{11}, a_{21}, \cdots, a_{n1}) \in F^n$$

$$v_2 = (a_{12}, a_{22}, \cdots, a_{n2})$$

$$\vdots$$

$$v_m = (a_{1m}, a_{2m}, \cdots, a_{nm}) \in F^n$$

$$\{v_1, \cdots, v_m\} \subset F^n.$$

$$\dim_n F^n = n \quad ; \quad m > n.$$

By Theorem ⓐ,

$\{v_1, \cdots, v_m\}$ must be linearly dependent.

$\exists \ (x_1, \cdots, x_m) \neq (0, \cdots 0)$ such that

$$x_1 v_1 + x_2 v_2 + \cdots + x_m v_m = 0$$

$$\Updownarrow$$

$(*)$ holds with $(x_1, \cdots, x_m)$. $\square$

Example $V = F[x]$ = the space of polynomials $/F$.

Then $\dim_F V = \infty$.

If $\dim_F V < \infty$, then $\dim_F V = n$ for some n.

By Theorem ☆, $\{1, x, x^2, \cdots, x^n\}$ must be linearly dependent. Namely, $\exists$ non-zero polynomial $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \cdot 1 = 0$

$\Rightarrow f(x) := a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \cdot 1$ has infinitely many roots in F. $\not\Rightarrow$.

Theorem ☆☆

Let V be a finite dimensional vector space $/F$.

Let $W \subset V$ be a subspace.

Then · W is finite dimensional and

· $\dim_F W \le \dim_F V$.

· If $\dim_F W = \dim_F V$, then $W = V$.

~~proof~~ Suppose W is not finite dimensional.

Then we can choose a sequence of subspaces of W.

$$\{0\} \subsetneq \operatorname{span}_F\{w_1\} \subsetneq \operatorname{span}_F\{w_1, w_2\} \subsetneq \cdots$$

$\neq W$

$$\subsetneq \operatorname{span}_F\{w_1, w_2, w_3, \ldots w_n\} \subsetneq \cdots \subset W$$

$(*)$ $w_n \notin \operatorname{span}_F\{w_1, w_2, \ldots, w_{n-1}\}$

$\forall n = 1, 2, \ldots$

Claim $\{w_1, \ldots, w_n\}$ is linearly independent.

$\because$ If $\{w_1, \cdots, w_n\}$ is <u>linearly dependent</u>,

then

$$\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_i w_i = 0 \qquad \underline{\alpha_i \neq 0}$$

$i \leq n.$

$\alpha_1, \alpha_2, \cdots, \alpha_i \in F$

$$\Rightarrow w_i = \left(-\frac{\alpha_1}{\alpha_i}\right) \cdot w_1 + \left(-\frac{\alpha_2}{\alpha_i}\right) \cdot w_2 + \cdots + \left(-\frac{\alpha_{i-1}}{\alpha_i}\right) \cdot w_{i-1}$$

$$\Rightarrow w_i \in \mathrm{span}_F\{w_1, w_2, \cdots, w_{i-1}\}$$

$\rightarrow\!\leftarrow$ $(*)$

This shows $\{w_1, \cdots, w_n\}$ is linearly independent $\forall n$

Replacement lemma $\Rightarrow \dim_F V \geq n \quad \forall n$

$\Rightarrow \dim_F V = \infty$ $\rightarrow\!\leftarrow$.

$\Rightarrow W$ is finite dimensional.

Let $\dim_F W = m$. Then W has a basis

$\{w_1, \cdots, w_m\}$ which is linearly indep $/F$

$\Rightarrow$ $\underset{\underset{\dim_F W}{\shortparallel}}{m} \leq \dim_F V$ by Theorem $\star$.

Now we prove that if $\dim_F W = \dim_F V$,

then $W = V$.

Suppose $W \subsetneq V$. $\exists\, v \in V$, $v \notin W$.

$W = \mathrm{span}_F\{w_1, \cdots, w_m\}$, $m = \dim_F W$

$= \dim_F V$.

$v \notin W \Rightarrow \{w_1, \cdots, w_m, v\}$ is linearly

independent $/F$.

$\Rightarrow \underset{\underset{m}{\shortparallel}}{\dim_F V} \geq m+1$ $\times$.

This implies $W = V$. $\square$

Sum of subspaces

W_1, W_2 : two subspaces of V

Define $W_1 + W_2 := \{ w_1 + w_2 \mid w_i \in W_i,\ i=1,2 \}$

$W_1 + W_2 \subset V$.

Then ① $W_1 + W_2$ is also subspace of V.

$W_1 + W_2$ contains W_1 and W_2.

(HW 2)

② $W_1 \cap W_2$ is also a subspace of V

$W_1 \cap W_2$ is contained in W_1 and W_2

HW2

$$\dim_F (W_1 + W_2) + \dim_F (W_1 \cap W_2) = \dim_F W_1 + \dim_F W_2$$

② Linear transformation.

Let V and W be two vector spaces.

We consider "morphisms" between V and W.

* "morphism" = maps from V to W which respect the structures of V and W as vector spaces $/F$.

Definition

A map $T: V \longrightarrow W$ is a linear transformation (homomorphism or morphism) if

(for all)

(i) $T(v_1 + v_2) = T(v_1) + T(v_2) \quad \forall v_1, v_2 \in V$

(ii) $T(\alpha \cdot v) = \alpha \cdot T(v) \quad \forall \alpha \in F, v \in V$.

