6th homework
Due date: 10/28

There are seven problems in total. Let F be either $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$. Recall that for $A \in M_{m\times n}(F)$, we let $\operatorname{rank} A$ be the rank of the linear transformation

$$A : F^n \to F^m, \quad v \mapsto A \cdot v.$$

In particular, if we write $A = [v_1 v_2, \dots v_n]$ with $v_n \in F^m$, then $\operatorname{rank} A = \dim_F \operatorname{span}_F \{v_1, v_2, \dots, v_n\}$.

Likewise, if $T : V \to V$ is a linear transformation, fixing a basis $\mathscr{A}$ of V, we define the characteristic polynomial of T by

$$\operatorname{ch}_T(x) := \operatorname{ch}_{[T]_{\mathscr{A}}}(x).$$

By $\operatorname{ch}_{C^{-1}AC}(x) = \operatorname{ch}_A(x)$, this definition of $\operatorname{ch}_T(x)$ does not depend on the choice of basis of V.

Exercise 1. Use Cramer's rule to solve the following system of linear equations

$$\begin{aligned} 2x_1 + 6x_2 + 4x_3 =& 2 \\ 3x_1 + 3x_2 + x_3 =& 1 \\ 5x_1 + 8x_2 + 4x_3 =& 3. \end{aligned}$$

Exercise 2. Find $a \in \mathbf{R}$ such that $(3, a, a)$ belongs to the space spanned by $(-1, -3, 6)$ and $(6, 8, 1)$.

Exercise 3. Let $A \in M_{m\times n}(F)$. Show that

$$\operatorname{rank} A = \operatorname{rank} A^t.$$

(Hint: use the Gauss elimination).

Exercise 4. Let $A \in M_m(F)$ be an invertible matrix and let $B \in M_{m\times n}(F)$, $C \in M_{n\times m}(F)$ and $D \in M_n(F)$. Show that if

$$\operatorname{rank} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \operatorname{rank} A,$$

then $D = CA^{-1}B$. (Use the Gauss elimination).

Exercise 5. Let V be a finite dimensional vector space of $\dim_F V = n$. Let $T : V \to V$ be a linear transformation such that

$$T^2 = 5T - 4 \cdot \mathrm{Id}_V$$

and let $\mathrm{ch}_T(x)$ be the characteristic polynomial (recall Id_V is the identity map). Let

$$V_1 = \{v \in V \mid T(v) = v\}, \quad V_2 = \{v \in V \mid T(v) = 4v\}$$

be subspaces of V. Show that

(1) $V = V_1 + V_2$ and $V_1 \cap V_2 = \{0\}$.
(2) $\mathrm{ch}_T(x) = (x-1)^r(x-4)^{n-r}$. Here $r = \dim_F V_1$.

Exercise 6. Let $A \in M_n(F)$. Suppose that $\mathrm{rank}\, A = 1$. Show that

$$\det(I_n + A) = \mathrm{Tr}(A) + 1.$$

Here $\mathrm{Tr}(A)$ is the trace of A.

Exercise 7. Show that

$$\det \begin{pmatrix} x_1 & a & a & \cdots & a \\ b & x_2 & a & \cdots & a \\ b & b & x_3 & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & x_n \end{pmatrix} = \frac{af(b) - bf(a)}{a-b},$$

where $f(x) = (x_1 - x)(x_2 - x) \cdots (x_n - x)$.
(Hint: Use Exercise 6 or prove by induction on n).

Exercise 8. Let Let

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 3 & -4 & 1 \\ 3 & -8 & 5 \end{pmatrix}.$$

Find an invertible $P \in M_3(\mathbf{R})$ such that $P^{-1}AP$ is diagonal.