10th homework
Due date: 5/26

Exercise 1. Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbf{C}$ be $2n$ complex numbers. Let

$$f(x) = \prod_{i=1}^{n}(x - \alpha_i), \quad g(x) = \prod_{j=1}^{n}(x - \beta_j)$$

be two polynomials of degree n. Let $R_{f,g}$ be the resultant of f and g. Show that

$$\det R_{f,g} = \pm \prod_{i=1}^{n}\prod_{j=1}^{n}(\alpha_i - \beta_j).$$

Exercise 2. Let $f(X)$ and $g(X)$ be polynomials in $\mathbf{C}[x]$ with $n = \deg f = \deg g$. Let $r(x)$ be the remainder of the division of $f(x)$ by $g(x)$. Prove that

$$\det R_{f,g} = (-1)^n \det R_{g,r}.$$

Here $R_{f,g}, R_{g,r} \in \mathrm{M}_{2n}(\mathbf{C})$.

Exercise 3. Let $A \in \mathrm{M}_n(\mathbf{C})$ and $B \in \mathrm{M}_n(\mathbf{C})$. Let $f(x) = \mathrm{ch}_A(x)$ and $g(x) = \mathrm{ch}_B(x)$ be the characteristic polynomials of A and B respectively. Consider the linear transformation $T : \mathrm{M}_{n\times n}(\mathbf{C}) \to \mathrm{M}_{n\times n}(\mathbf{C})$ defined by $T(X) := AX - XB$. Show that

$$\det R_{f,g} = \pm \det T.$$

The following exercises are the applications of Jacobi-Darboux Theorem discussed in the classroom:

Exercise 4. Let $f(x) = x^3 + bx + c \in \mathbf{C}[x]$. Show that $f(x)$ has no multiple roots if and only if $4b^3 + 27c^2 \neq 0$.

Exercise 5. Find the number of $(x, y) \in \mathbf{C}^2$ such that

$$xy^2 - y + x^2 + 1 = 0, \quad x^2y^2 + y - 1 = 0.$$