8th homework
Due date: 11/18

There are six problems in total. Let F be either $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$ and let n be a positive integer.

Exercise 1. Let $T : V \to V$ be a diagonalizable linear transformation and let

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_r}$$

be the decomposition of V as a driect sum of eigenspaces E_{λ_i} with distinct eigenvalues λ_i ($i = 1, 2, \ldots, r$). Let $v_1, v_2, \ldots, v_r$ be eigenvectors of T with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ respectively ($\iff v_i \in E_{\lambda_i}$ for $i = 1, 2, \ldots r$). Let W be a T-invariant subspace of V.

(1) Prove that if $v_1 + v_2 + \cdots + v_r \in W$, then $v_i \in W$ for every $i = 1, 2, \ldots, r$.
(2) Prove that $W = W_1 \oplus \cdots \oplus W_r$ with $W_i = W \cap E_{\lambda_i}$. This shows that $T|_W$ is also diagonalizable

Exercise 2. Let $V = M_2(F)$ be a four dimensional vector space. Let $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and define a linear transformation $T : V \to V$ by

$$T(B) = AB - BA.$$

Compute the characteristic polynomial $\mathrm{ch}_T(x)$ and show that T is diagonalizable.

Exercise 3. Let V be a vector space with $\dim_F V = n$. Let $T : V \to V$ be a linear transformation.

(1) Let $f \in F[x]$. If λ is an eigenvalue of T, show that $f(\lambda)$ is an eigenvalue of $f(T)$.
(2) If $\operatorname{rank} T = 1$, prove that either T is diagonalizable or $T^n = 0$.

Exercise 4. Let $a, b \in \mathbf{R}$. Prove that the matrix

$$\begin{pmatrix} 1 + a & b \\ 1 & 1 - a \end{pmatrix}$$

is diagonalizable in $M_2(\mathbf{R})$ if and only if $a^2 + b > 0$.

Exercise 5. Let $A \in M_3(\mathbf{R})$. Suppose that $\mathrm{ch}_A(x)$ has exactly one root in $\mathbf{R}$ and has no multiple roots. Show that there exists an invertible $P \in M_3(\mathbf{R})$ such that

$$P^{-1}AP = \begin{pmatrix} a & d & e \\ 0 & 0 & b \\ 0 & 1 & c \end{pmatrix}$$

with $c^2 + 4b < 0$.

Exercise 6. Let

$$A = \begin{pmatrix} 3 & 1 & 0 & -1 \\ -1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

(1) Find the minimal polynomial $\mathrm{m}_A(x)$ of A.
(2) Show that there exists an invertible $P \in M_4(\mathbf{R})$ such that

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & * & * \\ 0 & 2 & * & * \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$