Linear algebra 0914

$F = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$

Def. A set V is a vector space over F if

V is equipped with

ⓐ Addition : $V \times V \longrightarrow V$

$$(v, w) \longrightarrow v + w$$

① $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$ (結合律)

② $v_1 + v_2 = v_2 + v_1$ (交換性)

③ There exists distinguished element 0 such that
($\exists$)

$$v + 0 = v \quad \forall\, v \in V.$$ (原點或零向量的存在性)

④ $\forall\, v \in V$, $\exists\, (-v)$ such that

$$v + (-v) = 0$$ (減法)

We can write $v - w$ for $v + (-w)$.

(b) Scalar product of F

$$F \times V \longrightarrow V$$
$$(\alpha, v) \longmapsto \alpha \cdot v$$

① $\alpha \cdot (v+w) = \alpha \cdot v + \alpha \cdot w \quad \forall\ v, w \in V$

② $(\alpha+\beta) \cdot v = \alpha \cdot v + \beta \cdot v \quad \forall\ \alpha, \beta \in F$

(分配律)

③ $(\alpha\beta) \cdot v = \alpha \cdot (\beta \cdot v)$

④ $1 \cdot v = v$

- We often write αv for $\alpha \cdot v$ if $\alpha \in F$, $v \in V$

- $2v = \underline{2 \cdot v} = (1+1) \cdot v = 1 \cdot v + 1 \cdot v = \underline{v + v}$.

$N : \geq 0$ integer

$$N v = \underbrace{v + v + \cdots + v}_{N\text{個}}$$

$\underset{\text{scalar}}{\underline{0}} \cdot v = \underline{0}$ ← vector $\in V$

F

$0 \cdot v + v$

$\overset{\vee}{=} (0+1) \cdot v$

$\overset{\vee}{=} 1 \cdot v \overset{\vee}{=} v$

$\Rightarrow \quad 0 \cdot v + (v + (-v)) = v + (-v)$

$\Rightarrow \quad 0 \cdot v = 0$.

· By definition, $\underline{-v} = (-1)\cdot v$

$$(\ v + (-v) = 0$$

$$v + (-1)\cdot v = 0$$

$$\Rightarrow\ -v = (-1)\cdot v.$$

Example ① $V = F^n = F \times \cdots \times F$

$$= \{(x_1, \ldots, x_n) \mid x_i \in F\}$$

zero vector $0 = (0, 0, \cdots, 0)$.

② $V = \mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$ $\quad x \boxplus y$

定義 $x, y \in V$, $\quad x \boxplus y := x \cdot y \in \mathbb{R}_{>0}$

加法

係數積 , $\quad \alpha \cdot x := x^{\alpha} \in \mathbb{R}_{>0}$, $\alpha \in \mathbb{R}$

$$(= e^{\alpha \log x})$$

Check $(V, \boxplus, \cdot)$ is a vector space over $\mathbb{R}$.

zero vector of $V = 1$ $\quad\Big|\quad (\alpha+\beta)\cdot x = x^{\alpha+\beta}$

$x \boxplus 1 = x$; $(-x) = x^{-1}$. $\quad\Big|\quad = x^{\alpha} \cdot x^{\beta} = (\alpha \cdot x) \boxplus (\beta \cdot x)$

Def Given $\{v_1, \cdots, v_n\} \in V$, and $v \in V$,

we say v is a linear combination of

$\{v_1 \cdots v_n\}$ (over F) if

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \quad \text{for}$$

some $\alpha_1, \cdots, \alpha_n \in F$.

$$(= \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \cdots + \alpha_n \cdot v_n)$$

Def $S = \{v_1, \cdots, v_n\} \subset V$.

$$\operatorname{span}_F S := \{ \alpha_1 v_1 + \cdots + \alpha_n v_n \mid \alpha_1, \cdots, \alpha_n \in F \}$$

↳ the subspace spanned by S.

- $\operatorname{span}_F S$ is a vector space over F

with the same addition and scalar product of V.

We need to verify $\left(\begin{array}{l} \text{"+"} \ \operatorname{span}_F S \times \operatorname{span}_F S \to \operatorname{span}_F S \\ F \times \operatorname{span}_F S \to \operatorname{span}_F S. \end{array} \right)$

① if $w_1, w_2 \in \operatorname{span}_F S$, then $w_1 + w_2 \in \operatorname{span}_F S$

② if $w \in \operatorname{span}_F S$, $\alpha \in F$, then $\alpha \cdot w \in \operatorname{span}_F S$.

③ $0 \in \mathrm{Span}_F S$ $\left(0 = 0 \cdot \alpha_1 + \cdots + 0 \cdot a_n \right)$

Def. Let V be a vector space over F.

A subset $W \subset V$ is called a

subspace of V

If W is a vector space with the same addition and scalar product with V.

