7th homework
Due date: 5/5

Exercise 1. Find the singular values of the matrix

$$A = \left(\begin{array}{c|ccc} 1 & 1 & \cdots & 1 \\ \hline 1 & & & \\ \vdots & & -I_n & \\ 1 & & & \end{array}\right)$$

What are the eigenvalues of A?

Exercise 2. Let $A \in \mathrm{M}_{m\times n}(\mathbf{R})$ and $B \in \mathrm{M}_{r\times s}(\mathbf{R})$ and $C \in \mathrm{M}_{m\times s}(\mathbf{R})$ The matrix equation

$$AXB = C$$

has a solution $X \in \mathrm{M}_{n\times r}(\mathbf{R})$ if and only if $AA^{\dagger}CB^{\dagger}B = C$. Moreover, the solutions of $AXB = C$ are of the form

$$X = A^{\dagger}CB^{\dagger} + Y - A^{\dagger}AYBB^{\dagger},$$

where $Y \in \mathrm{M}_{n\times r}(\mathbf{R})$

For $x = (x_1, \ldots, x_n)^{\mathrm{t}} \in \mathbf{C}^n$, define

$$\|x\| := \sqrt{|x_1|^2 + \ldots |x_n|^2}.$$

For $A \in \mathrm{M}_n(\mathbf{C})$, define the spectral norm

$$\|A\|_{\mathrm{sp}} := \sup_{x\in\mathbf{C}^n, \|x\|=1} \|Ax\| = \sup_{x\in\mathbf{C}^n, x\neq 0} \frac{\|Ax\|}{\|x\|}.$$

Recall that $\rho(A)$ denotes the spectral radius of a square matrix A.

Exercise 3. (1) For $A, B \in \mathrm{M}_n(\mathbf{C})$, prove

$$\|AB\|_{\mathrm{sp}} \leq \|A\|_{\mathrm{sp}}\|B\|_{\mathrm{sp}}; \quad \|A+B\|_{\mathrm{sp}} \leq \|A\|_{\mathrm{sp}} + \|B\|_{\mathrm{sp}}.$$

(2) Show that $\|A\|_{\mathrm{sp}} = \sqrt{\rho(AA^*)}$ the maximal singular value of A.

Exercise 4. Let $A \in \mathrm{M}_n(\mathbf{R})$ be a stochastic matrix. Namely, $A = (a_{ij})$ with

$$a_{ij} \geq 0, \quad \sum_{i=1}^{n} a_{ij} = 1 \text{ for all } j = 1, \ldots, n.$$

Show that the spectral radius $\rho(A) = 1$.

Exercise 5 (15pts)**.** Let

$$A = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{pmatrix} \in \mathrm{M}_3(\mathbf{R}).$$

(1) Find a vector $u = (u_1, u_2, u_3)^{\mathrm{t}} \in \mathbf{R}^3$ such that $Au = u$, $u_i > 0$ and $u_1 + u_2 + u_3 = 1$.

(2) Let $w = (w_1, w_2, w_3)^{\mathrm{t}} \in \mathbf{R}^3$ be an eigenvector of the transpose A^{t} with the eigenvalue 1. Let $v = (v_1, v_2, v_3)^{\mathrm{t}} \in \mathbf{R}^3$ be any vector. Show that

$$\lim_{k\to\infty} A^k v = \lambda u \text{ for some } \lambda \in \mathbf{R}.$$

(3) Show that λ is non-zero if and only if

$$v_1 w_1 + v_2 w_2 + v_3 w_3 \neq 0.$$