Recall

Let $(V, \langle , \rangle)$ be a Hilbert space $/ \mathbb{F}$.

Let $W \subset V$ be a closed subspace

Theorem: $\mathrm{Proj}_W(x)$ exists $\forall x \in V$.

$$\Leftrightarrow \quad V = W \oplus W^{\perp}$$

proof: ① $W \cap W^{\perp} = \{0\}$ $\left(W^{\perp} = \{ x \in V \mid \langle x, W \rangle = 0 \} \right)$

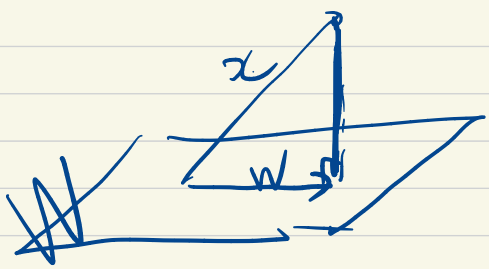
$x \in W \cap W^{\perp} \Rightarrow \|x\| = \langle x, x \rangle^2 = 0 \Rightarrow x = 0$

② $V = W + W^{\perp} \Leftrightarrow \forall x \in V$

there exists $w \in W$

such that $x - w \in W^{\perp}$

$\Leftrightarrow \langle x - w, W \rangle = 0$



(☆) If such w exists,

$\|x - w\|$ shoud be "minimum"

in $\{ \|x - y\|, y \in W \}$.

proof ②

$$d = \inf_{z \in W} \| z - x \|$$

$\forall\, n = 1, 2, 3, \cdots$

$\exists\, z_n \in W$ such that

$$d \le \| z_n - x \| < d + \frac{1}{n}.$$

Claim $\{z_n\}$ is a Cauchy sequence.

proof: $\forall\, n, m$

$$2\left(\| z_n - x \|^2 + \| z_m - x \|^2 \right) = \| z_n + z_m - 2x \|^2 + \| z_n - z_m \|^2$$

$$\Rightarrow \| z_n - z_m \|^2 = 2\left[\| z_n - x \|^2 + \| z_m - x \|^2 \right] - 4 \left\| \frac{z_n + z_m}{2} - x \right\|^2$$

$$< 2\left[\left(d + \frac{1}{n}\right)^2 + \left(d + \frac{1}{m}\right)^2 - 2 \left\| \frac{z_n + z_m}{2} - x \right\|^2 \right]$$

W

$$< 2\left[\left(d + \frac{1}{n}\right)^2 + \left(d + \frac{1}{m}\right)^2 - 2d^2 \right] \quad \left(\left\| \frac{z_n + z_m}{2} - x \right\| \ge d \right)$$

$$= \boxed{2\left[\frac{2d}{n} + \frac{2d}{m} + \frac{1}{n^2} + \frac{1}{m^2} \right]}$$

$\Rightarrow \| z_n - z_m \| \to 0$ as $n, m \to \infty$ $\square$.

V is Hilbert $\Rightarrow$ $z := \lim_{n\to\infty} z_n$ exists

W is closed $\Rightarrow$ z as a limit of vectors in W belongs to W.

$(z \in W)$ $\Downarrow$

$\|x - z\| = \lim_{n\to\infty} \|x - z_n\| = d$

Claim $x - z \in W^\perp$

Suppose $x - z \notin W^\perp$. Then there exists $y \in W$.

$$\langle x - z, y \rangle = 1$$

Consider $x - z + ty$, $t \in \mathbb{R}$.

$$= x - (\underbrace{z - ty}_{\in W})$$

$\cdot$ $\|x - z + ty\| \geq \|x - z\| = d$ $\forall t \in \mathbb{R}$

$$\|x - z + ty\|^2 \geq \|x - z\|^2$$

$\Downarrow$

$$t \cdot \langle x - z, y \rangle + t \cdot \langle y, x - z \rangle + t^2 \|y\|^2 \geq 0$$

$\Downarrow$

$t^2 \cdot \|y\|^2 + 2t \geq 0$ $\forall t \in \mathbb{R}$ $\to\!\!\!\!\!\times$

(For example; $t = -\frac{1}{\|y\|^2}$, $t^2\|y\|^2 + 2t$

$$= -\frac{1}{\|y\|^2} < 0)$$

This is a contradiction. So $x - z \in W^{\perp}$ $\square$.

<u>Remark</u>: We have seen there is an injective map:

$$\overline{V} \hookrightarrow V^{\vee}$$
$$v \longmapsto l_v : x \mapsto \langle x, v \rangle$$

If $\dim_F V < \infty$, then $\overline{V} \simeq V^{\vee}$.

$$(\dim V = \dim V^{\vee})$$

Let $(V, \langle , \rangle)$ be a Hilbert space.

$$V^{\vee}_{cont} := \{ l \in V^{\vee} \mid l : V \to F \text{ is continuous} \}$$

$$\Updownarrow$$

$$:= \{ l \in V^{\vee} \mid l|_{\{x \in V \mid \|x\| = 1\}} \text{ is bounded} \}$$

$$\Updownarrow$$

$$= \{ l \in V^{\vee} \mid \exists\, C > 0 \text{ s.t. } \forall x \in V,\ \|x\| = 1,\ |l(x)| \le C \}$$

$V^\vee_{cont}$: continuos linear functionals.

