3rd homework
Due date: 10/07

As usual, F denotes either $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$, and V denotes a finite dimensional vector space over F.

Exercise 1.

(1) Let $T : F^2 \to F^3$ be a linear transformation such that
$$T(1,2) = (1,1,1);\ T(2,1) = (1,-1,3).$$
Find $T(5,2)$.

(2) Show that there does NOT exist a linear transformation $T : F^3 \to F^2$ such that
$$T(1,0,3) = (2,1),\ T(1,-1,-5) = (1,4);\ T(2,-1,-2) = (3,9).$$

Exercise 2. Let W be the vector space over F defined by
$$W = \left\{(x,y,z,w) \in F^4 \mid 2x+3y+z+w = 0,\ -2x-3y+3z-w = 0\right\}.$$
Find the dimension and a basis of W.

In the following two exercises, you may need to use the definitions or results in Exercise 5 in the 2nd homework.

Exercise 3. Let $T : V \to V$ be a linear transformation. Show that $V = \operatorname{Ker} T + \operatorname{Im} T$ if and only if $\operatorname{Ker} T \cap \operatorname{Im} T = \{0\}$.

Exercise 4. Let $T : V \to V$ be a linear transformation such that $T^2 = T$. Let
$$U = \{v \in V \mid T(v) = v\}, \quad W = \{v \in V \mid T(v) = 0\}.$$
Show that

(1) U and W are subspaces of V,
(2) $U \cap W = \{0\}$,
(3) $V = U + W$.

Exercise 5. Let $P_3(F)$ be the space of polynomials with coefficients in F of degree ≤ 3. Let $T : P_3(F) \to P_3(F)$ be the linear transformation defined by
$$T(f(x)) := f(2x+3).$$
Let $\mathcal{A} = \{1,\ 1+x,\ 1+x^2,\ 2x+x^3\}$ be a basis of $P_3(F)$. Write down the matrix of T with respect to the basis $\mathcal{A}$.

Exercise 6. Let $T : V \to V$ be a linear transformation. Assume that

$$\operatorname{rank} T = \operatorname{rank} T^2.$$

Show that $\operatorname{Ker} T \cap \operatorname{Im} T = \{0\}$.

Exercise 7. Let $T : V \to V$ be a linear transformation. Suppose that $T^m = 0$ for some positive integer m. Show that $T^n = 0$, where $n = \dim_F V$.