7th homework
Due date: 11/11

Let F be either $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$.

Exercise 1. Let $\lambda_1, \dots \lambda_n \in \mathbf{C}$. Put

$$(0.1)\quad f(x) := (1-\lambda_1 x)(1-\lambda_2 x)\dots(1-\lambda_n x) = 1 + c_1 x + \cdots + c_n x^n.$$

For $k = 1, 2, \dots,$ define

$$p_k := \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k.$$

Prove Newton's identity: for every $1 \le j \le n$

$$-jc_j = c_{j-1}p_1 + c_{j-2}p_2 + \cdots c_1 p_{j-1} + p_j \quad (c_0 := 1).$$

(Hint: Take the derivative of $f(x)$ on both sides of (0.1). Show the left-hand side becomes

$$f(x)\sum_{j=0}^{n} \frac{-\lambda_i}{1-\lambda_i x} = \sum_{j=1}^{n} jc_j x^{j-1}.$$

Then use the identity $\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k$ and compare the x^{j-1}-coefficients of both sides.)

Exercise 2. Let $A \in M_n(F)$. Use Exercise 1 to show that if

$$\mathrm{Tr}(A) = \mathrm{Tr}(A^2) = \cdots = \mathrm{Tr}(A^n) = 0,$$

then $A^n = 0$.

Exercise 3. Let $A \in M_n(F)$ such that $A^n = 0$ but $A^{n-1} \neq 0$.

(1) Show that there exists $v \in F^n$ such that $\{v, Av, A^2v, \dots, A^{n-1}v\}$ is a basis of F^n.

(2) If $B \in M_n(F)$ such that $AB = BA$, prove that

$$B = a_0 I_n + a_1 A + a_2 A^2 + \cdots a_{n-1}A^{n-1}$$

for some $a_0, \dots, a_{n-1} \in F$.

Exercise 4. Let $A, B \in M_n(F)$. Suppose that $\mathrm{ch}_A(x)$ has n distinct roots in F and $AB = BA$. Show that B is a polynomial in A. Namely, show that there exists $a_1, a_2, \dots, a_n \in F$ such that

$$B = a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1}A + a_n I_n.$$

Exercise 5. Let W be a finite dimensional vector space over F. If W_1 is a subspace of W, show that there is a subspace W_2 such that $W = W_1 \oplus W_2$ is a direct sum. (Hint: use the *extension lemma* in 2nd homework).