(bounded)

(Riesz representation theorem) $\subset V^\vee$.

<u>Theorem</u> $\overline{V} \xrightarrow{\sim} V^\vee_{cont} \subset V^\vee$ $((V, \langle , \rangle)$: Hilbert space$)$

$\Longleftrightarrow$ $\forall\, l \in V^\vee_{cont}$, there exists $y \in V$

such that $l(x) = \langle x, y \rangle \ \forall\, x \in V$.

<u>proof</u> Let $0 \neq l \in V^\vee_{cont}$

Consider $N := \{ x \in V \mid l(x) = 0 \}$

$= \ker l \quad (l : V \to F)$

l is continuous $\overset{\dagger}{\Longrightarrow}$ $N := \ker l$ is closed

$\boxed{\begin{array}{l} f: X \to Y \text{ continuous} \\ Z \subset Y \text{ is closed} \\ \Rightarrow f^{-1}(Z) \text{ is closed} \end{array}}$

$\Rightarrow$ N is a closed subspace.

$\Rightarrow$ $V = N \oplus N^\perp$.

$l \neq 0 \Rightarrow N^\perp \neq \{0\}$ $\quad \exists\, v \neq 0 \in N^\perp$

$l(v) \neq 0 \quad \left(\begin{array}{l} l(v) = 0 \Rightarrow v \in N \\ \Rightarrow v \in N \cap N^\perp = \{0\} \end{array} \right)$

$\forall\ x \in V \quad \ell(x) = \alpha \cdot \ell(v)$ for $\alpha \in F$

$$\left(\alpha = \frac{\ell(x)}{\ell(v)}\right)$$

$$\Rightarrow \ell(x - \alpha v) = 0$$

$$\Rightarrow x - \alpha v \in N \quad (N = \ker \ell)$$

$$\Rightarrow \langle x - \alpha v, v \rangle = 0 \quad (v \in N^\perp)$$

$$\Rightarrow \alpha \cdot \langle v, v \rangle = \langle x, v \rangle$$

$$\Rightarrow \ell(x) = \frac{\ell(v)}{\|v\|^2} \cdot \langle x, v \rangle$$

$$= \left\langle x, \frac{\overline{\ell(v)}}{\|v\|^2} \cdot v \right\rangle$$

$$y :=$$

$\square$.

$\dagger$ To see N is closed, we need to show if $x = \lim_{n\to\infty} x_n$, $x_n \in N$, then $x \in N$

$\Leftrightarrow \ell(x) = 0$

$\ell(x) = 0 \Leftarrow |\ell(x)| = 0 \Leftarrow |\ell(x)| \leq C \cdot \|x - x_n\|$, $\forall n$

ℓ is cont. $\Leftrightarrow \exists C > 0$ $|\ell(y)| \leq C\ \forall \|y\| = 1$

$$\Rightarrow \boxed{|\ell(y)| \leq C \cdot \|y\|,\ \forall y \in V} \Rightarrow |\ell(x - x_n)| \leq C \cdot \|x - x_n\|.$$

$$|\ell(x)| = |\ell(x - x_n)|$$

Rmk Riesz rep'n Thm. characterizes the image of $\overline{V}$ inside $V^\vee$ under $v \mapsto \ell_v$.

2 Adjoint of linear transformation.

$(V, \langle , \rangle_V)$, $(W, \langle , \rangle_W)$ inner product space.

$T : V \to W$ linear transformation.

Def. A linear transformation $T^* : W \to V$ is called adjoint of T if

$$\boxed{\langle T(v), w \rangle_W = \langle v, T^*(w) \rangle_V}$$

$\forall\ v \in V,\ w \in W.$

Example : V, W : finite dim'l. $\dim V = n$, $\dim W = m$.

Let $\mathcal{A} = \{v_1 \cdots v_n\}$ be an orthonormal basis of V

$\mathcal{B} = \{w_1 \cdots w_m\}$ " of W

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i, \quad [T]_{\mathcal{A},\mathcal{B}} = (a_{ij})$$

If T^* exists, say $[T^*]_{B,A} = (b_{ij})$,

$$\langle T^*(w), v \rangle = \langle w, T(v) \rangle$$

$$\Leftrightarrow \langle T^*(w_i), v_j \rangle = \langle w_i, T(v_j) \rangle \quad \forall i, j$$

$$\Leftrightarrow \left\langle \sum_{k=1}^{n} b_{ki} v_k, v_j \right\rangle = \left\langle w_i, \sum_{l=1}^{m} a_{lj} w_l \right\rangle$$

$$\Leftrightarrow b_{ji} = \overline{a_{ij}}$$

$$\Leftrightarrow \boxed{[T^*]_{B,A} = [T]_{A,B}^*}$$

T^*: exists if V, W are finite dim'l.