Def. We say V is **finite dimensional** if there exists a finite subset $S \subset V$ such that

$$V = \mathrm{span}_F S.$$

Example ① F^n is finite dimensional

$$F^n = \mathrm{span}_F \{ \underbrace{(1,0,\cdots 0), (0,1,0 \cdots 0), \cdots, (0,0,\cdots,1)}_{n} \}$$

② $V = \{ (x, y, z) \in F^3 \mid x + y + 2z = 0 \}$.

is also finite dimensional.

$$V = \{ (x, -x-2z, z) \mid x, z \in F \}$$

$$= \{ x \cdot (1, -1, 0) + z \cdot (0, -2, 1) \mid x, z \in F \}$$

$$= \mathrm{span}_F \{ (1, -1, 0), (0, -2, 1) \}.$$

<u>Def.</u> A subset $\{ v_1, \cdots, v_n \}$ of V - sof. is linearly independent

if the only solution to

$$x_1 \cdot v_1 + x_2 \cdot v_2 + \cdots + x_n v_n = 0$$

$$x_i \in F$$

is $x_1 = x_2 = \cdots = x_n = 0$.

If $\{ v_1, \cdots, v_n \}$ is <u>NOT</u> linearly independent,

$\exists$ $x_1, \cdots, x_n \in F$, not all zero

(some $x_i \neq 0$)

s.t. $x_1 v_1 + x_2 v_2 + \cdots + x_i v_i + \cdots + x_n v_n = 0$

$$\Rightarrow v_i = \left(-\frac{x_1}{x_i}\right) \cdot v_1 + \left(-\frac{x_2}{x_i}\right) v_2 + \cdots + \left(-\frac{x_{i-1}}{x_i}\right) v_{i-1}$$

$$+ \left(-\frac{x_{i+1}}{x_i}\right) v_{i+1} + \cdots + \left(-\frac{x_n}{x_i}\right) v_n$$

$$\Rightarrow v_i \in \underline{\mathrm{Span}_F \{v_1, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n\}} = \mathrm{span}_F\{v_1, v_2, \cdots, v_i, v_n\}$$

if $V = \mathrm{Span}_F\, S$,

$S := \{v_1 \cdots v_n\}$ is NOT linearly independent,

then V can be spanned by

a proper subset $S' \subsetneq S$

We often say $\{v_1 \cdots v_n\}$ is <u>linearly dependent</u>

(線性相依)

if $\{v_1, \cdots, v_n\}$ is NOT linearly independent.

V : vector space over F

A non-empty subset $W \subset V$ is a subspace if

① $w_1 + w_2 \in W \quad \forall\ w_1, w_2 \in W$ (加法封閉性)

② $\alpha \cdot w \in W \quad \forall\ \alpha \in F,\ w \in W$

$(0 = 0 \cdot w \in W)$

Def. We say $\{v_1, \dots, v_n\} \subset V$ is a basis (基底) of V if

① $V = \text{span}_F\{v_1, \dots, v_n\}$

② $\{v_1, \dots, v_n\}$ is linearly independent (over F)

Example ① $V = F^3$, $e_1 = (1,0,0)$, $e_2 = (0,1,0)$

$e_3 = (0,0,1)$

$\{e_1, e_2, e_3\}$ is a basis of V

② $V = \{(x, y, z) \in F^3 \mid 3x + 2y + z = 0\}$

$\{v_1 = (1, 0, -3)\ ;\ v_2 = (0, 1, -2)\}$ is a basis of V.

$\boxed{\text{If } V \text{ is finite dimensional, then } V \text{ must have a basis.}}$

$\because V = \mathrm{span}_F\{v_1, v_2, \dots, v_n\}$

If $\{v_1, \dots, v_n\}$ is linearly dependent, then

some v_i, say v_n, belongs to $\mathrm{span}_F\{v_1, \dots, v_{n-1}\}$

$\Rightarrow V = \mathrm{span}_F\{v_1, \dots, v_{n-1}\}$

$\vdots$

Eventually, we can find a subset

of $\{v_1 \dots v_n\}$, which is linearly independent

and spans V.

Theorem Let $\{v_1, \dots, v_n\}$ be a basis of V.

Then every vector $v \in V$ can be written

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \qquad \alpha_i \in F$$

uniquely

proof. $V = \operatorname{span}_F \{v_1, \ldots, v_n\}$.

$\Rightarrow v \in V$, $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ for some $\alpha_i \in F$.

