4th homework
Due date: 10/14

There are five problems in total. Let F be either $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$.

Exercise 1. Let $A, B \in M_{n\times m}(F)$. Show that

$$\operatorname{rank}(A+2B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

Exercise 2. Let $T : F^3 \to F^3$ be the linear transformation given by the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 1 & -2 & 4 \end{pmatrix}.$$

Let

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}$$

be a basis of F^3.

(1) Compute the matrix representation $[T]_{\mathcal{A}} \in M_3(F)$ of T with respect to $\mathcal{A}$;

(2) Find a invertible matrix $P \in M_3(F)$ such that $[T]_{\mathcal{A}} = P^{-1}AP$.

Exercise 3. Let $T : F^4 \to F^4$ be the linear transformation given by the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 1 & -1 \\ -1 & 1 & 0 & -3 \\ 1 & -2 & -1 & 1 \end{pmatrix}.$$

(1) Find the the nullity and rank of T;

(2) determine a basis of the kernel and the image of T.

Exercise 4. Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 4 & -1 \\ 0 & 1 & -1 \end{pmatrix} \in M_3(\mathbf{R}).$$

(1) Find elementary matrices $E_1, E_2, \ldots, E_n$ such that

$$E_n E_{n-1} \cdots E_1 A = I_3.$$

(2) Use (1) to find a matrix Q such that $QA = I_3$ (so $Q = A^{-1}$).

Note that this exercise exhibits a method to find the inverse of an invertible matrix by the Gauss elimination.

Exercise 5. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{R})$ be a matrix such that

$$\mathrm{Tr}(A) := a + d = 0.$$

Suppose that $A \neq 0$.

(1) Show that there exists $v \in \mathbf{R}^2$ such that $\{v, Av\}$ is a basis of $\mathbf{R}^2$.

(2) Use (1) to show there exists an invertible $P \in M_2(\mathbf{R})$ with

$$P^{-1}AP = \begin{pmatrix} 0 & e \\ 1 & 0 \end{pmatrix}, \quad e = a^2 + bc.$$