In what follows, we assume V and W are finite dim'l.

休息到 11:35.

⚠ The definition of T^* depends on the choices of inner products on V & W.

By definition, $T: V \to W$

- $T^{**} = T$
- $(\operatorname{Im} T)^{\perp} = \ker T^*$

$$\left(\begin{aligned} & w \in (\operatorname{Im} T)^{\perp} \\ \Leftrightarrow\ & \langle w, T(v) \rangle = 0 \quad \forall v \\ \Leftrightarrow\ & \langle T^*(w), v \rangle = 0 \quad \forall v \\ \Leftrightarrow\ & T^*(w) = 0 \end{aligned} \right)$$

- $(\ker T)^{\perp} = \operatorname{Im} T^*$.

保長變換

Def. ① $T: V \to W$ is an <u>isometry</u> if $T^* T = Id_V$

$$\Leftrightarrow \quad \| T(v) \| = \| v \| \quad \forall v \in V$$

$$\Leftrightarrow \quad \langle T(x), T(y) \rangle_W = \langle x, y \rangle_V \quad \forall x, y \in V.$$

② $T: V \to V$ is <u>unitary</u> if T is an isometry from V to itself.

③ $T: V \to V$ is <u>self-adjoint</u> if $T = T^*$

④ $T: V \to V$ is normal if

$$TT^* = T^*T$$

Lemma $T: V \to V$ is normal $\Leftrightarrow$ $\|T(x)\| = \|T^*(x)\|$ $\forall x \in V$

proof If $TT^* = T^*T$, then

$$\langle T(x), T(x) \rangle \overset{\checkmark}{=} \langle T^*T(x), x \rangle$$

$$= \langle TT^*(x), x \rangle$$

$$= \langle T^*(x), T^*(x) \rangle$$

$$\Rightarrow \|T(x)\|^2 = \|T^*(x)\|^2$$

$$\Rightarrow \|T(x)\| = \|T^*(x)\|.$$

Conversely, if $\|T(x)\| = \|T^*(x)\|$ $\forall x \in V$,

then $\forall x, y \in V$, $\|T(x+y)\|^2 = \|T^*(x+y)\|^2$

$$- \langle T(x+y), T(x+y) \rangle = \langle T^*(x+y), T^*(x+y) \rangle$$

$$\langle T(x), T(x)\rangle = \langle T^*(x), T^*(x)\rangle$$

$$\langle T(y), T(y)\rangle = \langle T^*(y), T^*(y)\rangle$$

$$\Rightarrow \langle T(x), T(y)\rangle + \overline{\langle T(x), T(y)\rangle} = \langle T^*(x), T^*(y)\rangle + \overline{\langle T^*(x), T^*(y)\rangle}$$

- $F = \mathbb{R}$, $\forall x, y \in F$

$$\Rightarrow \langle T(x), T(y)\rangle = \langle T^*(x), T^*(y)\rangle \quad \forall x, y \in V$$

$$\langle T^*T(x), y\rangle \qquad\qquad \langle TT^*(x), y\rangle$$

$$\Rightarrow T^*T = TT^*.$$

- $F = \mathbb{C}$, $\langle T(i\cdot x), T(y)\rangle + \overline{\langle T(i\cdot x), T(y)\rangle}$

$$= \langle T^*(i\cdot x), T^*(y)\rangle + \overline{\langle T^*(ix), T^*(y)\rangle}$$

$$\Rightarrow \langle T(x), T(y)\rangle - \overline{\langle T(x), T(y)\rangle}$$

$$= \langle T^*(x), T^*(y)\rangle - \overline{\langle T^*(x), T^*(y)\rangle}$$

$$\Rightarrow \langle T(x), T(y)\rangle = \langle T^*(x), T^*(y)\rangle$$

$\Rightarrow \langle T^*T(x), y\rangle = \langle TT^*(x), y\rangle$

$\Rightarrow \quad T^*T = TT^*. \quad \square$

<u>Rmk</u>

$T: V \to V$

(unitary) $\subset$ (normal)

(self-adjoint) $\subset$

<u>Theorem</u> (Spectral Theorem for normal operators)

If $T: V \to V$ is normal linear transformation,

then T is diagonalizable.

(In terms of matrices, given $A \in M_n(\mathbb{C})$,
$AA^* = A^*A$, there exists invertible
$P \in M_n(\mathbb{C})$ s.t. $PAP^{-1} = \begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$.
We may choose P to be unitary. $PP^* = I_n$)