6th homework
Due date: 4/28

Exercise 1. Let V be a finite dimensional vector space over $\mathbf{Q}$ and let $\langle\, , \,\rangle : V \times V \to \mathbf{Q}$ be an inner product. Put

$$\mathrm{O}(V) = \{ \text{ linear transformation } T : V \to V \mid \|T(v)\| = \|v\| \text{ for every } v \in V\}\,.$$

Prove that if $x, y \in V$ with $\|x\| = \|y\|$, then there exists $T \in \mathrm{O}(V)$ such that $T(x) = y$.

Exercise 2 (Bonus)**.** If $A \in M_n(\mathbf{Q})$ such that $A^{\mathrm{t}} = -A$, show that $\det A \in \mathbf{Q}^2$. Namely, $\det A$ is a square in $\mathbf{Q}$.

Exercise 3 (20pts)**.** Find the singular value decomposition and the Moore-Penrose inverse for the following matrices:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}.$$

Exercise 4. Let $\sigma_1, \ldots, \sigma_n$ be the singular values of $A \in M_n(\mathbf{C})$. Prove that the eigenvalues of $\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ are equal to $\sigma_1, \ldots, \sigma_n, -\sigma_1, \ldots, -\sigma_n$.

Exercise 5. Let $A \in M_n(\mathbf{C})$. Show that AA^* is similar to A^*A.