If $v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$, $\beta_i \in F$

$$= \alpha_1 v_1 + \cdots + \alpha_n v_n$$

$$\Rightarrow (\alpha_1 - \beta_1) \cdot v_1 + (\alpha_2 - \beta_2) \cdot v_2 + \cdots + (\alpha_n - \beta_n) \cdot v_n = 0$$

$\{v_1, \ldots, v_n\}$ is linearly indep. / F

$$\Rightarrow \alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \cdots = \alpha_n - \beta_n = 0$$

$$\Rightarrow \alpha_1 = \beta_1, \ldots, \alpha_n = \beta_n$$

This proves the uniqueness $\square$.

Rmk

A basis can be viewed as a "coordinate" of V.

(座標)

$$V \text{ "=" } F^n$$

$$v \longmapsto (\alpha_1, \cdots, \alpha_n) \quad (v = \alpha_1 v_1 + \cdots + \alpha_n v_n)$$

$$(F^n = \{ (x_1, \cdots, x_n) \mid x_i \in F \}$$

$$e_1 = (1, 0, \cdots 0), \quad e_2 = (0, 1, 0 \cdots 0)$$

$$\cdots\cdots)$$

Q $\{v_1 \cdots v_n\}$ & $\{w_1, \cdots w_m\}$ are two bases of V

Is $n = m$?? The answer is Yes!

Steinitz replacement lemma

$$V = \operatorname{span}_F \{u_1, u_2, \cdots, u_n\} \quad n \geq 1$$

$\{v_1, \cdots, v_k\} \subset V$ is linearly indep. $/F$.

$(v_i \neq 0)$

Then

① $k \leq n$

② $V = \mathrm{span}_F\{v_1, \cdots, v_k, u_{k+1}, \cdots, u_m\}$

for a subset $\{u_{k+1}, \cdots, u_m\} \subset \{u_1 \cdots u_n\}$

(possibly reordering).

~~proof~~: Prove by induction (數學歸納法) on k

If $k = 1$, $1 \leq n$, $\overset{\neq 0}{v_1} \in V$

$$\Rightarrow v_1 = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n, \quad \alpha_i \in F$$

some $\alpha_i \neq 0$, say $\alpha_1 \neq 0$

$$\Rightarrow u_1 = \frac{1}{\alpha_1} \cdot v_1 + \left(-\frac{\alpha_2}{\alpha_1}\right) u_2 + \cdots + \left(-\frac{\alpha_n}{\alpha_1}\right) u_n$$

$$\Rightarrow u_1 \in \mathrm{span}_F\{v_1, u_2, \cdots, u_n\}$$

$\Rightarrow \quad V = \operatorname{span}_F\{v_1, u_2, \ldots, u_n\}.$

$\square$

Suppose that $\{v_1, \ldots, v_{k+1}\}$ is linearly indep.

$$V = \operatorname{span}_F\{v_1, \ldots, v_k, u_{k+1}, u_{k+2}, \ldots, u_n\}$$

$v_{k+1} \in V$

$$\Rightarrow \quad v_{k+1} = \alpha_1 v_1 + \cdots + \alpha_k v_k$$
$$+ \alpha_{k+1} u_{k+1} + \cdots + \alpha_n u_n.$$

Some $\alpha_i \neq 0$ for $k+1 \leq i \leq n$ $(\Rightarrow k+1 \leq n)$

, say $\alpha_{k+1} \neq 0$

$$\Rightarrow \quad u_{k+1} = \left(-\frac{\alpha_1}{\alpha_{k+1}}\right) v_1 + \cdots + \left(-\frac{\alpha_k}{\alpha_{k+1}}\right) \cdot v_k + \frac{1}{\alpha_{k+1}} \cdot v_{k+1}$$
$$+ \left(-\frac{\alpha_n}{\alpha_{k+1}}\right) \cdot u_n$$

$$\Rightarrow \quad u_{k+1} \in \operatorname{span}_F\{v_1, \ldots, v_k, v_{k+1}, u_{k+2}, \ldots, u_n\}.$$

$\Rightarrow V = \mathrm{span}_F \{ v_1, \dots, v_{k+1}, u_{k+2}, \dots, u_n \}$ $\square$.

Corollary (推論)

If $\{v_1 \dots v_n\}$ & $\{w_1 \dots w_m\}$ is a basis of V, then $\underline{\underline{n = m}}$.

<u>proof</u> $V = \mathrm{span}_F \{ v_1 \dots v_n \}$

$\{w_1 \dots w_m\}$ is linearly indep.

By Replacement lemma, $m \le n$.

By symmetry, $n \le m \Rightarrow n = m$.

<u>Def.</u> If V is finite dimensional,

define $\dim_F V :=$ the number of a basis of V.

$\dim_F V^{<\infty}$ is <u>independent of the choice of bases.</u>

If V is not finite dimensional, then

define $\dim_F V = \infty$.

Example ① $V = F^n$, $\dim_F V = n$

② $V = \{ (x, y, z) \in F^3 \mid 3x + 2y + z = 0 \}$

$\dim_F V = 2$.