8th homework
Due date: 5/12

Exercise 1. If A and B are stochastic matrices, then show that AB is also a stochastic matrix.

Exercise 2. Let

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix} \in \mathrm{M}_4(\mathbf{R}).$$

Show that A is a regular matrix. Find an eigenvector $u \in \mathbf{R}^4_{>0}$ of A with eigenvalue $\rho(A)$.

Exercise 3. Let A be an irreducible matrix in $\mathrm{M}_n(\mathbf{R})$ (so A is non-negative by definition). Let $B \neq A$ be a matrix in $\mathrm{M}_n(\mathbf{R})$ such that $B - A$ is non-negative. Show that $\rho(B) > \rho(A)$.

Recall that a matrix $A \in \mathrm{M}_n(\mathbf{R})$ is irreducible if A is non-negative and for any $1 \leq i, j \leq k$ there exists k such that the (i, j)-entry of A^k is positive.

Exercise 4. Let $A \in \mathrm{M}_n(\mathbf{R})$ be an irreducible matrix. Show that $(A + \epsilon I_n)^{n-1}$ is positive for any $\epsilon > 0$.

Exercise 5. Let A be an irreducible matrix in $\mathrm{M}_n(\mathbf{R})$. If $\lambda \in \mathbf{C}$ is an eigenvalue of A with $|\lambda| = \rho(A)$, show that

$$\lambda^N = \rho(A)^N$$

for some positive integer N (Hint: You may use Exercise 6).

Exercise 6. Let $A = (a_{ij}) \in \mathrm{M}_n(\mathbf{R})$ be an irreducible matrix and let $B = (b_{ij}) \in \mathrm{M}_n(\mathbf{R})$ be a matrix such that $|b_{ij}| \leq a_{ij}$ for all $1 \leq i, j \leq n$. Let λ be an eigenvalue of B such that $\lambda = e^{i\theta}\rho(A)$. Show that $|b_{ij}| = a_{ij}$ and $B = e^{i\theta} \cdot DAD^{-1}$, where $D = \mathrm{diag}(d_1, \ldots, d_n)$ and $|d_i| = 1$.