By definition, $T(0) = 0$

$$(\because T(0+0) = T(0) + T(0)$$

$$\Rightarrow T(0) = T(0) + T(0)$$

$$)$$

$$\Rightarrow T(0) = 0$$

Examples

① $V = F^3$, $W = F^2$

$T : V \to W$

$$T(a_1, a_2, a_3) := (a_1 + a_2 + 3a_3, 4a_1, -a_3).$$

Then T is a linear transformation.

In terms of the standard basis

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

$f_1 = (1,0)$, $f_2 = (0,1)$

$T(a_1 e_1 + a_2 e_2 + a_3 e_3)$

$= a_1 \cdot (1, 4) + a_2 \cdot (1, 0) + a_3 \cdot (3, -1)$

$= a_1 \cdot (f_1 + 4 f_2) + a_2 \cdot f_1 + a_3 \cdot (3 f_1 - f_2)$

$=$

② $V = F[x]$, $W = F[x]$.

$T : V \longrightarrow W$

$T(f(x)) := x \cdot f(x)$ is also a linear transformation.

$T : V \to W$

$T(f(x)) := f'(x)$ is a linear transformation.

③ W is a subspace of V.

Then $W \hookrightarrow V$

the inclusion map $(w \longmapsto w)$

The inclusion map is a linear transformation.

Conversely, if $W \subset V$ is a subset

and W is a vector space $/F$,

then W is a subspace of V

$\Longleftrightarrow$ the inclusion map $W \hookrightarrow V$

is a linear transformation.

(4) $V = \mathbb{C}$ as vector space $/ \mathbb{R}$.

$\mathbb{C} = \operatorname{span}_{\mathbb{R}}\{1, \sqrt{-1}\}$

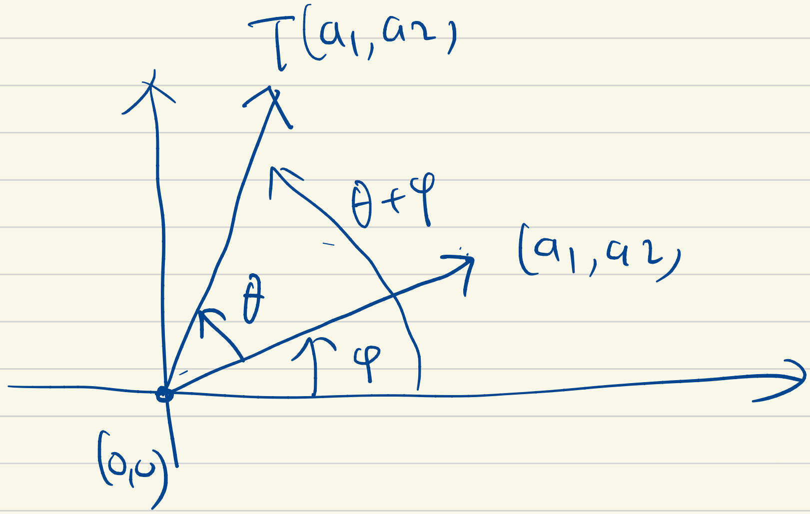
$T: \mathbb{C} \longrightarrow \mathbb{C}$

$z \longmapsto \bar{z}$: complex conjugation is a linear transformation.

⑤ $V = W = \mathbb{R}^2$, $0 \leq \theta \leq 2\pi$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (counterclockwise)

$(a_1, a_2) \longmapsto T(a_1, a_2) :=$ the rotation of (a_1, a_2) by θ around the origin $(0,0)$



T is also a linear transformation.

We use the polar coordinate.

$a_1 = r\cos\varphi$, $a_2 = r\sin\varphi$

$$T(a_1, a_2) = (r\cos(\varphi+\theta), \ r\sin(\varphi+\theta))$$

$$= (r(\cos\theta\cos\varphi - \sin\theta\sin\varphi), \ r(\cos\varphi\sin\theta + \sin\varphi\cos\theta))$$

$$= (r\cos\theta\cos\varphi - r\sin\theta\sin\varphi, \ r\cos\varphi\sin\theta + r\sin\varphi\cos\theta)$$

$$= \cos\theta\cdot(\underbrace{r\cos\varphi}_{= a_1}, \underbrace{r\sin\varphi}_{= a_2}) + \sin\theta\cdot(-\underbrace{r\sin\varphi}_{= a_2}, \underbrace{r\cos\varphi}_{= a_1})$$

$$\Rightarrow T(a_1, a_2) = \cos\theta\cdot(a_1, a_2) + \sin\theta\cdot(-a_2, a_1)$$

Then one can easily verfy T is a linear transformation.

⑥ $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(a_1, a_2) = (a_1^2, 0)$ is NOT

a linear transformation

$\because$

$T(2,0) = (4,0)$

$\nparallel$ $(2,0) = (1,0) + (1,0)$

$T(1,0) + T(1,0) = (1,0) + (1,0) = (2,0)$

- Isomorphism (同構 = 相同結構)

A linear transformation $T: V \to W$

is an isomorphism if

$\exists$ $S: W \to V$ linear transformation

such that $S \circ T = Id_V$ (:= the identity map $: V \to V$, $v \mapsto v$)

and $T \circ S = Id_W$ (identity map of W)