9th homework
Due date: 11/25

There are five problems in total. Let F be either $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$.

Exercise 1. Recall that the $m \times m$ Jordan block J corresponding to λ is defined by

$$J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} \in M_m(\mathbf{R}).$$

Show that $(J - \lambda I_m)^m = 0$ and that if $r \geq m$, then

$$J^r = \begin{pmatrix} \lambda^r & r\lambda^{r-1} & \frac{r(r-1)}{2!}\lambda^{r-2} & \cdots & \frac{r(r-1)\cdots(r-m+2)}{(m-1)!}\lambda^{r-m+1} \\ 0 & \lambda^r & r\lambda^{r-1} & \cdots & \frac{r(r-1)\cdots(r-m+3)}{(m-2)!}\lambda^{r-m+2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^r \end{pmatrix}.$$

If $|\lambda| < 1$, find the limit $\lim_{r\to\infty} J^r$.

Exercise 2. Let

$$A = \begin{pmatrix} 3 & 4 & 6 & -2 \\ -1 & -2 & -5 & 2 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

Find the Jordan form J of A and an invertible $P \in M_4(\mathbf{Q})$ such that $P^{-1}AP = J$.

Exercise 3. Let $A \in M_n(\mathbf{C})$. Suppose that $\mathrm{ch}_A(x) = x^5(x-2)$ and that $\operatorname{rank} A^2 = \operatorname{rank} A^3$. Determine all possible Jordan forms of A.

Exercise 4. Let $A, B \in M_6(\mathbf{R})$ such that $A^6 = B^6 = 0$. Suppose that A and B have the same degree of the minimal polynomials and the same nullity ($\deg \mathrm{m}_A(x) = \deg \mathrm{m}_B(x)$ and $\dim \operatorname{Ker} A = \dim \operatorname{Ker} B$). Show that A and B are similar.

Exercise 5. Let V be a finite dimensional vector space over F. Let $T, S : V \to V$ be two linear transforms. Assume that T and S are both diagonalizable and that $ST = TS$. Prove that T and S are *simultaneously* diagonalizable. In other words, prove that there exist a basis $\mathscr{A}$ of V such that $[T]_{\mathscr{A}}$ and $[S]_{\mathscr{A}}$ are diagonal matrices. (Hint: First decompose V into a direct sum of eigenspaces of T and show that each T-eigenspace is S-invariant. Then use the fact that if S is diagonalizable, the restriction of $S|_W$ to any S-invariant subspace W is also diagonalizable.)